2. Study of p-adic analytic families via cohomology groups

We studied the duality theorem between Hecke algebra and the space of modular forms in [LEC] Proposition 2.18. The duality is useful to recover the space of cusp forms $S_2(\Gamma_0(N), A)$ as the linear dual $\mathrm{Hom}_A(h(N,A),A)$ and the set of Hecke eigenforms is in bijection with $\mathrm{Spec}(h(N;A))(A) = \mathrm{Hom}_{A\text{-alg}}(h(N,A),A)$. We try to construct a Hecke algebra $\mathbb{H}^{ord}(N;\Lambda)$ (which is a Λ-algebra) so that $M^{ord}(N;\Lambda)$ is the Λ-linear dual of $\mathbb{H}^{ord}(N;\Lambda)$. Then we study the ring structure of this algebra which eventually lead to a proof of Theorem 1.37.

The idea of getting $\mathbb{H}^{ord}(N;\Lambda)$ is as follows. Taking the limit increasing the level by p-power, we create a big module, call it X. Recall $Y_1(Np^r) = X_1(Np^r) - \{\text{cusps}\} = \Gamma_1(Np^r)\backslash\mathfrak{H}$. The module X can be the projective limit $\varprojlim_n X_n$ for cohomology groups $X_n = H_1^{ord}(Y_1(Np^n), \mathbb{Z}_p)$ or its Pontryagin dual $\varinjlim_n H^1_{ord}(Y_1(Np^n), \mathbb{Q}_p/\mathbb{Z}_p)$ or $\varprojlim_n X_n$ for $X_n = S_2^{ord}(\Gamma_1(Np^\infty); \mathbb{Z}_p)$. Here the subscript or superscript "*ord*" indicate the image of the projector $e = \lim_{n\to\infty} U(p)^{n!}$. Note that $(\mathbb{Z}/Np^r\mathbb{Z})^\times$ acts on X_n by the diamond operator $\langle\cdot\rangle$, and $\Gamma/\Gamma^{p^{r-1}} \hookrightarrow (\mathbb{Z}/p^r\mathbb{Z})^\times \subset (\mathbb{Z}/Np^r\mathbb{Z})^\times$. Thus $W[[\Gamma]] := \varprojlim_r W[\Gamma/\Gamma^{p^{r-1}}]$ acts on X. Note that this algebra $W[[\Gamma]]$ is isomorphic to Λ by sending $\gamma \in \Gamma \subset W[[\Gamma]]$ to $1+T$. Then we define $\mathbb{H}^{ord}(N;\Lambda)$ to be the Λ-subalgebra of $\mathrm{End}_\Lambda(X)$ generated by Hecke operators. An important point is that the algebra $\mathbb{H}^{ord}(N;\Lambda)$ does not depends on the choice of X_n.

2.1. **Fundamental group.** By the theory of fundamental group $\pi_1(X)$ for a Riemann surface X, we have a universal covering $\pi : U \twoheadrightarrow X = X \cong \pi_1(X)\backslash X$ for a simply connected space U. The projection is unramified, and U is determined uniquely (up to isomorphisms; see, for example, [AFC] Chapter 3). Since $\mathfrak{H}$ is simply connected, if a subgroup $\Delta \subset PSL_2(\mathbb{Z})$ of finite index is torsion-free, $\Delta \cong \pi_1(\Delta\backslash\mathfrak{H})$.

Here is a brief sketch of the construction of U and the fundamental group: A path $\gamma : x \to y$ on X is a continuous map γ from the interval $[0,1]$ into X (under the Euclidean topology on X) such that $\gamma(0) = x$ and $\gamma(1) = yx$. Two paths $\gamma, \gamma' : x \to x$ are homotopy equivalent (for which we write $\gamma \approx \gamma'$) if there is a bi-continuous map $\varphi : [0,1]\times[0,1] \to X$ such that $\varphi(0,t) = \gamma(t)$ and $\varphi(1,t) = \gamma'(t)$. Fix an origin $x \in X$. Let U be the set of all equivalence classes of paths emanating from x.

More generally, for each complex manifold M, we can think of the space $U = U(M)$ of homotopy classes of paths emanating from a fixed point $x \in M$. An open neighborhood U of x is called *simply connected* if $U(U) \cong U$ by projecting $(\gamma : x \to y)$ down to y. For example, if U is homeomorphic to an open disk with center x, it is simply connected (that is, every loop is equivalent to x). If $\gamma : x \to y$ and $\gamma' : y \to z$ are two paths, we define their product path $\gamma\gamma' : x \to z$ by

$$\gamma\gamma'(t) = \begin{cases} \gamma(2t) & \text{if } 0 \le t \le 1/2 \\ \gamma'(2t-1) & \text{if } 1/2 \le t \le 1. \end{cases}$$

By this multiplication, $\pi_1(M) = \pi_1(M,x) = \{\gamma \in U(M) | \gamma : x \to x\}$ becomes a group called *the fundamental group* of M. Taking a fundamental system of neighborhoods $\mathcal{U}_y$

of $y \in M$ made of simply connected open neighborhoods of y, we define a topology on $U(M)$ so that a fundamental system of neighborhoods of $\gamma : x \to y$ is given by $\{\gamma U | U \in \mathcal{U}_x\}$. Then $\pi_1(M)$ acts on $U(M)$ freely without fixed points. By definition, we have a continuous map $\pi : \pi_1(M)\backslash U(M) \to M$ given by $\pi(\gamma : x \to y) = y$, which is a local isomorphism. Since $(\pi)^{-1}(x) = \{x\}$, $\pi : \pi_1(M)\backslash U(M) \cong M$ is a homeomorphism. Since $\pi : U(M) \to M$ is local isomorphism, we can regard $U(M)$ as a complex manifold. This space $U(M)$ is called a universal covering space of M.

Fix a base point $x \in X$ and cut X by 1-cycles α_i passing through x so that the resulting space is a simply connected polygon P, α_i generates $\pi_1(X)$ (basically by definition). If X is compact, and writing the edges of P counterclockwise as $\alpha_1, \beta_1, \alpha_2, \beta_2, \ldots, \alpha_g, \beta_g$, then $\pi_1(X)$ is generated by these paths with only relation

$$\prod_{j=1}^{g}(\alpha_j, \beta_j) = (\alpha_1, \beta_1)(\alpha_2, \beta_2)\cdots(\alpha_g, \beta_g) = 1, \tag{2.1}$$

where $(\alpha, \beta) = \alpha\beta\alpha^{-1}\beta^{-1}$ (a commutator). If $Y = X - S$ for a finite sets of points (outside the fixed x), ordering $S = \{s_1, \ldots, s_c\}$ according to the orientation of P (from left to right), we can take a path π_j starting at x encircling s_j, cutting P by π_j, we get a new polygon of $2g + c$ sides. In this case, $\pi_1(Y)$ is generated by $\{\pi_i, \alpha_j, \beta_j\}_{i,j}$ with only relation

$$\prod_{i=1}^{c}\pi_i \prod_{j=1}^{g}(\alpha_j, \beta_j) = \pi_1\pi_2\cdots\pi_c(\alpha_1, \beta_1)(\alpha_2, \beta_2)\cdots(\alpha_g, \beta_g) = 1. \tag{2.2}$$

2.2. **Cohomology groups.** Suppose that $Y = X$ (so $S = \emptyset$) is a compact Riemann surface or an open Riemann surface of the form $Y = X - S$ as in (2.2) for a compact Riemann surface X. Then by (2.1) and (2.2), we have

(2.3)

$$\pi_1^{ab}(Y) = \frac{\pi_1(Y)}{(\pi_1(Y), \pi_1(Y))} \cong \bigoplus_{j=1}^{g}(\mathbb{Z}[\alpha_j] \oplus \mathbb{Z}[\beta_j]) \cong H_1(X, \mathbb{Z}), \quad \text{if } Y = X \text{ is compact,}$$

$$\pi_1^{ab}(Y) \cong \left(\bigoplus_{j=1}^{c-1}\mathbb{Z}[\pi_j]\right) \oplus \left(\bigoplus_{j=1}^{g}\mathbb{Z}[\alpha_j] \oplus \mathbb{Z}[\beta_j]\right) \cong H_1(Y, \mathbb{Z}), \quad \text{otherwise.}$$

Thus the genus of X is given by g.

Exercise 2.1. *Prove* (2.3).

