1. Introduction

In this introduction, first, without going into technical details, we describe a prototypical example of a $p$-adic analytic families of modular forms. Starting with the third week (or slightly earlier), we start to justify our construction cohomologically. The examples we describe are from [LFE] Chapter 7.

1.1. $p$-Adic $L$-functions as a power series. We start with a general fact on the Kubota-Leopoldt $p$-adic $L$-functions. We consider the binomial formula:

\[(1 + T)^s = \sum_{n=0}^{\infty} \binom{s}{n} T^n.\]

Since $s \mapsto \binom{s}{n} = \frac{s(s-1)(s-2)\cdots(s-n+1)}{n!}$ is a polynomial in $s$ and has integer value over natural numbers, it is a polynomial on $\mathbb{Z}_p$ with values in $\mathbb{Z}_p$. Thus if $\gamma \equiv 1 \mod p$, we have the $p$-adic power $\gamma^s = \sum_{n=0}^{\infty} \binom{s}{n} (\gamma - 1)^n \in \mathbb{Z}_p$ (convergent $p$-adically) for all $s \in \mathbb{Z}_p$.

Let $K$ be a finite extension of $\mathbb{Q}_p$ with $p$-adic integer ring $W = \{w \in K \, | \, |w|_p \leq 1\}$. Let $\varphi$ be a $p$-adic measure on $\mathbb{Z}_p$ with values in $W$ (so it is a bounded measure). Since the power series ring $W[[T]]$ is a Banach algebra under the norm $\|\sum_{n=1}^{\infty} a_n T^n\| = \sup_{n} |a_n|_p$, we can integrate any continuous function $\phi : \mathbb{Z}_p \to W[[T]]$ under $d\varphi$. In other words, we approximate $\phi$ by step functions $\phi_n : \mathbb{Z}_p \to W[[T]]$ factoring through $(\mathbb{Z}/p^n\mathbb{Z})$ so that $\lim_{n \to \infty} \phi_n = \phi$ under the norm $\| \cdot \|$ and define

$$\int_{\mathbb{Z}_p} \phi d\varphi = \lim_{n \to \infty} \int_{\mathbb{Z}_p} \phi_n d\varphi \in W[[T]].$$

**Exercise 1.1.** Prove that

$$\int_{\mathbb{Z}_p} (1 + T)^s d\varphi(s) = \sum_{n=0}^{\infty} \binom{s}{n} \phi(s) T^n = : \Phi_{\varphi}(T).$$

Let $\Gamma = 1 + p\mathbb{Z}_p$ and $z \mapsto \langle z \rangle = \omega(z)^{-1} z$ be the projection of $\mathbb{Z}_p^\times$ onto $\Gamma$, where $\omega$ is the Teichmüller character defined in [LEC] Theorem 1.33. By the existence of a primitive root, for an odd prime $p$, the multiplicative group $(\mathbb{Z}/p^n\mathbb{Z})^\times$ is a cyclic group, and hence its subgroup $\{x \in (\mathbb{Z}/p^n\mathbb{Z})^\times \, | \, x \equiv 1 \mod p\}$ is cyclic generated by $\gamma = 1 + p$.

**Exercise 1.2.** Let $\Gamma_n = \{u^n \mid u \in \Gamma\} \subset \Gamma$ and $p$ be an odd prime. Prove the following facts

1. $\Gamma_n = \Gamma$ if $p \nmid n$;
2. $\Gamma/\Gamma_n^{-1} \cong \{x \in (\mathbb{Z}/p^n\mathbb{Z})^\times \, | \, x \equiv 1 \mod p\}$ by sending $u\Gamma_n^{-1}$ to $u \mod p^n$. In particular, $[\Gamma : \Gamma_n^{-1}] = p^n$;
3. $\Gamma \cong \mathbb{Z}_p$ by $\gamma^s \mapsto s \in \mathbb{Z}_p$ for $\gamma = 1 + p$;
4. $1 + 4\mathbb{Z}_2 \cong \mathbb{Z}_2$ by $\gamma^s \mapsto s \in \mathbb{Z}_2$ for $\gamma = 5$.

We have a projection $\langle \cdot \rangle : \mathbb{Z}_p^\times \to \Gamma$. Thus we can define a bounded measure $\langle \varphi \rangle$ on $\Gamma$ by $\int_{\Gamma} \phi d\langle \varphi \rangle = \int_{\mathbb{Z}_p^\times} \phi(\langle z \rangle) d\varphi$. Identifying $\Gamma$ with $\mathbb{Z}_p$ by $\gamma^s \mapsto s \in \mathbb{Z}_p$, consider $\Phi_{\langle \varphi \rangle}(T) \in W[[T]]$. 


