EXTRA HOMEWORK EXERCISES NO.1

1. Homework set No.1 (in addition to Exercises in the notes)

The due date of this first set is April 18, 2008. The exercises in Section 1 of the lecture notes are also included.

(1) Show that $S_A : B\text{-}ALG \to SETS$ is a covariant functor.
(2) Show that $A \mapsto S_A$ is a contravariant functor from the category of B-algebras into the category of covariant functors from $B\text{-}ALG$ into $SETS$.
(3) For any ring A, there is at most one morphism $S_A \to S_{\mathbb{Z}}$ of functors.
(4) Show that $S_{\mathbb{Q}} \to S_{\mathbb{Z}}$ is flat but not finite.
(5) Show that $S_{\mathbb{F}_p} \to S_{\mathbb{Z}}$ is finite but not flat.
(6) Show that $S_{\mathbb{Q}} \to S_{\mathbb{Z}}$ is not faithfully flat.
(7) For $A = \mathbb{Z}[X]/(X^2)$, compute $A[\frac{1}{X}]$.
(8) For a ring A, show that, if $f \in A$ is outside any proper ideal of A, $f \in A^\times$.
(9) For an element $0 \neq f$ of a $\mathbb{Z}$-algebra A, show that $S_A(R) = S_{\overline{A}}(R) \sqcup S_{A_f}(R)$ if R is a field.
(10) Let $K = \mathbb{Q}[\sqrt{-1}]$, and consider $K \otimes_{\mathbb{Q}} K$ as K-algebra by $K \ni a \mapsto a \otimes 1 \in K \otimes_{\mathbb{Q}} K$. Prove that $K \otimes_{\mathbb{Q}} K \cong K \oplus K$ as K-algebras.
(11) Let $K = \mathbb{F}_9$ (the field of 9 elements), and consider $K \otimes_{\mathbb{F}_3} K$ as K-algebra by $K \ni a \mapsto a \otimes 1 \in K \otimes_{\mathbb{Q}} K$. Prove that $K \otimes_{\mathbb{F}_3} K \cong K \oplus K$ as K-algebras.
(12) Let $\mathbb{F}_2$ be the field of 2 elements. Let $k = \mathbb{F}_2(x^2)$ inside $\mathbb{F}_2(x)$ (rational function field). Consider $K \otimes_k K$ by $K \ni a \mapsto a \otimes 1$. Compute $\dim_k K$ and $\dim_K(K \otimes_k K)$. Prove or disprove that $K \otimes_k K \cong K \oplus K$ as K-algebras (hint: consider $(x \otimes 1 - 1 \otimes x)^2$).
(13) Determine the $\mathbb{Q}$-algebra structure of $K \otimes_{\mathbb{Q}} K$ if $K/\mathbb{Q}$ is a cubic extension of $\mathbb{Q}$.
(14) Take $\mathbb{G}_{a/\mathbb{Q}}$ (so, $B = \mathbb{Q}$ and $\mathbb{G}_a = S_{\mathbb{Q}[X]}$). Let $K = \mathbb{Q}[\sqrt{-1}]$, and consider $\mathbb{G}_{a/K} := S_K \times_{\mathbb{Q}} \mathbb{G}_a$.
   (a) Show that $\mathbb{G}_{a/K} = S_{K[X]}$ taking $B = K$.
   (b) Since $\mathbb{G}_{a/\mathbb{Q}}(R) = R = \mathbb{G}_{a/K}(R)$ for any K-algebra R, we can think of $\sqrt{-1} \in \mathbb{G}_{a/K}(K) = \mathbb{G}_{a/\mathbb{Q}}(K)$. Since $\sqrt{-1} \in \mathbb{G}_{a/K}(K) = \mathrm{Hom}_{K\text{-alg}}(K[X], K)$, taking K-algebra homomorphism $\underline{\phi} : K[X] \to K$ corresponding to $\sqrt{-1}$, we have an inclusion $\phi : S_{K/K} \subset \mathbb{G}_{a/K}$ $(B = K)$. Show that ϕ is a closed immersion. How many elements are there in $S_{K/K}(K)$?
   (c) Take $B = \mathbb{Q}$. Consider $\underline{\Phi} \in \mathrm{Hom}_{\mathbb{Q}\text{-alg}}(K[X], K)$ corresponding to $\sqrt{-1} \in \mathbb{G}_{a/\mathbb{Q}}(K) = \mathrm{Hom}_{\mathbb{Q}\text{-alg}}(\mathbb{Q}[X], K)$ and $\Phi : S_{K/\mathbb{Q}} \to \mathbb{G}_{a/\mathbb{Q}}$ induced by Φ. Is Φ a closed immersion? How many elements are there in $S_{K/\mathbb{Q}}(K)$?

Date: April 4, 2008.