In particular, we have

Lemma 2.2. *We have*

$$H^1(Y, A) = \mathrm{Hom}(\pi_1(Y), A).$$

Take an abstract group G to introduce group cohomology. For a ring A, take a $A[G]$-module M (where $A[G]$ is the group ring of G). We define a general group cohomology $H^q(G,M)$ for $q=0,1,2$. We have

$$H^0(G,M)=M^G=\{m\in M|gm=m \text{ for all } g\in G\},$$

and if M is finite, the first cohomology is defined by

$$H^1(G,M)=\frac{\{G\xrightarrow{c}M|c(\sigma\tau)=\sigma c(\tau)+c(\sigma)\text{ for all }\sigma,\tau\in G\}}{\{G\xrightarrow{b}M|b(\sigma)=(\sigma-1)x\text{ for }x\in M\text{ independent of }\sigma\}}.$$

As for the second cohomology, 2-cocycles $c:G\times G\to M$ are functions satisfying the following relation:

$$c(\alpha,\beta)+c(\alpha\beta,\gamma)=\alpha\cdot c(\beta,\gamma)+c(\alpha,\beta\gamma)$$

for all $\alpha,\beta,\gamma\in G$. For any function $b:G\to M$, we define

$$\partial b(\alpha,\beta)=b(\alpha\beta)-\alpha b(\beta)-b(\alpha). \tag{2.4}$$

Then ∂b is easily checked to be a 2-cocycle by computation. Such 2-cocycles obtained from $b:G\to M$ is called a 2-coboundary. Then

$$H^2(G,M)=\frac{\{\text{2-cocycles with values in }M\}}{\{\text{2-coboundaries with values in }M\}}.$$

Exercise 2.3. *Check that ∂b as above is a 2-cocycle.*

We can define cohomology group $H^q(G,M)$ of general degree q, but we apply this theory for a torsion-free subgroup Δ of finite index in $SL_2(\mathbb{Z})$, and in this case, we know that $H^q(\Delta,M)=0$ for $q\geq 2$ (see [CGP] Example 5 page 217 or [LFE] Proposition 6.1.1 for $q=2$); so, we do not touch in details the general definition (which can be found in [CGP]).

Exercise 2.4. *Prove that $\Gamma_1(N)$ is torsion-free if $N\geq 4$.*

Let $\overline{\Gamma}_0(N)$ be the image of $\Gamma_0(N)$ in $PSL_2(\mathbb{Z})$. Now we assume that $\Delta\subset\Gamma_0(N)$, and let $L(n,\chi;A)$ be a space of homogeneous polynomials in (x,y) of degree n. Choose a Dirichlet character $\chi:(\mathbb{Z}/N\mathbb{Z})^\times\to A^\times$, and let $\gamma=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\Gamma_0(N)$ on $L(n,\chi;A)$ by

$$\gamma\cdot P(x,y)=\chi(d)P((x,y)^t\gamma^\iota),$$

where $\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)^\iota=\left(\begin{smallmatrix}d&-b\\-c&a\end{smallmatrix}\right)$.

Exercise 2.5. *Prove the following facts:*

1. $\gamma^\iota\delta^\iota=(\delta\gamma)^\iota$ *for any* $\gamma,\delta\in M_2(A)$,
2. $\gamma^\iota+\gamma=\mathrm{Tr}(\gamma)$ *and* $\gamma^\iota\gamma=\gamma\gamma^\iota=\det(\gamma)$,
3. $L(n,\chi;A)$ *is a left* $\Gamma_0(N)$*-module,*
4. *If* $\chi(-1)=(-1)^n$*, the action of* $\Gamma_0(N)$ *on* $L(n,\chi;A)$ *factors through* $\overline{\Gamma}_0(N)$.

Let $f\in M_{k,\chi}(\Gamma_0(N))$ $(k\geq 2)$, and define an $L(k-2,\chi;\mathbb{C})$-valued differential form $\omega(f)$ by $\omega(f)=f(z)(x-zy)^n dz$ for $n=k-2$. For the complex conjugate $\overline{S}_{k,\chi^{-1}}(\Gamma_0(N))$ of $S_{k,\chi^{-1}}(\Gamma_0(N))$ and $f\in\overline{S}_{k,\chi^{-1}}(\Gamma_0(N))$, we similarly define $\omega(f)=f(z)(x-\overline{z}y)^n d\overline{z}$.

Exercise 2.6. *Prove the following facts:*

1. $\omega(f) \circ \gamma(z) = \gamma \cdot \omega(f)(z)$ *for all* $\gamma \in \Gamma_0(N)$,
2. *Fix a point* $z \in \mathfrak{H}$. *The map* $\gamma \mapsto c_f(\gamma) = \int_z^{\gamma(z)} \omega(f) \in L(k-2, \chi; \mathbb{C})$ *is a* 1*-cocycle,*
3. *The cohomology class of* c_f *in* $H^1(\Gamma_0(N), L(k-2, \chi; \mathbb{C}))$ *is independent of the choice of* z.

We take the following theorem for granted (cf. [IAT] Chapter 8 or [LFE] Chapter 6):

Theorem 2.7 (Eichler–Shimura)**.** *By* $f \mapsto [c_f]$, *we get an isomorphism:*

$$M_{k,\chi}(\Gamma_0(N)) \oplus \overline{S}_{k,\chi^{-1}}(\Gamma_0(N)) \cong H^1(\overline{\Gamma}_0(N), L(k-2, \chi; \mathbb{C})).$$

This isomorphism is equivariant under the Hecke operators (defined in the following section on cohomology groups). Similarly we have a Hecke equivariant isomorphism:

$$M_k(\Gamma_1(N)) \oplus \overline{S}_k(\Gamma_1(N)) \cong H^1(\Gamma_1(N), L(k-2, \chi; \mathbb{C}))$$

(for $N > 2$*).*

Exercise 2.8. *Check that if* $k = 2$*, the above theorem is just a restatement of* Corollary 2.19 *in the lecture notes of the course in the previous quarter.*

2.3. **Hecke operators on cohomology groups.** Let M be $A[\Delta_0(N)^\iota]$-module. Let $\Delta = \Gamma_0(N)$. Then in particular M is a Δ-module, because $\Delta \subset \Delta_0(N)^\iota$. For any left coset decomposition $\Delta\alpha\Delta = \bigsqcup_j \Delta\alpha_j$, we have $\alpha_j\gamma = \gamma_j\alpha_{\gamma^*(j)}$ with $\gamma_j \in \Delta$ for a permutation $j \mapsto \gamma^*(j)$ of indices. Then for a 1-cocycle c: we define a map $c|[\Delta\alpha\Delta](\gamma) = \sum_j \alpha_j^\iota c(\gamma_j)$.

Lemma 2.9. *If* $c : \Delta \to M$ *is a* 1*-cocycle,* $c|[\Delta\alpha\Delta]$ *is* 1*-cocycle, and the cohomology class of* $c|[\Delta\alpha\Delta]$ *is uniquely determined by the cohomology class of* c.

Proof. Let $\gamma, \delta \in \Delta$, and write $c' = c|[\Delta\alpha\Delta]$. Then $\alpha_j\gamma\delta = \gamma_j\alpha_{\gamma^*(j)}\delta = \gamma_j\delta_{\gamma^*(j)}\alpha_{\delta^*\gamma^*(j)}$. Note that $\alpha_j\gamma = \gamma_j\alpha_{\gamma^*(j)} \Leftrightarrow \gamma_j^{-1}\alpha_j = \alpha_{\gamma^*(j)}\gamma^{-1}$. We then compute

$$\begin{aligned}
c'(\gamma\delta) &= \sum_j \alpha_j^\iota c(\gamma_j\delta_{\gamma^*(j)}) = \sum_j \{\alpha_j^\iota \gamma_j c(\delta_{\gamma^*(j)}) + \alpha_j^\iota c(\gamma_j)\} = \sum_j (\gamma_j^{-1}\alpha_j)^\iota c(\delta_{\gamma^*(j)}) + c'(\gamma) \\
&= \sum_j (\alpha_{\gamma^*(j)}\gamma^{-1})^\iota c(\delta_{\gamma^*(j)}) + c'(\gamma) \overset{\gamma^*(j) \mapsto j}{=} \gamma c'(\delta) + c'(\gamma).
\end{aligned}$$

If $c(\gamma) = (\gamma - 1)m$ for $m \in M$, again by $\gamma_j^{-1}\alpha_j = \alpha_{\gamma^*(j)}\gamma^{-1}$, we have

$$\begin{aligned}
c'(\gamma) &= \sum_j \alpha_j^\iota(\gamma_j - 1)m = \sum_j (\gamma_j^{-1}\alpha_j)^\iota m - \sum_j \alpha_j^\iota m = \sum_j (\alpha_{\gamma^*(j)}\gamma^{-1})^\iota m - \sum_j \alpha_j^\iota m \\
&= \gamma \sum_j \alpha_{\gamma^*(j)}^\iota m - \sum_j \alpha_j^\iota m = (\gamma - 1)\sum_j \alpha_j^\iota m.
\end{aligned}$$

Thus $c \mapsto c|[\Delta\alpha\Delta]$ preserves the subspace of 1-coboundaries. Thus it induces a unique linear endomorphism of $H^1(\Delta, M)$. □

We can let $\delta = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Delta_0(N)^\iota$ on $L(k-2,\chi;A)$ by $\delta \cdot P(x,y) = \chi(d)P((x,y)^t\delta^\iota)$, and this action coincides with the original action of $\Gamma_0(N)$ on $\Gamma_0(N) \subset \Delta_0(N)^\iota$. Thus by the above lemma, we have a well defined Hecke operator $T(n)$ on $H^1(\Gamma_0(N), L(k-2,\chi;A))$. In other words, splitting $T_n = \bigsqcup_\alpha \Delta\alpha\Delta$ and define $T(n) = \sum_\alpha [\Delta\alpha\Delta]$.