Lemma 1.3. We have \( \int_{\Gamma} u^s d\langle \varphi \rangle(u) = \Phi(\varphi)(\gamma^s - 1) \).

Proof. For the isomorphism \( \iota : \Gamma \cong \mathbb{Z}_p \) with \( \iota(\gamma^z) = z \), we can define a measure \( \varphi_+ \) on \( \mathbb{Z}_p \) by \( \int_{\mathbb{Z}_p} \phi d\varphi_+ = \int_{\Gamma} \phi \circ d\langle \varphi \rangle \). Then we have \( \Phi_\varphi = \Phi_{\varphi_+} \), and \( \Phi_{\varphi_+}(T) = \int_{\mathbb{Z}_p}(1 + T)^z d\varphi_+(z) \).

Replacing \( T \) by \( \gamma^s - 1 \) and writing \( u = \gamma^z \), we get

\[
\Phi_{\varphi_+}(\gamma^s - 1) = \int_{\mathbb{Z}_p} \gamma^z d\varphi_+(z) = \int_{\Gamma} u^s d\langle \varphi \rangle(u),
\]

which shows the assertion. \( \square \)

Exercise 1.4. Define a Dirac measure \( \delta_z \) for \( z \in \mathbb{Z}_p \) by \( \int_{\mathbb{Z}_p} \phi d\delta_z = \phi(z) \). Prove that \( \Phi_{\delta_z}(T) = (1 + T)^z \).

Let \( N \) be a positive integer prime to \( p \). We defined in [LEC] Theorem 1.33 the \( p \)-adic Dirichlet \( L \)-function for each primitive odd character \( \chi \) modulo \( Np^s \) (with values in \( K \)). Reformulating the result there (by making a variable change \( \chi \mapsto \chi \omega^{-1} \); so now \( \chi \) is even), by the above lemma, we thus get

Theorem 1.5. Let \( N \) be a positive integer prime to \( p \) and \( \chi \) with \( \chi(-1) = 1 \) be a Dirichlet character modulo \( Np \). Suppose that \( \chi_N \) is primitive modulo \( N \). Then there exists a power series \( \Phi_{\chi}(T) \in \mathbb{W}[[T]] \) such that \( L_p(1 - s, \chi) = \Phi_{\chi}(\gamma^s - 1) \) if \( \chi_N \neq 1 \) and \( L_p(1 - s, 1) = \frac{\Phi_{\chi}(\gamma^s - 1)}{\gamma^s - 1} \).

Exercise 1.6. Give a detailed proof of the above theorem.

A \( p \)-adic analytic function on \( \mathbb{Z}_p \) of the form \( s \mapsto \Phi(\gamma^s - 1) \) for a power series \( \Phi(T) \in \mathbb{W}[[T]] \) is called an Iwasawa function. Iwasawa functions form a special subclass of \( p \)-adic analytic functions on \( \mathbb{Z}_p \).

1.2. Eisenstein series. Let \( \chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \overline{\mathbb{Q}}^\times \) be a primitive Dirichlet character. We consider the Eisenstein series of weight \( 0 < k \in \mathbb{Z} \)

\[
E'_k(z, s) = \sum_{(m, n) \in \mathbb{Z}^2 - \{(0, 0)\}} \chi^{-1}(n)(mNz + n)^{-k}|mNz + n|^{-2s},
\]

where \( z \in \mathfrak{H} \) and \( s \in \mathbb{C} \). When \( N = 1 \), \( \chi \) is the trivial character \( 1 \). For the following exercise, see [MFM] Section 2.6 and Chapter 7.

Exercise 1.7. Prove

1. \( E'_k(z, s) \) converges absolutely and locally uniformly with respect to \( (z, s) \in \mathfrak{H} \times \mathbb{C} \) if \( \Re(2s + k) > 2 \);
2. \( E'_k(z, s) = 0 \) if \( \chi(-1) \neq (-1)^k \) (assuming convergence);
3. \( E'_k(z) = E'_k(z, 0) \) is a holomorphic function of \( z \) if \( k < 2 \) (this fact is actually true if \( k = 2 \) and \( \chi \neq 1 \) for the limit \( E'_k(z) = \lim_{s \to 0} E'_k(z, s) \));
4. \( E'_k(\gamma(z)) = \chi(d)(cz + d)^kE'_k(z) \) for \( \gamma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma_0(N) \).

