EXTRA HOMEWORK EXERCISES NO.2

1. Homework set No.2 (in addition to Exercises in the notes)

The due date of this second set is May 9, 2008. The exercises in Sections 2 and 3 of the lecture notes are also included (exercises in the lecture notes are easy, and some of the exercises here is slightly more demanding).

(1) If $\phi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves, prove that $U \mapsto \mathrm{Ker}(\phi(U))$ is a sheaf.

(2) If $\phi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves inducing an isomorphism at all stalks, prove that ϕ is an isomorphism.

(3) Let P and P' be two points of $\mathrm{Spec}(A)$ (P could be equal to P'). If $A/P \cong A/P'$, prove that there exists an affine scheme $\mathrm{Spec}(A')$ containing two copies $X \cong \mathrm{Spec}(A)$ and $Y \cong \mathrm{Spec}(A)$ of $\mathrm{Spec}(A)$ inside $\mathrm{Spec}(A')$ as closed subschemes such that $X \cap Y$ is exactly isomorphic to $\mathrm{Spec}(A/P) \cong \mathrm{Spec}(A/P')$ and $X \cup Y = \mathrm{Spec}(A')$.

(4) Let $\mathbb{G}_m = \mathrm{Spec}(\mathbb{Z}[t, t^{-1}])$. As a functor $\mathbb{G}_m : ALG \to AB$, $\mathbb{G}_m$ is an affine group with $\mathbb{G}_m(R) = R^\times$. The multiplication $m : \mathbb{G}_m \times \mathbb{G}_m \to \mathbb{G}_m$ given by $m_R : \mathbb{G}_m(R) \times \mathbb{G}_m(R) \to \mathbb{G}_m(R)$ sending $(a, b) \in R^\times \times R^\times$ to its product ab is a functor morphism. Note here $\mathrm{Hom}_{SCH}(\mathbb{G}_m \times \mathbb{G}_m, \mathbb{G}_m) \cong \mathrm{Hom}_{alg}(\mathbb{Z}[t, t^{-1}], \mathbb{Z}[t, t^{-1}] \otimes_{\mathbb{Z}} \mathbb{Z}[t, t^{-1}])$ canonically. Write this isomorphism $\phi \mapsto \phi^\#$. What is the corresponding algebra homomorphism $m^\# : \mathbb{Z}[t, t^{-1}] \to \mathbb{Z}[t, t^{-1}] \otimes_{\mathbb{Z}} \mathbb{Z}[t, t^{-1}]$? In other words, find the value $m^\#(t) \in \mathbb{Z}[t, t^{-1}] \otimes_{\mathbb{Z}} \mathbb{Z}[t, t^{-1}]$.

(5) Let $\mathbb{G}_{m/\mathbb{F}_2} = \mathrm{Spec}(\mathbb{F}_2[T, T^{-1}])$ for a variable T. Consider the scheme automorphism group $\mathrm{Aut}_{SCH}(\mathbb{G}_{m/\mathbb{F}_2})$ and the scheme endomorphism semi-group $\mathrm{End}_{SCH}(\mathbb{G}_{m/\mathbb{F}_2})$. For the multiplication map $m_R : \mathbb{G}_m(R) \times \mathbb{G}_m(R) \to \mathbb{G}_m(R)$ given by $m_R(a, b) = a \cdot b$, define the addition on $\mathrm{End}_{SCH}(\mathbb{G}_m)$ by $\phi + \psi = m \circ (\phi \times \psi)$ and multiplication by $\phi \circ \psi$. Is $\mathrm{End}_{SCH}(\mathbb{G}_m)$ a commutative ring? Determine $\mathrm{Aut}(\mathbb{G}_{m/\mathbb{F}_2})$ and $\mathrm{End}_{SCH}(\mathbb{G}_{m/\mathbb{F}_2})$. If we replace the base ring $\mathbb{F}_2$ in the above definition by $\mathbb{Z}$, is $\mathrm{End}_{SCH}(\mathbb{G}_{m/\mathbb{Z}})$ still a commutative ring?

(6) Let $\mathbb{G}_a = \mathrm{Spec}(\mathbb{C}[t])$ for a variable t. Prepare one more copy $\mathbb{G}'_a = \mathrm{Spec}(\mathbb{C}[s])$, and identify

$$\mathbb{G}_a - V((t)) = \mathrm{Spec}(\mathbb{C}[t, t^{-1}]) \cong \mathrm{Spec}(\mathbb{C}[s, s^{-1}]) = \mathbb{G}'_a - V((s))$$

via the isomorphism: $t \mapsto s^{-1}$. Write the resulting scheme $\mathbf{P}^1_{/\mathbb{C}} = \mathbb{G}_a \cup \mathbb{G}'_a$. Compute $\Gamma(\mathbf{P}^1_{/\mathbb{C}}, \mathcal{O}_{\mathbf{P}^1})$.

(7) Show that $\mathbf{P}^1$ is not affine.

(8) Show that the topological space of the image of f for $f \in \mathrm{Hom}_{\mathbb{C}-SCH}(\mathbf{P}^1, \mathrm{Spec}(A))$ is a closed subset made of a single point for any $\mathbb{C}$-algebra A. What happens if we replace $\mathbb{C}$ by $\mathbb{Z}$ in the definition of $\mathbf{P}^1$ and take any commutative ring A instead of a $\mathbb{C}$-algebra A?

(9) Regarding $\mathbf{P}^1$ as a functor from $\mathbb{C}$-ALG into $SETS$, show $\mathbf{P}^1(\mathbb{C}[T]) = \mathbf{P}^1(\mathbb{C}(T))$ for a variable T, where $\mathbb{C}[T]$ is the polynomial ring and $\mathbb{C}(T)$ is the field of rational functions of T (that is, the field of fractions of $\mathbb{C}[T]$). In other words, show

$$\mathbf{P}^1(\mathbb{C}[T]) = \mathrm{Hom}_{\mathbb{C}-SCH}(\mathrm{Spec}(\mathbb{C}[T]), \mathbf{P}^1) = \mathrm{Hom}_{\mathbb{C}-SCH}(\mathrm{Spec}(\mathbb{C}(T)), \mathbf{P}^1) = \mathbf{P}^1(\mathbb{C}(T)).$$

Date: April 21, 2008.