Exercise 2.10. *Check that the isomorphism of* Theorem 2.7 *is equivariant under Hecke operators $T(n)$.*

The above action of $\Delta_0(N)^\iota$ induces an action of $\Delta_1(N)^\iota$ on $L(k-2,\chi;A)$. Since χ is modulo N, the action of $\Delta_1(N)^\iota$ does not depend on χ; so, we write simply $L(k-2;A)$ for this $A[\Delta_1(N)^\iota]$-module $L(k-2,\chi;A)$. Then in the same manner as above, replacing $\Delta_0(N)$ by $\Delta_1(N)$, we have a natural action of $T(n)$ on $H^1(\Gamma_1(N), L(k-2;A))$.

We can generalize a bit the above definition. Let $\Delta, \Delta' \subset \Gamma_0(N)$ be two subgroups of finite index. Decomposition $\Delta\alpha\Delta' = \bigsqcup_j \Delta\alpha_j$ and defining $\alpha_j\gamma' = \gamma_j\alpha_{\gamma'^*(j)}$ for $\gamma' \in \Delta'$ and $\gamma_j \in \Delta$, we can define $c|[\Delta\alpha\Delta'](\gamma') = \sum_j \alpha_j^\iota c(\gamma_j)$. This linear map induces, by the same proof as above,

$$[\Delta\alpha\Delta'] : H^1(\Delta, M) \to H^1(\Delta', M). \tag{2.5}$$

If $\Delta \subset \Delta'$, the double coset $\Delta' = \Delta 1 \Delta'$ induces a map $[\Delta'] : H^1(\Delta, M) \to H^1(\Delta', M)$ which is called a trace map $\mathrm{Tr} = \mathrm{Tr}_{\Delta'/\Delta}$.

Exercise 2.11. *Define the restriction map* $\mathrm{Res} : H^1(\Delta', M) \to H^1(\Delta, M)$ *by restricting cocycle of Δ' to Δ (if $\Delta' \supset \Delta$). Prove* $\mathrm{Tr} \circ \mathrm{Res}([c]) = [\Delta' : \Delta] \cdot [c]$. *Using this fact, prove that $H^1(G, M) = 0$ for a finite group G and a finite G-module M if $|G|$ is prime to $|M|$.*

The fact $H^q(G, M) = 0$ for a finite group G and a finite G-module M with $(|G|, |M|) = 1$ is true in general (and one can give a similar proof as above using a higher degree version of Tr and Res).

For $\alpha \in \Delta_1(N)$, put $\Delta_\alpha = \Delta \cap \alpha^{-1}\Delta\alpha$ and ${}_\alpha\Delta = \Delta \cap \alpha\Delta\alpha^{-1}$ (which are subgroups of fiite index of Δ. Then for a 1-cocycle $c : {}_\alpha\Delta \to M$, define $c|[\alpha] : \Delta_\alpha \to M$ by $c|[\alpha](\delta) = \alpha^\iota c(\alpha\delta\alpha^{-1})$.

Exercise 2.12. *Prove the following facts:*

(1) *$c|[\alpha]$ is a 1-cocycle of Δ_α;*
(2) *the above operation induces a linear map $[\alpha] : H^1({}_\alpha\Delta, M) \to H^1(\Delta_\alpha, M)$;*
(3) $[\Delta\alpha\Delta] = \mathrm{Tr}_{\Delta/\Delta_\alpha} \circ [\alpha] \circ \mathrm{Res}_{\Delta/{}_\alpha\Delta}$.

2.4. **Hecke operators and level.** Fix N so that $\Gamma_1(N)$ is torsion-free, and put $\Gamma_r = \Gamma_1(Np^r)$ for $r > 0$.

Lemma 2.13. *Let $\Phi_{r,m} = \Gamma_r \cap \Gamma_0(p^m)$ for $m \geq r > 0$. Then we have*

$$\Phi_{r,m} \left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right) \Gamma_r = \bigsqcup_{u=0}^{p-1} \Phi_{r,m} \left(\begin{smallmatrix} 1 & u \\ 0 & p \end{smallmatrix}\right).$$

Proof. Let $\alpha = \left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right)$. Since $\alpha^{-1}\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\alpha = \begin{pmatrix} a & bp \\ c/p & d \end{pmatrix}$, we have

$$\alpha^{-1}\Phi_{r,m}\alpha \cap \Gamma_r = \left\{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1 \middle| a \equiv 1 \mod p^r,\ c \equiv 0 \mod p^{\max(r,m-1)},\ b \equiv 0 \mod p\right\}.$$

Thus we have

$$\Gamma_r = \bigcup_{u=0}^{p-1} (\alpha^{-1}\Phi_{r,m}\alpha \cap \Gamma_r)\left(\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix}\right).$$

Multiplying $\alpha^{-1}\Phi_{r,m}\alpha$ from the left side, we have

$$\alpha^{-1}\Phi_{r,m}\alpha\Gamma_r = \bigcup_{u=0}^{p-1} \alpha^{-1}\Phi_{r,m}\alpha\left(\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix}\right).$$

This implies

$$\text{(U)} \qquad \Phi_{r,m}\alpha\Gamma_r = \bigcup_{u=0}^{p-1} \Phi_{r,m}\alpha\left(\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix}\right).$$

Since $\Gamma_0(p)\left(\begin{smallmatrix} 1 & u \\ 0 & p \end{smallmatrix}\right) \cap \Gamma_0(p)\left(\begin{smallmatrix} 1 & v \\ 0 & p \end{smallmatrix}\right) = \emptyset$ if $u \not\equiv v \mod p$, the union in (U) has to be disjoint. Note that $\left(\begin{smallmatrix} 1 & u \\ 0 & p \end{smallmatrix}\right) = \alpha\left(\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix}\right)$, and the result follows. □

Exercise 2.14. *Prove*

$$\Gamma_n = \bigsqcup_{u=0}^{p-1} (\alpha^{-1}\Phi_{r,m}\alpha \cap \Gamma_r)\left(\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix}\right).$$

Restricting cocycle $c : \Delta' \to M$ to a subgroup $\Delta \subset \Delta'$, we get the restriction morphism Res : $H^1(\Delta', M) \to H^1(\Delta, M)$. Applying Lemma 2.13 to $m = r$, we have the following commutative diagrams.

Corollary 2.15. *If $m \geq r > 0$, the following two diagrams are commutative:*

$$\begin{array}{ccc} H^1(\Gamma_r, L(k-2;A)) & \xrightarrow{\text{Res}} & H^1(\Gamma_m, L(k-2;A)) \\ {\scriptstyle U(p)}\downarrow & & \downarrow{\scriptstyle U(p)} \\ H^1(\Gamma_r, L(k-2;A)) & \xrightarrow{\text{Res}} & H^1(\Gamma_m, L(k-2;A)) \end{array}$$

and

$$\begin{array}{ccc} M_k(\Gamma_r; A) & \xrightarrow{\subset} & M_k(\Gamma_m; A) \\ {\scriptstyle U(p)}\downarrow & & \downarrow{\scriptstyle U(p)} \\ M_k(\Gamma_r; A) & \xrightarrow{\subset} & M_k(\Gamma_m; A). \end{array}$$

More generally, by the decomposition given in [LEC] Lemma 2.14 and its version for $\Gamma_1(Np^r)$ in [IAT] Proposition 3.36 (or [MFM] Section 4.5), actually we have

Corollary 2.16. *If $m \geq r > 0$, the following two diagrams are commutative for all positive integer n:*

$$\begin{array}{ccc} H^1(\Gamma_r, L(k-2;A)) & \xrightarrow{\text{Res}} & H^1(\Gamma_m, L(k-2;A)) \\ {\scriptstyle T(n)}\downarrow & & \downarrow{\scriptstyle T(n)} \\ H^1(\Gamma_r, L(k-2;A)) & \xrightarrow{\text{Res}} & H^1(\Gamma_m, L(k-2;A)) \end{array}$$

and

$$\begin{array}{ccc} M_k(\Gamma_r;A) & \xrightarrow{\subset} & M_k(\Gamma_m;A) \\ {\scriptstyle T(n)}\downarrow & & \downarrow{\scriptstyle T(n)} \\ M_k(\Gamma_r;A) & \xrightarrow{\subset} & M_k(\Gamma_m;A). \end{array}$$