A holomorphic function \( f : \mathfrak{H} \to \mathbb{C} \) is called a modular form on \( \Gamma_0(N) \) of weight \( k \) with character \( \chi \) if \( f \) satisfies the following conditions:
Exercise 1.9. Prove that $f$ is finite at all cusps of $X_0(N)$; in other words, for all $\alpha = (a \ b \ c \ d) \in SL_2(\mathbb{Z})$, $f|_{k,\alpha}(z) = f(\alpha(z))(cz + d)^{-k}$ has Fourier expansion of the form

$$\sum_{0 \leq n \in \mathbb{N}^{-1}} a(n, f|_{k,\alpha}) \exp(2\pi inz) \quad \text{with } a(n, f|_{k,\alpha}) \in \mathbb{C}.$$  

The above condition means that the function $f$ is finite at the cusp $\alpha(\infty)$ of $X_0(N)$ (whose value at the cusp is $a(0, f|_{k,\alpha})$). We write $M_{k,\alpha}(\Gamma_0(N))$ for the space of functions satisfying (M1–2). Replace (M2) by

$$(S) \ f \ is \ vanishing \ at \ all \ cusps \ of \ X_0(N) \ (that \ is, \ a(n, f|_{k,\alpha}) = 0 \ for \ all \ \alpha \in SL_2(\mathbb{Z}) \ and \ n \leq 0),$$

we define subspace $S_{k,\alpha}(\Gamma_0(N)) \subset M_{k,\alpha}(\Gamma_0(N))$ by imposing (S). Functions in the space $S_{k,\alpha}(\Gamma_0(N))$ are called holomorphic cusp forms on $\Gamma_0(N)$ of weight $k$ with character $\chi$.

Exercise 1.8. Prove that $M_{0,\chi}(\Gamma_0(N))$ is either $\mathbb{C}$ (constants) or 0 according as $\chi = 1$ or not.

Exercise 1.9. Prove that $M_{k,\chi}(\Gamma_0(N)) = 0$ if $\chi(-1) \neq (-1)^k$.

Proposition 1.10. Let $\chi$ be a primitive Dirichlet character modulo $N$. The Eisenstein series $E'_{k,\chi}(z, s)$ for $0 < k \in \mathbb{Z}$ can be meromorphically continued as a function of $s$ for a fixed $z$ giving a real analytic function of $z$ if $E'_{k,\chi}(z, s)$ is finite at $s \in \mathbb{C}$. If $\chi \neq 1$ or $k \neq 2$, $E'_{k,\chi}(z) = E'_{k,\chi}(z, 0)$ is an element in $M_{k,\chi}(\Gamma_0(N))$.

We only prove the last assertion for $k > 2$, since the proof of the other assertions require more preparation from real analysis. See [LFE] Chapter 9 (or [MFM] Chapter 7) for a proof of these assertions not proven here.

Proof. Suppose $k > 2$. Then $E'_{k,\chi}$ is absolutely and locally uniformly convergent by the exercise above, and hence $E'_{k,\chi}$ is a holomorphic functions in $z \in \mathbb{H}$. Thus we need to compute its Fourier expansion. Since the computation is basically the same for all cusps, we only do the computation at the cusp $\infty$. We use the following partial fraction expansion of cotangent function (can be found any advanced Calculus text or [LFE] (2.1.5-6) in page 28):

$$\pi \cot(\pi z) = \pi i \frac{\exp(2\pi iz) + 1}{\exp(2\pi iz) - 1} = \frac{1}{z} + \sum_{n=1}^{\infty} \left( \frac{1}{z + n} + \frac{1}{z - n} \right)$$

$$\pi \cot(\pi z) = \pi i \frac{\exp(2\pi iz) + 1}{\exp(2\pi iz) - 1} = \pi i \left( -1 - 2 \sum_{n=1}^{\infty} q^n \right), \quad q = \exp(2\pi iz).$$

The two series converge locally uniformly on $\mathbb{H}$ and periodic on $\mathbb{C}$ by definition. Applying the differential operator $(2\pi i)^{-1} \frac{\partial}{\partial z}$ to the formulas in (1.2) term by term, we get

$$S_k(z) = \sum_{n \in \mathbb{Z}} \frac{1}{(z + n)^k} = \frac{(-2\pi i)^k}{(k - 1)!} \sum_{n=1}^{\infty} n^{k-1} q^n.$$
Form this, assuming $\chi(-1) = (-1)^k$, we have

$$E'_{k,\chi}(z) = 2 \sum_{n=1}^{\infty} \chi(n)^{-1} n^{-k} + 2 \sum_{r=1}^{N} \chi^{-1}(r) \sum_{m=1}^{\infty} \sum_{n \in \mathbb{Z}} N^{-k}(mz + \frac{r}{N} + n)^{-k}$$

(1.4)

$$= 2L(k, \chi^{-1}) + 2 \sum_{r=1}^{N} \chi^{-1}(r) \sum_{m=1}^{\infty} S_k(mz + \frac{r}{N})$$

$$= \sum_{r=1}^{N} \chi^{-1}(r) \sum_{m=1}^{\infty} \sum_{n \in \mathbb{Z}} N^{-k}(mz + \frac{r}{N})$$

(1.3)

By the functional equation (see [LFE] Theorem 2.3.2), we have, if $\chi(-1) = (-1)^k$,

$$L(k, \chi^{-1}) = G(\chi^{-1}) \frac{(-2\pi i)^k}{N^k(k-1)!} L(1-k, \chi),$$

(1.5)

where $G(\psi)$ for a primitive character $\psi$ modulo $C$ is the Gauss sum $\sum_{r=1}^{C} \psi(r) \exp(2\pi i \frac{nr}{N})$.

We have $\sum_{r=1}^{N} \chi^{-1}(r) \exp(2\pi i \frac{nr}{N}) = \begin{cases} \chi(n) G(\chi^{-1}) & \text{if } n \text{ is prime to } N, \\ 0 & \text{otherwise,} \end{cases}$ and we get the formula

$$E'_{k,\chi}(z) = G(\chi^{-1}) \frac{2(-2\pi i)^k}{N^k(k-1)!} E_{k,\chi}(z)$$

(1.6)

for

$$E_{k,\chi}(z) = 2^{-1} L(1-k, \chi) + \sum_{n=1}^{\infty} \sigma_{k-1,\chi}(n) q^n$$

for $\sigma_{k-1,\chi}(n) = \sum_{0<d|n} \chi(d) d^{k-1}$. Here we used the convention that $E_{k,\chi}(z) = 0$ if $\chi(-1) \neq (-1)^k$.

**Exercise 1.11.** Give a proof of

$$\sum_{r=1}^{N} \chi^{-1}(r) \exp(2\pi i \frac{nr}{N}) = \begin{cases} \chi(n) G(\chi^{-1}) & \text{if } n \text{ is prime to } N, \\ 0 & \text{otherwise.} \end{cases}$$

**Exercise 1.12.** Let $p$ be a prime, and write $1_p$ for the imprimitive identity character of $(\mathbb{Z}/p\mathbb{Z})^\times$. Prove that

$$E_{k,1}(z) - p^{k-1} E_{k,1}(pz) = 2^{-1}(1 - p^{k-1}) \zeta(1-k) + \sum_{n=1}^{\infty} \sigma_{k-1,1}^{(p)}(n) q^n$$

for $\sigma_{k-1,1}^{(p)}(n) = \sum_{0<d|n,p\not|d} d^{k-1}$. More generally, if $N$ is prime to $p$, prove that

$$E_{k,\chi}(z) - \chi(p) p^{k-1} E_{k,\chi}(pz) = 2^{-1}(1 - \chi(p) p^{k-1}) L(1-k, \chi) + \sum_{n=1}^{\infty} \sigma_{k-1,\chi}^{(p)}(n) q^n$$

for $\sigma_{k-1,\chi}^{(p)}(n) = \sum_{0<d|n,p\not|d} \chi(d) d^{k-1}$. 
1.3. Eisenstein family. We continue to fix a positive integer \( N \) prime to \( p \) and a Dirichlet character \( \chi \) modulo \( Np \) with \( \chi(-1) = (-1)^k \). We know by a work of Shimura (recalled in [LEC] Proposition 2.18 when \( k = 2 \) and \( \chi = 1 \)) that \( M_{k,\chi}(\Gamma_0(Np^r); \mathcal{A}) \otimes_{\mathbb{A}} \mathbb{C} = M_{k,\chi}(\Gamma_0(Np)) \) for any algebra \( \mathcal{A} \subset \mathbb{C} \) containing the values of \( \chi \), where

\[
M_{k,\chi}(\Gamma_0(Np^r); \mathcal{A}) = \left\{ f \in M_{k,\chi}(\Gamma_0(Np^r)) \mid a(n, f) \in \mathcal{A} \quad \text{for all } n \geq 0 \right\}.
\]