Let $\mathbb{H}_k(Np^r;A)$ be the A-subalgebra of $\mathrm{End}_A(M_k(Np^r;A))$ generated by Hecke operators $T(n)$ for all n. Then by the above corollary, a Hecke operator $h \in \mathbb{H}_k(Np^m;A)$ restricted to $M_k(\Gamma_r;A)$ belongs to $\mathbb{H}_k(Np^r;A)$, getting a surjective A-algebra homomorphism $\mathbb{H}_k(Np^m;A) \twoheadrightarrow \mathbb{H}_k(Np^r;A)$ sending $T(n)$ to $T(n)$ as long as $m \geq r > 0$. We then take a projective limit:

$$\mathbb{H}_k(Np^\infty;A) = \varprojlim_m \mathbb{H}_k(Np^m;A).$$

Since the transition map takes $U(p)$ to $U(p)$, the projector $e = \lim_{n\to\infty} U(p)^{n!}$ of level Np^m is sent to e of level Np^r. Thus we have the projector $e = \lim_{n\to\infty} U(p)^{n!}$ well defined in $\mathbb{H}_k(Np^\infty;W)$. We define $\mathbb{H}_k^{ord}(Np^\infty;W) = e(\mathbb{H}_k(Np^\infty;A))$. We know the following facts (cf. [H86b] (1.7)):

Theorem 2.17. *If $k \geq 2$, we have unique isomorphisms: $\mathbb{H}_k(Np^\infty;W) \cong \mathbb{H}_2(Np^\infty;W)$ and $\mathbb{H}_k^{ord}(Np^\infty;W) \cong \mathbb{H}_2^{ord}(Np^\infty;W)$ taking $T(n)$ to $T(n)$ for all n.*

Taking this theorem granted, we put $\mathbb{H}^{ord}(Np^\infty;W) := \mathbb{H}_2^{ord}(Np^\infty;W)$. We find that $\ell\langle\ell\rangle_2 = T(\ell)^2 - T(\ell^2)$ of weight 2 is sent to $\ell^{k-1}\langle\ell\rangle_k = T(\ell)^2 - T(\ell^2)$ of weight k in $\mathbb{H}_k^{ord}(Np^\infty;W)$ (see Exercise 1.28). Anyway, $\Gamma \ni \ell \mapsto \ell^k\langle\ell\rangle_k = \langle\ell\rangle \in \mathbb{H}_k(Np^\infty;W)$ gives rise to a continuous character $\iota : \Gamma \to \mathbb{H}_k(Np^\infty;W)^\times$, which induces $W[[\Gamma]]$-algebra structure on $\mathbb{H}_k(Np^\infty;W)$ and $\mathbb{H}_k^{ord}(Np^\infty;W)$ independent of $k \geq 2$.

2.5. **Control theorem.** We want to prove the following result which basically implies Theorem 1.37:

Theorem 2.18. *As Λ-module, the Pontryagin dual $H^1_{ord}(Y_1(Np^\infty), K/W)^*$ of the cohomology group $H^1_{ord}(Y_1(Np^\infty), K/W)$ (which is isomorphic to $H_1^{ord}(Y_1(Np^\infty), W) = \varprojlim_r H_1^{ord}(Y_1(Np^r), W)$ by Poincaré's duality) is free of finite rank. Moreover the quotient $H_1^{ord}(Y_1(Np^\infty), W) \otimes_\Lambda \Lambda/((1+T)^{p^r} - \gamma^{p^r k})$ is canonically isomorphic to the dual $H^1_{ord}(\Gamma_1(Np^r), L(n, K/W))^*$ of $H^1_{ord}(\Gamma_1(Np^r), L(n, K/W))$. This isomorphism commutes with Hecke operators $T(n)$ for all n.*

Note here thar if we use the Λ-algebra structure $\Gamma \ni \ell \mapsto \langle \ell \rangle_k$ instead of $\Gamma \ni \ell \mapsto \langle \ell \rangle = \ell^k \langle \ell \rangle_k$, then the isomophism in the theorem can be written as

$$H_1^{ord}(Y_1(Np^\infty), W) \otimes_\Lambda \Lambda/((1+T)^{p^r} - 1) \cong H^1_{ord}(\Gamma_1(Np^r), L(n, K/W))^*.$$

We have a long way to go to prove this theorem. However, this basically tells us that $\mathbb{H}_k^{ord}(Np^\infty; W) \cong \mathbb{H}^{ord}(Np^\infty; W)$ by the Eichler-Shimura isomorphism (Theorem 2.7), because taking $r = \infty$

$$\mathbb{H}_k^{ord}(Np^\infty; W) \subset \mathrm{End}_\Lambda(H^1_{ord}(\Gamma_1(Np^\infty), L(n, K/W))^*) = \mathrm{End}_\Lambda(H^1_{ord}(Y_1(Np^\infty), K/W)^*)$$

generated by $T(n)$ for all n over Λ), and moreover $\mathbb{H}^{ord}(Np^\infty; W)/((1+T) - \gamma^k)$ is almost isomorphic to the level Np weight k Hecke algebra $\mathbb{H}_k^{ord}(Np; W)$ by an isomorphism sending $T(n)$ to $T(n)$. Then by duality between (the χ-part of) the Hecke algebra and $M_\chi(N; \Lambda)$, we get Theorem 1.37. This is our scheme of proving Theorem 1.37.

Here is a review of Pontryagin's duality: Let $\mathbb{T}_p = \mathbb{Q}_p/\mathbb{Z}_p$. For any abelian p-profinite compact or p-torsion discrete module X, we define the Pontryagin dual module X^* by $X^* = \mathrm{Hom}_{cont}(X, \mathbb{T}_p)$ and give X^* the topology of uniform convergence on every compact subgroup of X. By Pontryagin's duality theory (cf. [LFE] Section 8.3), we have $(X^*)^* \cong X$ canonically.

Exercise 2.19. *Show that $X^* \cong X$ noncanonically if X is finite.*

Exercise 2.20. *Prove that X^* is a discrete module if X is p-profinite and X^* is compact if X is discrete* (e.g., [LFE] Lemma 8.3.1).

By this fact, if X^* is the dual of a profinite module $X = \varprojlim_n X_n$ for finite modules X_n with surjections $X_m \twoheadrightarrow X_n$ for $m > n$, $X^* = \bigcup_n X_n^*$ is a discrete module which is a union of finite modules X_n^* and vice versa. We quote the following fact:

Exercise 2.21. *If X is a profinite A-module, then by the duality, we have, for an ideal $\mathfrak{a}$ of A, $X^*[\mathfrak{a}] := \{x \in X^* | ax = 0$ for all $a \in \mathfrak{a}\} \cong (X/\mathfrak{a}X)^*$ naturally.*

2.6. **Restriction and Inflation.** We start preparing the proof of Theorem 2.18. Let U be a normal subgroup of a group G. Let M be a G-module. For a 1-cocycle $u : U \to M$ and $g \in G$, ${}^g u : (g_1) \mapsto gu(g^{-1}g_1 g)$ is again a 1–cocycle of U, and the cohomology class of ${}^g u$ is equal to that of u if $g \in U$, as easily verified by computation. Thus the quotient group G/U acts on $H^1(U, M)$ by $[u] \mapsto [{}^g u]$. We now prove

Theorem 2.22. *Let U be a normal subgroup of G. Then the following sequence is exact:*

$$0 \to H^1(G/U, M^U) \xrightarrow{\text{Inf}} H^1(G, M) \xrightarrow{\text{Res}} H^0(G/U, H^1(U, M)) \xrightarrow{\text{Trans}} H^2(G/U, M^U).$$

We shall give a definition of the *transgression* Trans, due to Hochschild and Serre, in the following proof of the theorem.

Proof. For the projection $\pi : G \to G/U$ (and a cocycle $c : G/U \to M^U$), $\mathrm{Inf}(c) = c \circ \pi$ and $\mathrm{Res}(c) = c|_U$. For these two maps, it is easy to show the exactness by a simple computation (see Exercise 2.23).

We prove the exactness at $H^0(G/U, H^1(U, M))$. Let $c : U \to M$ be a 1-cocycle representing a class $[c]$ in $H^0(G/U, H^1(U, M))$. Then $gc(g^{-1}ug) - c(u) = (u-1)a(g)$ for a function $a : G \to M$, because $g[c] = [c]$. If $g \in U$, by cocycle relation, we see

$$gc(g^{-1}ug) - c(u) = c(ug) + gc(g^{-1}) - c(u) = uc(g) - c(g) = (u-1)c(g).$$

Thus we may take the function a to be c on U and hence may assume that $a(u) = c(u)$ for all $u \in U$. Then we have

$$ga(g^{-1}ug) - a(u) = (u-1)a(g).$$

Let F be the space of functions $f : U \to M$. Then we make F into a G-module by the following G–action: $gf(u) = gf(g^{-1}ug)$. Note that $(g-1)f(u) = gf(g^{-1}ug) - f(u)$. We then consider the space of functions: $C_j(G, F) = \{G^j \to F\}$. We pave differential $\partial : C_1(G, F) \to C_2(G, F)$ defined in (2.4). But note that $\partial(g \mapsto (g-1)f) = 0$, and by applying ∂ to $ga(g^{-1}ug) - a(u) = (u-1)a(g)$, we have

$$\begin{aligned}0 &= \partial(x \mapsto (x-1)a(u)) = \partial(x \mapsto (u-1)a(x))(g,h)\\ &= g(g^{-1}ug - 1)a(h) - (u-1)a(gh) + (u-1)a(g) = (u-1)(ga(h) - a(gh) + a(g)).\end{aligned}$$

Now we put $b(g,h) = \partial(a)(g,h) = ga(h) - a(gh) + a(g)$. Then the above equation becomes:

$$(u-1)b(g,h) = 0.$$

Thus the 2–cocycle $b : G \times G \to M$ actually has values in M^U.