Here we write the \( q \)-expansion of \( f \) as \( f = \sum_{n=0}^{\infty} a(n, f) q^n \). Then we take \( \mathcal{A} = W \cap \mathbb{Q} \) and define \( M_{k,\chi}(\Gamma_0(Np^r); W) = M_{k,\chi}(\Gamma_0(Np^r); \mathcal{A}) \otimes_{\mathcal{A}} W \) and

\[
M_{k,\chi}(\Gamma_0(Np^r); W) = M_{k,\chi}(\Gamma_0(Np^r); \mathcal{A}) \otimes_{\mathcal{A}} W.
\]

By definition, \( M_{k,\chi}(\Gamma_0(Np^r); \mathcal{A}) \hookrightarrow \mathcal{A}[[q]] \) via \( q \)-expansion.

**Definition 1.13.** A \( p \)-adic analytic family of modular forms of character \( \chi \) (modulo \( Np \)) with coefficients in \( \Lambda = W[[T]] \) is a formal \( q \)-expansion \( F(T) = \sum_{n=0}^{\infty} a(n, F(T)) q^n \in \Lambda[[q]] \) such that for all sufficiently large integers \( k \gg 0 \), \( F(\gamma^k - 1) \) is the \( q \)-expansion of an element in \( M_{k,\chi \cdot \omega^{-k}}(\Gamma_0(Np); W) \) for the Teichmüller character \( \omega(z) = \lim_{n \to \infty} z^{p^n} \) for \( z \in \mathbb{Z}_p \) (which factors through \( \mathbb{Z}/p\mathbb{Z} \)).

**Exercise 1.14.** Prove that the limit \( \omega(z) = \lim_{n \to \infty} z^{p^n} \) exists in \( \mathbb{Z}_p \) and that it gives rise to a Dirichlet character modulo \( p \).

**Exercise 1.15.** Prove that \( \log_p(z) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(z-1)^n}{n} \) converges \( p \)-adically for \( z \in \Gamma \) to an element in \( p \mathbb{Z}_p \) and satisfies \( \log_p(zw) = \log_p(z) + \log_p(w) \) and \( \gamma^{\log_p((n))/\log_p(\gamma)} = \langle n \rangle \) for all integer \( n \) prime to \( p \) (cf. [LFE] Section 1.3). Similarly, prove that \( \exp_p(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \) converges to an element in \( \Gamma \) \( p \)-adically over \( p \mathbb{Z}_p \) and show that \( \exp_p \circ \log_p \) and \( \exp_p \circ \log_p \) are the identity maps.

Define \( \Phi_{\chi}(T) \in W[[T]] \) by

\[
\Phi_{\chi}(\gamma^s - 1) = \begin{cases} 2^{-1}L_p(1 - s, \chi) & \text{if } \chi \neq 1 \\ 2^{-1}(\gamma^s - 1)L_p(1 - s, \chi) & \text{otherwise} \end{cases}
\]

for \( s \in \mathbb{Z}_p \) and

\[
a(n, \mathcal{E}_\chi)(T) = \begin{cases} \sum_{0 < d | n, p \not| d} \chi(d)(1 + T)^{\log_p((d))/\log_p(\gamma)} & \text{if } \chi \neq 1, \\ T \sum_{0 < d | n, p \not| d} (1 + T)^{\log_p((d))/\log_p(\gamma)} & \text{if } \chi = 1. \end{cases}
\]

**Exercise 1.16.** Using Theorem 1.5, prove the existence and uniqueness of \( \Phi_{\chi}(T) \in W[[T]] \) if \( \chi_N \) is primitive modulo \( N \).

**Theorem 1.17.** Let \( \chi \) be an even Dirichlet character modulo \( Np \) with primitive \( \chi_N \). Then the \( q \)-expansion \( \mathcal{E}_\chi = \Phi_{\chi}(T) + \sum_{n=1}^{\infty} a(n, \mathcal{E}_\chi) q^n \) gives a \( p \)-adic analytic family of modular form with character \( \chi \). Moreover \( E_\chi(\gamma^k - 1) \in M_{k,\chi \cdot \omega^{-k}}(\Gamma_0(Np); W) \) for \( k \geq 2 \) except for the case where \( \chi = 1 \). When \( N = 1 \) and \( \chi = 1 \), \( E_1(\gamma^k - 1) \in M_{k,\omega^{-k}}(\Gamma_0(Np); W) \) if \( k = 0 \) or \( k > 3 \).
Proof. We prove the result assuming \( \chi \neq 1 \), since the case of \( \chi = 1 \) and \( N = 1 \) is similar. By computation, we have
\[
    a(n, \mathcal{E}_\chi)(\gamma^k - 1) = \sum_{0 < d | n, p \not| n} \chi(d) \gamma^{k \log_p(d) / \log_p(\gamma)}
\]
\[
    = \sum_{0 < d | n, p \not| n} \chi(d) \exp_p(\log_p(\gamma))^{k \log_p(d) / \log_p(\gamma)} = \sum_{0 < d | n, p \not| n} \chi(d) \langle n \rangle^k
\]
\[
    = \sum_{0 < d | n, p \not| n} \chi \omega^{-k}(d) d^k = \sigma_{k, \chi}^{(p)}(n).
\]
Similarly by definition, \( \Phi_\chi(\gamma^k - 1) = 2^{-1} L_p(1-k, \chi \omega^{-k}) \). Thus we have from Exercise 1.12
\[
    \mathcal{E}_\chi(\gamma^k - 1) = \begin{cases} 
        E_{k, \chi \omega^{-k}}(\gamma) & \text{if } \chi \omega^{-k} \text{ is primitive modulo } Np, \\
        E_{k, \chi}(\gamma) - \chi(N) p^k E_{k, \chi}(p \gamma) & \text{otherwise}.
    \end{cases}
\]
This finishes the proof. \( \square \)

Exercise 1.18. Give a detailed proof of the above theorem when \( \chi = 1 \) and \( N = 1 \).

The collection of all \( p \)-adic analytic families of modular forms with character \( \chi \) form a \( \Lambda \)-module \( M_\chi(N; \Lambda) \). If \( F \in M_\chi(N; \Lambda) \) specializes to a cusp form \( F(\gamma^k - 1) \in S_{k, \chi \omega^{-k}}(\Gamma_0(N); W) \) for all sufficiently large \( k \gg 0 \), \( F \) is called a \( p \)-adic analytic cuspidal family. The correction of all cuspidal families is written as \( S_\chi(N; \Lambda) \). For a given modular form \( f \in M_{\ell, \psi}(\Gamma_0(pN); W) \), we can define a convoluted product \( f \ast \mathcal{E}_\chi \) by
\[
    f \ast \mathcal{E}_\chi(T) = f \mathcal{E}_\chi(\gamma^{-\ell}(1 + T) - 1)).
\]
Then \( f \ast \mathcal{E}_k \in \Lambda[[q]] \) and by computation, we have \( f \ast \mathcal{E}_\chi(\gamma^k - 1) = f \cdot \mathcal{E}(\gamma^{k - \ell} - 1) \). Since \( \mathcal{E}(\gamma^{k - \ell} - 1) \in M_{k - \ell, \psi \omega^{-k}}(\Gamma_0(N); W) \), we find \( f \ast \mathcal{E}(\gamma^{k - \ell} - 1) \in M_{k, \psi \omega^{-k}}(\Gamma_0(N); W) \) if \( k \geq \ell + 2 \). This shows

Corollary 1.19. We have \( f \ast \mathcal{E}_\chi \in M_{\psi \omega^c}(\Gamma_0(N); \Lambda) \) if \( f \in M_{\ell, \psi}(\Gamma_0(pN); W) \). If \( f \in S_{\ell, \psi}(\Gamma_0(pN); W) \), we have \( f \ast \mathcal{E}_\chi \in S_{\psi \omega^c}(\Gamma_0(N); \Lambda) \).

In this way, we can produce a lot of \( p \)-adic analytic families.

Exercise 1.20. Prove that \( f \ast \mathcal{E}_\chi \in S_{\psi \omega^c}(\Gamma_0(N); \Lambda) \) if \( f \in S_{\ell, \psi}(\Gamma_0(pN); W) \).