Note that

$$\begin{aligned}(u-1)(ua(g) + a(u)) &= u(ga(g^{-1}ug) - a(u)) + ua(u) - a(u)\\ &= uga(g^{-1}ug) - a(u) = (u-1)a(ug).\end{aligned}$$

Thus fixing a complete representative set R for $U\backslash G$ so that $1 \in R$, we may normalize a so that $a(ug) = ua(g) + a(u)$ for all $u \in U$ and all $g \in R$. Since $a|_U$ is a 1–cocycle, by computation, we conclude that $a(ug) = ua(g) + a(u)$ for all $u \in U$ and all $g \in G$ (not just in R). Then for all $u \in U$ and $g, h \in G$, we see $b(u,g) = 0$, and 2–cocycle relation is

$$ub(g,h) - b(ug,h) + b(u,gh) - b(u,g) = 0.$$

This shows that $b(g,h) = ub(g,h) = b(ug,h)$. Similarly, we can show $b(g,uh) = b(g,h)$. Thus b factors through G/U.

If $a' : G \to M$ satisfies the same properties as a, that is, $ga'(g^{-1}ug) - a'(u) = (u-1)a'(g)$ and $a' = c$ on U, then

$$(u-1)(a(g) - a'(g)) = ga(g^{-1}ug) - a(u) - (ga'(g^{-1}ug) - a'(u)) = 0,$$

because $a = c = a'$ on U. This shows that $d(g) = a(g) - a'(g) \in M^U$. Then $b - b' = \partial(d) \in \mathrm{Im}(C_1(G/U, M^U) \xrightarrow{\partial} C_2(G/U, M^U))$, and hence we have the identity of the cohomology classes:

$$[b] = [b'] \in H^2(G/U, M^U)$$

for $b' = \partial(a')$. We then define $\mathrm{Trans}([c])$ by the cohomology class of $[b]$ in $H^2(G/U, M^U)$.

Suppose that $\mathrm{Trans}([c]) = 0$. Then choosing a 1–cochain $d : G/U \to M^U$ such that $\partial(d) = b$, we see that $a' = a - d$ agrees with c on U and $\partial(a') = 0$; so, a' is a 1–cocycle of G inducing c. This shows

$$\mathrm{Ker}(\mathrm{Trans}) \supset \mathrm{Im}(\mathrm{Res}).$$

By definition, if $c \in \mathrm{Im}(\mathrm{Res})$, we take a to be the 1–cocycle of G restricting c on U. Thus, $\mathrm{Im}(\mathrm{Res}) \supset \mathrm{Ker}(\mathrm{Trans})$. This proves the desired exactness for the degree 1 cohomology group. □

Exercise 2.23. *Prove the exactness at $H^1(G, M)$ of the sequence in the above theorem.*

If $0 \to M_1 \xrightarrow{\iota} M_2 \xrightarrow{\pi} M_3 \to 0$ is an exact sequence of G-modules, we have a corresponding long exact sequence:

$$0 \to M_1^G \xrightarrow{\iota} M_2^G \xrightarrow{\pi} M_3^G \xrightarrow{\delta_0} H^1(M_1) \xrightarrow{\iota_*} H^1(M_2) \xrightarrow{\pi_*} H^1(M_3) \xrightarrow{\delta} H^2(M_1) \to \cdots,$$

writing $H^q(M_j)$ for $H^q(G, M_j)$. The maps π_* and ι_* are given by composition of π and ι with 1-cocycles. The map δ_0 is given as follows. For $x \in M_3^G$, pick $\widehat{x} \in M_2^G$ with $\pi(\widetilde{x}) = x$. Then $(g-1)\widetilde{x} \in M_1$ because $\pi((g-1)\widetilde{x}) = (g-1)x = 0$. Then $\delta_0(x)(g) = (g-1)\widetilde{x}$ as a 1-cocycle. The map δ can be defined similarly. Pick a class $[c] \in H^1(M_3)$ represented by a 1-cocycle $c : G \to M_3$. Take any function $\widetilde{c} : G \to M_2$ such that $\pi(\widetilde{c}(g)) = c(g)$. Then $\pi(\partial\widetilde{c}) = \partial c = 0$, and hence $\partial\widetilde{c}$ has values in M_1 and is a 2-cocycle. Then $\delta([c]) = [\partial\widetilde{c}] \in H^2(M_1)$.

Exercise 2.24. *Prove the exactness of the long sequence up to degree 2 cohomology groups.*

2.7. **Freeness and divisibility.** As we already remarked below Exercise 2.3, second cohomology vanishes: $H^2(\Delta, M) = 0$ for all Δ-module M if Δ is a torsion-free subgroup of finite index in $\Gamma_r = \Gamma_0(Np^r)$. We consider the following short exact sequence $0 \to L(n, \chi; W) \xrightarrow{\iota} L(n, \chi; K) \xrightarrow{\pi} L(n, \chi; K/W) \to 0$ for a character χ modulo Np^r and the corresponding long exact sequence. Since $H^2(\Delta, L(n, \chi; W)) = 0$, $\pi_* : H^2(\Delta, L(n, \chi; K)) \to H^1(\Delta, L(n, \chi; K/W))$ is surjective; so, $H^1(\Delta, L(n, \chi; K/W))$ is divisible (because $H^1(\Delta, L(n, \chi; K))$ is a K-vector space). Since Δ acts trivially on $L(0; A) = A$, $H^1(\Delta, W)$ is mapped injectively into $H^1(\Delta, K)$; so, $H^1(\Delta, W)$ is W-free (which actually follows from the expression in (2.3)).

Proposition 2.25. *If Δ is torsion-free, the cohomology group $H^1(\Delta, L(n, \chi; K/W))$ is p-torsion and divisible, and the cohomology group $H^1(\Delta, W)$ is W-free of finite rank.*

The torsionness follows from that fact that $\mathbb{Q}_p/\mathbb{Z}_p$ is p-torsion and a 1-cocycle is determined by its value at the (finitely many) generators of Δ.

Exercise 2.26. *Find an example of an integer $n > 0$ such that $H^1(\Delta, L(n; W))$ has non-trivial torsion.* (Hint: Find n such that $L(n; K/W)^\Delta/\pi(L(n; K)^\Delta) \neq 0$, and use the long exact sequence.)

Write $H^1(\Delta, L(n,\chi;K/W))[p^n]$ is the submodule of $H^1(\Delta, L(n,\chi;K/W))$ killed by p^n. By the above proposition and Theorem 2.7,

$$H^1(\Delta, L(n,\chi;K)) = \left(\varprojlim_n H^1(\Delta, L(n,\chi;K/W))[p^n]\right) \otimes_W K,$$

and hence by Theorem 2.7, the Hecke algebra $\mathbb{H}_k(Np^r;W) \subset \mathrm{End}_W(M_k(\Gamma_1(Np^r);W))$ generated by Hecke operators $T(n)$ is identical to the subalgebra generated by Hecke operators $T(n)$ of $\mathrm{End}_W(H^1(\Gamma_1(Np^r), L(k-2,\chi;K/W)))$. Thus we get

Corollary 2.27. *The Hecke algebra $\mathbb{H}_k(Np^r;W)$ is isomorphic to the W-subalgebra of $\mathrm{End}_W(H^1(\Gamma_1(Np^r), L(k-2,\chi;K/W)))$ generated by Hecke operators $T(n)$ for all n.*

Exercise 2.28. *Give a detailed proof of the above corollary.*

Recall $\Phi_{r,m} = \Gamma_0(p^m) \cap \Gamma_r$ and $\Gamma_r = \Gamma_1(p^r N)$ for $m \geq r > 0$. Then Γ_m is a normal subgroup of $\Phi_{r,m}$.

We define the action of $U(p)$ on $H^q(\overline{\Phi}, L(n,\chi;K/W)^{\Gamma_m})$ in the following way: Note that $i : \overline{\Phi} \subset (\mathbb{Z}/p^m\mathbb{Z})^\times$ by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto (a^{-1} \mod p^m) = (d \mod p^m)(\mathbb{Z}/p^m\mathbb{Z})^\times$. Take q-cocycle $c : \overline{\Phi}^q \to L(n,\chi;K/W)^{\Gamma_m}$. We decompose $\Phi_{r,m}\alpha_0\Phi_{r,m} = \bigsqcup_{j=0}^{p-1} \Phi_{r,m}\alpha_j$ for $\alpha_j = \left(\begin{smallmatrix} 1 & j \\ 0 & p \end{smallmatrix}\right)$. Note that $\alpha_j^\iota \in \Delta_0(pN)^\iota$ acts trivially on $L(n,\chi;A)$. For $\gamma, \ldots, \in \Phi_{r,m}$, write $\alpha_j\gamma = \gamma_j\alpha_{\gamma^*(j)}$ for $\gamma_j \in \Phi_{r,m}$. Then $c|U(p)(\gamma) = \sum_{j=0}^{p-1} c(\gamma_j)$ for 1-cocycle c and $c|U(p)(\gamma,\delta) = \sum_{j=0}^{p-1} c(\gamma_j, \delta_j)$ for 2-cocycle c. Note that $i(\gamma_j) = i(\gamma)$, and therefore, we find that $c|U(p) = p \cdot c$. By the construction of Inf and Trans in the proof of Theorem 2.22, the maps $\mathrm{Inf}, \mathrm{Res}, \mathrm{Trans}$ in the exact sequence of Theorem 2.22 can be easily checked to be $U(p)$-linear. Thus we get the following result:

Lemma 2.29. *The above operator $U(p)$ on $H^q(\overline{\Phi}, L(n,\chi;K/W)^{\Gamma_m})$ is well defined, and the group $H^q(\overline{\Phi}, L(n,\chi;K/W)^{\Gamma_m})$ ($q = 1,2$) is killed by $U(p)^M$ for sufficiently large integer $M > 0$.*

Exercise 2.30. *Give a detailed proof of the above lemma.*

By the above lemma, we get

Proposition 2.31. *Suppose that $\Phi_{r,m}$ is torsion-free (for integers $m \geq r > 0$). We have*

$$H^1_{ord}(\Phi_{r,m}, K/W) \cong H^1_{ord}(\Gamma_m, K/W)^{\overline{\Phi}},$$

and more generally,

$$H^1_{ord}(\Phi_{r,m}, L(0,\chi;K/W) \cong H^1_{ord}(\Gamma_m, K/W)[\chi],$$

where χ is any character modulo Np^m and

$$H^1_{ord}(\Gamma_m, K/W)[\chi] = \left\{ c \in H^1_{ord}(\Gamma_m, K/W) \middle| c|\langle z \rangle = \chi(z)c \ \text{ for } z \in \overline{\Phi} \hookrightarrow (\mathbb{Z}/Np^m\mathbb{Z})^\times \right\}.$$

In particular, all these modules a divisible of finite corank.

Proof. The assertion follows from the above lemma. Indeed, by the above lemma, $H^q_{ord}(\overline{\Phi}, L(n,\chi;K/W)^{\Gamma_m}) = 0$. Then applying e to the exact sequence in Theorem 2.22, we get an isomorphism $H^1_{ord}(\Phi_{r,m}, L(0,\chi;K/W)) \overset{\mathrm{Res}}{\cong} H^1_{ord}(\Phi_{r,m}, L(0,\chi;K/W))^{\overline{\Phi}}$. Though we have $L(0,\chi;A) = A$ as Γ_m-modules, by the definition of action of $\overline{\Phi}$ on the cohomology group $H^1(\Gamma_m, L(0,\chi;A))$, we find

$$H^1(\Gamma_m, L(0,\chi;A))^{\overline{\Phi}} = H^1(\Gamma_m, A)[\chi].$$

The divisibility then follows from Proposition 2.25. □

Proposition 2.32. *Let $\Gamma = 1+p\mathbb{Z}_p$ act on $H^1_{ord}(\Gamma_1(Np^m), K/W)$ via diamond operators. Then for $0 < r \le m$, the restriction map induces an isomorphism*

$$H^1_{ord}(\Gamma_1(Np^m), K/W)^{\Gamma^{p^r}} \cong H^1_{ord}(\Gamma_1(Np^r), K/W)$$

for $r = 1, 2, \ldots, \infty$.

Proof. By Lemma 2.13 and Corollary 2.15, we have the following commutative diagram:

$$\begin{array}{ccc} H^1(\Gamma_r, L(k-2;K/W)) & \xrightarrow{\mathrm{Res}} & H^1(\Phi_{r,m}, L(k-2;K/W)) \\ U(p)\downarrow & \swarrow u & \downarrow U(p) \\ H^1(\Gamma_r, L(k-2;K/W)) & \xrightarrow{\mathrm{Res}} & H^1(\Phi_{r,m}, L(k-2;K/W)) \end{array}$$

for $u = [\Gamma_r \alpha \Phi_{r,m}]$ with $\alpha = \left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right)$. Since $U(p)$ is bijective on $H^1_{ord}(\Gamma_r, L(k-2;K/W))$ and $H^1_{ord}(\Phi_{r,m}, L(k-2;K/W))$, we conclude

$$H^1_{ord}(\Gamma_r, L(k-2;K/W)) \cong H^1_{ord}(\Phi_{r,m}, L(k-2;K/W)).$$

Then the assertion for finite r follows from Proposition 2.31. Then passing to the limit, we get the assertion for $r = \infty$. □

2.8. **Co-freeness over the group algebra.** We start with a lemma (cf. [GME] Theorem 1.9.7):

Lemma 2.33. *Let A be a reduced local W-algebra free of finite rank over W. If M be a W-free module of finite rank with an action of A such that for any algebra homomorphism $\lambda : A \to \overline{\mathbb{Q}}_p$, $M \otimes_A (A/\mathrm{Ker}(\lambda))$ is W-free, then M is A-free.*

Here the reducedness means that there is no nontrivial nilpotent element in A.

Proof. Let $\mathfrak{m}_A$ be the maximal ideal of A with residue field $\mathbb{F}$. Then by Nakayama's lemma (see [CRT] Theorem 2.2), $m = \dim_{\mathbb{F}} M \otimes_A \mathbb{F}$ (which is finite $\le \mathrm{rank}_W M$) is the least number of generators of M over W. Pick a set of generators $\{x_1, \ldots, x_m\}$ of M over A. Then $\pi : A^m \ni (a_1, \ldots, a_m) \mapsto \sum_j a_j x_j \in M$ is a surjective A-linear map. Writing $A_\lambda = A/\mathrm{Ker}(\lambda) = \mathrm{Im}(\lambda)$, we first assume that $A_\lambda = W$ for all λ. By the transitivity of the tensor products, we have $(M \otimes_A A_\lambda) \otimes_{A_\lambda} \mathbb{F} = A \otimes_A \mathbb{F}$, and hence $M_\lambda = M \otimes_A A_\lambda$ is W-torsion-free and the least number of generators of M_λ is m; so, $M_\lambda \cong W^m$. This shows $\mathrm{Ker}(\pi) \subset \mathrm{Ker}(\lambda)^m$. By the reducedness, $\cap_\lambda \mathrm{Ker}(\lambda) = 0$, we find that π is an isomorphism. If $A_\lambda \supsetneq W$, we take sufficiently large extension K'/K such that K' contain the image of all λ and replace A by $A' = A \otimes_W W'$ and M by $M \otimes_W W'$ for the p-adic integer ring W' of K'. Then by the argument as above, we find that $M \otimes_W W'$ is A'-free of finite rank. Then M is A-free of finite rank (because W' is W-free of finite rank). □

Exercise 2.34. *Give a detailed proof of the above lemma. In particular the following points:*

(1) *There exists a finite extension K'/K in $\overline{\mathbb{Q}}_p$ which contain the image of all W-algebra homomorphism $\lambda : A \to \overline{\mathbb{Q}}_p$;*
(2) *Suppose that λ has values in W. The map: $(A/\mathrm{Ker}(\lambda))^m \to M/\mathrm{Ker}(\lambda)M$ induced by π is an isomorphism if $M/\mathrm{Ker}(\lambda)M \cong W^m$; so, $\mathrm{Ker}(\pi) \subset \mathrm{Ker}(\lambda)^m$;*
(3) *$\bigcap_\lambda \mathrm{Ker}(\lambda) = 0$ if A is reduced;*
(4) *M is A-free if $M \otimes_W W'$ is A'-free.*

Exercise 2.35. *Let G be a finite abelian p-group. Prove that the group algebra $W[G]$ is a local ring.*

Theorem 2.36. *Let $M = M_m$ be the Pontryagin dual of $M^1_{ord}(\Gamma_m, K/W)$, and assume that $\Phi_{r,m}$ is torsion-free. Then M is a $W[\Gamma^{p^{r-1}}/\Gamma^{p^{m-1}}]$-free module of finite rank. Moreover $M_m/(\gamma^{p^{r-1}} - 1)M_m \cong M_r$ for all $m \geq r > 0$ for a generator γ of $\Gamma = 1 + p\mathbb{Z}_p$.*

Proof. Note that $H^1_{ord}(\Gamma_m, K/W)^{\Gamma_r}$ is the subspace of $H^1_{ord}(\Gamma_m, K/W)$ killed by $\gamma^{p^{r-1}} - 1$ (by Proposition 2.32); so, by duality, we get $M_m/(\gamma^{p^{r-1}} - 1)M_m \cong M_r$.

Since any algebra homomorphism $\lambda : W[\Gamma^{p^{r-1}}/\Gamma^{p^{m-1}}] \to \overline{\mathbb{Q}}_p$ is induced by a character $\chi : \Gamma^{p^{r-1}}/\Gamma^{p^{m-1}} \to \overline{\mathbb{Q}}_p^\times$, by duality (and Proposition 2.31), we find M_λ which is then Pontryagin dual of $H^1_{ord}(\Gamma_m, K/W)[\chi]$ is W-free of finite rank. Therefore, M is $W[\Gamma^{p^{r-1}}/\Gamma^{p^{m-1}}]$-free module of finite rank. □

Corollary 2.37. *Let $M = M_m$ be the Pontryagin dual of $H^1_{ord}(\Gamma_m, K/W)$. The projective limit $M_\infty = \varprojlim_m M_m$ is a Λ-free module of finite rank, and $M_\infty/((1+T) - \chi(\gamma))M_\infty$ is isomorphic to the Pontryagin dual of $H^1_{ord}(\Gamma_0(Np^m), K/W)[\chi]$ for all characters $\chi : (\mathbb{Z}/Np^m\mathbb{Z})^\times \to W^\times$ factoring through Γ_m.*

Since $\Phi_{r,m}$ is torsion-free for sufficiently large r, we find that M_∞ is $W[[\Gamma^{p^{r-1}}]]$-free of finite rank. Then actually M_∞ is also Λ-free of finite rank. Thus in this corollary, we do not need to assume torsion-freeness of $\Phi_{r,m}$. We do not give a detailed proof of the last assertion.

Exercise 2.38. *Prove the above remark that a $W[[\Gamma]]$-module M (of finite type) is actually free if it is free over $W[[\Gamma^{p^r}]]$ for $r > 0$.* (Hint: Use that fact that if we identify $W[[\Gamma]]$ with $W[[T]]$, then $W[[\Gamma^{p^r}]]$ is identified with $W[[(1+T)^{p^r} - 1]]$, in particular, $W[[\Gamma]]$ is free of finite rank over $W[[\Gamma^{p^r}]]$.)

We have basically proven the following result for $k = 2$ (see [H86b] for the general case of $k \geq 2$). Here we note that the Λ-algebra structure on $\mathbb{H}^{ord}(Np^\infty; W)$ is the twist twice by the (identity inclusion) character $\Gamma \ni \delta \mapsto \delta \in W^\times$ of the action $\langle \delta \rangle_2$ on the cohomology groups.

Corollary 2.39. *The algebra $\mathbb{H}^{ord}(Np^\infty;W)$ is a torsion free Λ-module of finite type and has a canonical surjective homomorphism*

$$\pi_k : \mathbb{H}^{ord}(Np^\infty;W)\otimes_\Lambda \Lambda/((1+T)-\gamma^k) \to \mathbb{H}^{ord}_k(Np;W)$$

taking $T(n)$ to $T(n)$, where $\mathbb{H}^{ord}_k(Np;W)$ is the W-algebra in the endomorphism algebra $\mathrm{End}(H^1_{ord}(\Gamma_1(Np);L(k-2;K)))$ *generated by $T(n)$ over W. More over we have*

$$\pi_k : \mathbb{H}^{ord}(Np^\infty;W)\otimes_\Lambda \Lambda/((1+T)-\gamma^k)\otimes_W K \cong \mathbb{H}^{ord}_k(Np;K)$$

for all $k \geq 2$.

By Eichler-Shimura isomorphism, $\mathbb{H}^{ord}_k(Np;W)$ is the modular W-algebra in the endomorphism algebra $\mathrm{End}(M^{ord}_k(\Gamma_1(Np);K))$ generated by $T(n)$ over W. The last assertion can be proven, for example, by counting the dimension of the left and right-hand-side. Here is a sketch how to prove the higher weight $k>2$ case. Note that the evaluation of polynomial $L(k-2;W_m)\ni P\mapsto P(1,0)\in W_m(k-2)$ for $W_m = W/p^mW$ is a morphism of $\Phi_{0,m}$-modules, where $\gamma = \left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in\Gamma_0(Np^m)$ acts on $W_m(k-2) = W_m$ by $x\mapsto d^{k-2}x$. Thus we have a morphism $\iota : H^1(\Phi_{0,m}, L(k-2,W_m)) \to H^1(\Phi_{0,m}, W_m(k-2)) \cong H^1(\Gamma_m,W_m)[k-2]$, where $H^1(\Gamma_m,W_m)[k-2]$ is the submodule of $H^1(\Gamma_m,W_m)$ on which $\overline{\Phi} = \Phi_{0,m}/\Gamma_m \cong (\mathbb{Z}/p^m\mathbb{Z})^\times$ acts by $z\mapsto z^{k-2}$. Similarly, $W_m(2-k)\ni a\mapsto aY^{k-2}\in L(k-2,W_m)$ gives a morphism of $\Phi_{0,m}$-modules, which results $\pi_0 : H^1(\Phi_{0,m},W_m(2-k)) \to H^1(\Phi_{0,m},L(k-2,W_m)) \cong H^1(\Gamma_m,W_m)[k-2]$. For simplicity, suppose $N=1$. Then $\Phi_{0,m} = \Gamma_0(p^m)$. Since $\tau = \left(\begin{smallmatrix} 0 & -1\\ p^m & 0\end{smallmatrix}\right)$ normalizes $\Gamma_0(p^m)$, defining $c|[\tau](\sigma) = \tau^\iota c(\tau\sigma\tau^{-1})$ for 1-cocycles $c:\Gamma_0(p^m)\to L(k-2,W_m)$, we have an endomorphism $[\tau]$ of $H^1(\Gamma_0(p^m),L(k-2,W_m))$. Similarly, for 1-cocycles $c:\Gamma_0(p^m)\to W_m(k-2)$, we verify that $c|[\tau](\sigma) = c(\tau\sigma\tau^{-1})$ is a 1-cocycle with values in $W_m(2-k)$, since $\tau\left(\begin{smallmatrix} a & b\\ p^mc & d\end{smallmatrix}\right)\tau^{-1} = \left(\begin{smallmatrix} d & c\\ p^mb & a\end{smallmatrix}\right)$ and $ad\equiv 1 \mod p^m$. Thus we can define $[\tau] : H^1(\Gamma_0(p^m),W_m(k-2)) \to H^1(\Gamma_0(p^m),W_m(2-k))$, Put $\pi = [\Gamma_0(p^m)\delta\Gamma_0(p^m)]\circ\pi_0\circ[\tau]$. Then by computation, we can prove $\pi\circ\iota = U(p^m)$ on $H^1(\Gamma_0(p^m),L(k-2,W_m))$ and $\iota\circ\pi = U(p^m)$ on $H^1(\Gamma_m,W_m)[k-2]$. Thus taking limit, we find an isomorphism

$$\begin{aligned}(2.6)\quad I : H^1_{ord}(\Gamma_0(p^\infty),L(k-2,K/W)) &= \varinjlim_m H^1_{ord}(\Gamma_0(p^m),L(k-2,W_m))\\ &\cong \varinjlim_m H^1_{ord}(\Gamma_1(p^m),W_m)[k-2] = H^1_{ord}(\Gamma_1(p^\infty),K/W)[k-2],\end{aligned}$$

where $H^1_{ord}(\Gamma_1(p^\infty),K/W)[k-2]$ is the subspace of $H^1_{ord}(\Gamma_1(p^\infty),K/W)$ on which $\langle z\rangle_2$ for $z\in\mathbb{Z}_p^\times$ acts by the character: $\mathbb{Z}_p^\times\ni z\mapsto z^{k-2}\in W$. This morphism I can be checked to satisfy $T(n)\circ I = I\circ T(n)$, and in this way, we get the result of higher weight (see [H86b] Theorem 4.4 for more details).

Instead of $\varinjlim_m H^1_{ord}(Y_1(Np^m),K/W)$, taking the Pontryagin dual N_∞ of the limit cohomology $\varinjlim_m H^1_{ord}(X_1(Np^m),K/W)$ and its Hecke algebra

$$\mathbf{h}^{ord}(Np^\infty;W) = \Lambda[T(n)|n=1,2,\dots]\subset \mathrm{End}_\Lambda(N_\infty),$$

we can prove the cuspidal version of the above corollary:

Corollary 2.40. *The algebra* $\mathbf{h}^{ord}(Np^\infty;W)$ *is a torsion free* Λ*-module of finite type and has a canonical surjective homomorphism*

$$\pi_k : \mathbf{h}^{ord}(Np^\infty;W)\otimes_\Lambda \Lambda/((1+T)-\gamma^k\chi(\gamma)) \to \mathbf{h}_k^{ord}(Np^r;W)$$

taking $T(n)$ *to* $T(n)$*, where* $\mathbf{h}_k^{ord}(Np;W)$ *is the* W*-algebra in* $\mathrm{End}(S_k(\Gamma_1(Np);K))$ *generated by* $T(n)$ *over* W*. Moreover we have*

$$\pi_k : \mathbf{h}^{ord}(Np^\infty;W)\otimes_\Lambda \Lambda/((1+T)-\gamma^k\chi(\gamma))\otimes_W K \cong \mathbf{h}_k^{ord}(Np;K)$$

for all $k\ge 2$*.*

2.9. **Ordinary p-adic analytic families.** Write $\mathbf{h}=\mathbf{h}^{ord}(Np^\infty;W)$ simply. Consider $\mathrm{Hom}_\Lambda(\mathbf{h},\Lambda)$ and embed it into $\Lambda[[q]]$ by sending $F\in\mathrm{Hom}_\Lambda(\mathbf{h},\Lambda)$ to a q-expansion $\sum_{n=1}^\infty a(n,F)q^n$ with $a(n,F)=F(T(n))$. Since $\mathbf{h}\otimes_\Lambda\Lambda/((1+T)-\gamma^k)\otimes_W K\cong\mathbf{h}_k^{ord}(Np,K)$ and $\mathrm{Hom}_K(\mathbf{h}_k^{ord}(Np,K),K)=S_k^{ord}(\Gamma_1(Np),K)$, we can easily conclude that $F(\gamma^k-1)$ is the q-expansion of an element in $S_k^{ord}(\Gamma_1(Np),K)$. This shows that

Theorem 2.41. *The* Λ*-module* $\mathrm{Hom}_\Lambda(\mathbf{h},\Lambda)$ *is isomorphic to the* Λ*-module* $S^{ord}(N;\Lambda)$ *of ordinary analytic families of cusp forms of prime-to-p level* N*. Moreover if* $F\in S^{ord}(N;\Lambda)$*,* $F(\gamma^k-1)\in S_k^{ord}(\Gamma_1(Np);K)$ *for all* $k\ge 2$*.*

Exercise 2.42. *Give a detailed proof of the above theorem, and prove that* $S^{ord}(N;\Lambda)$ *is a free of finite rank over* Λ*.*

2.10. **Higher weight.** We briefly explain how to relate weight 2 and higher weight $k\ge 2$. Since $\gamma\in\Phi=\Phi_{0,m}=\Gamma_0(p^m)\cap\Gamma_1(N)$ is upper triangular modulo p^m, the evaluation of polynomials $P(X,Y)\in L(k-2;W/p^mW)$ at $(1,0)$ induces the following morphism of Φ-modules: $i: L(k-2;W/p^mW)\to W/p^mW(k-2)$, where $\gamma=\left(\begin{smallmatrix} a & b\\ cp^m & d\end{smallmatrix}\right)\in\Phi$ acts on $W/p^mW(k-2)\cong W/p^mW$ via multiplication by d^{k-2}. Thus we have a morphism of cohomology $i_*: H^1(\Phi,L(k-2;W/p^mW))\to H^1(\Phi,W/p^mW(k-2))$. In the same manner as in the proof of Proposition 2.31, we get

Proposition 2.43. *Suppose that* $\Phi_{r,m}$ *is torsion-free (for integers* $m\ge r>0$*). We have*

$$H^1_{ord}(\Phi_{r,m},W/p^mW(k-2))\cong H^1_{ord}(\Gamma_m,W/p^mW)[k-2],$$

where for $\overline{\Phi}=\Phi_{r,m}/\Gamma_m\hookrightarrow(\mathbb{Z}/Np^m\mathbb{Z})^\times$

$$H^1_{ord}(\Gamma_m,W/p^mW)[k-2]=\left\{c\in H^1_{ord}(\Gamma_m,W/p^mW)\big|c|\langle z\rangle=z^{k-2}c\ \text{ for } z\in\overline{\Phi}\right\}.$$

By Lemma 2.13 and Corollary 2.15, we have the following commutative diagram:

$$\begin{array}{ccc} H^1(\Gamma_r,L(k-2;K/W)) & \xrightarrow{\mathrm{Res}} & H^1(\Phi_{r,m},L(k-2;K/W)) \\ U(p)\downarrow & \swarrow u & \downarrow U(p) \\ H^1(\Gamma_r,L(k-2;K/W)) & \xrightarrow{\mathrm{Res}} & H^1(\Phi_{r,m},L(k-2;K/W)) \end{array}$$

for $u=[\Gamma_r\alpha\Phi_{r,m}]$ with $\alpha=\left(\begin{smallmatrix}1&0\\0&p\end{smallmatrix}\right)$. Since $U(p)$ is bijective on $H^1_{ord}(\Gamma_r,L(k-2;K/W))$ and $H^1_{ord}(\Phi_{r,m},L(k-2;K/W))$, we conclude

$$H^1_{ord}(\Gamma_r,L(k-2;K/W))\cong H^1_{ord}(\Phi_{r,m},L(k-2;K/W)).$$

Then passing to the limit now going to $m = \infty$, we get for $k > 2$

$$(2.7)\quad H^1_{ord}(\Gamma_1(N)\cap\Gamma_0(p), L(k-2;K/W)) \cong \varinjlim_m H^1_{ord}(\Phi_{0,m}, L(k-2; W/p^mW)) \xrightarrow{i_*} \varinjlim_m H^1_{ord}(\Phi_{0,m}, W/p^mW))[k-2] \cong \varinjlim_m H^1_{ord}(\Gamma_m, W/p^mW)[k-2].$$

We can make an inverse of this map i_*. We define $j : W/p^mW(2-k) \to L(k-2; W/p^mW)$ by $j(z) = zY^{k-2}$. Then j is a morphism of $\Phi_{0,m}$-modules, inducing a morphism $j_* : H^1(\Phi_{0,m}, W/p^mW(2-k)) \to H^1(\Phi_{0,m}, L(k-2; W/p^mW))$. For simplicity, we assume that $N = 1$. Take $\tau = \tau_m = \left(\begin{smallmatrix} 0 & -1 \\ p^m & 0 \end{smallmatrix}\right)$. Then $\tau \left(\begin{smallmatrix} a & b \\ cp^m & d \end{smallmatrix}\right) \tau^{-1} = \left(\begin{smallmatrix} d & -c \\ -bp^m & a \end{smallmatrix}\right)$. Thus for a 1-cocycle $c : \Phi_{0,m} \to W/p^mW(k-2)$, $c|[\tau](\gamma) = c(\tau\gamma\tau^{-1})$ is a 1-cocycle with values in $W/p^mW(2-k)$. Since $[\tau]^2 = 1$, we have an isomorphism

$$[\tau] : H^1(\Phi_{0,m}, W/p^mW(k-2)) \xrightarrow{\sim} H^1(\Phi_{0,m}, W/p^mW(2-k)).$$

Then we define $\pi : H^1(\Phi_{0,m}, W/p^mW(k-2)) \to H^1(\Phi_{0,m}, L(k-2, W/p^mW))$ by

$$\pi = [\Phi_{0,m}\tau_0\Phi_{0,m}] \circ j_* \circ [\tau_m].$$

Just by computation, we get (see [H86b] Theorem 4.4)

Theorem 2.44. *We have $\pi \circ i_* = U(p^m)$ on $H^1(\Phi_{0,m}, L(k-2; W/p^mW))$ and $i_* \circ \pi = U(p^m)$ on $H^1(\Phi_{0,m}, W/p^mW(k-2))$.*

By this theorem, we find that (2.7) is all isomorphism

$$H^1_{ord}(\Gamma_0(p), L(k-2; K/W)) \cong \varinjlim_m H^1_{ord}(\Gamma_m, W/p^mW)[k-2].$$

This shows that

$$M_\infty/(1+T-\gamma^k)M_\infty \cong \mathrm{Hom}(H^1_{ord}(\Gamma_0(p), L(k-2; K/W)), K/W)$$

actually as Hecke modules, where

$$M_\infty = \varprojlim_m H_1^{ord}(Y_1(p^m), W) \cong \mathrm{Hom}(\varinjlim_m H^1_{ord}(\Gamma_1(p^m), K/W), K/W).$$

When $N > 1$, we just take τ_m such that $\tau_m \equiv \left(\begin{smallmatrix} 0 & -1 \\ p^m & 0 \end{smallmatrix}\right) \mod p^{2m}$ and $\tau_m \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & p^m \end{smallmatrix}\right)$ mod N with $\det(\tau_m) = p^m$, and just do the same argument to conclude the above isomorphism for $\Gamma_1(N) \cap \Gamma_0(p)$.

Exercise 2.45. *Show the existence of τ_m as above if $N > 1$ prime to p.*