1.4. Hecke operators. Recall
\[
    \Delta_0(pN) = \left\{ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in M_2(\mathbb{Z}) \mid c \equiv 0 \mod pN, aZ + NpZ = Z, ad - bc > 0 \right\}.
\]
We define a character \( \chi_\Delta \) of \( \Delta_0(pN) \) by \( \chi_\Delta(\left( \begin{array}{cc} a & b \\ c & d \end{array} \right)) = \chi^{-1}(a) \).

Exercise 1.21. Prove that \( \chi_\Delta(\left( \begin{array}{cc} a & b \\ c & d \end{array} \right)) = \chi(d) \) if \( \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma_0(pN) \).

Define for \( \alpha = (\left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Delta_0(pN) \) and a function \( f : \mathfrak{H} \to \mathbb{C} \), a new function \( f|_{k, \chi} \alpha(z) \) by \( f|_{k, \chi} \alpha(z) = \det(\alpha)^{-k} f(\alpha(z)) \chi^{-1}_\Delta(\alpha)(cz + d)^{-k} \). Splitting \( T_n = \{ \alpha \in \Delta_0(pN) | \det(\alpha) = n \} \) into a disjoint union \( T_n = \bigsqcup_{\alpha} \Gamma_0(pN) \alpha \), we define \( f|T(n) = \sum_\alpha f|_{k, \chi} \alpha \). Then the same proof of Lemma 2.14 in [LEC] gives
Lemma 1.22.  (1) Write $T(n)$ for the operator corresponding to $T_n$. Then $T(n)$ gives a linear endomorphism of $M_{k,\chi}(\Gamma_0(pN))$.

(2) We get the following identity of Hecke operators for $f \in M_{k,\chi}(\Gamma_0(pN))$:

$$a(m, f|T(n)) = \sum_{0<d|(m,n)\cdot(d,pN)=1} \chi(d)d^{k-1} \cdot a\left(\frac{mn}{d^2}, f\right).$$

(3) $T(m)T(n) = T(n)T(m)$ for all integers $m$ and $n$.

When $m|pN$, we often write $U(m)$ for $T(m)$.

Exercise 1.23. Give a detailed proof of the above Lemma.

Corollary 1.24. If $k \geq 1$ and $A$ contains the values of $\chi$, the Hecke operators $T(n)$ preserves $M_{k,\chi}(\Gamma_0(pN), \chi)$.

Definition 1.25. We consider the operator $T_\Lambda(n)$ on $F = \sum_{n=0}^{\infty} a(n, F)(T)q^n\Lambda[[q]]$ defined by $a(m, F|T_\Lambda(n)) = \sum_{0<d|(m,n)\cdot(d,\Lambda)=1} \chi(d)d^{-1}(1+T)^{\log_p(d)/\log_p(\gamma)} \cdot a\left(\frac{mn}{d^2}, F\right)$.

Since $(1+T)^{\log_p(d)/\log_p(\gamma)}|_{T=\gamma^k-1} = \langle d \rangle^k = \omega^{-k}(d)d^k$, after specializing $T = \gamma^k-1$, we find $(F|T_\Lambda(n))(\gamma^k-1) = (F(\gamma^k-1)|T(n))$. Thus $T_\Lambda(n)$ preserves $M_{\chi}(N; \Lambda)$ and $S_{\chi}(N; \Lambda)$.

Proposition 1.26. We have a linear operator $T_\Lambda(n)$ defined by Definition 1.25 acting on $M_{\chi}(N; \Lambda)$ which preserves $S_{\chi}(N; \Lambda)$. In particular, $(F|T_\Lambda(n))(\gamma^k-1) = (F(\gamma^k-1)|T(n))$ for all $F \in M_k(N; \Lambda)$ and all $k \gg 0$ and $T_\Lambda(m)T_\Lambda(n) = T_\Lambda(n)T_\Lambda(m)$ and $T_\Lambda(m)T_\Lambda(n) = T_\Lambda(mn)$ if $m\mathbb{Z} + n\mathbb{Z} = \mathbb{Z}$.

There are a lot of questions we can ask for $p$-adic analytic families; for example,

(Q1) Is the $\Lambda$-module $M_{\chi}(N; \Lambda)$ finitely generated?

(Q2) Is the module $M_{\chi}(N; \Lambda)$ spanned by Hecke eigenforms (at least topologically if it is infinite rank)?

(Q3) What is $F(\zeta-1)$ for a general $\zeta \in \overline{\mathbb{Q}_p}$ with $|\zeta-1|_p < 1$?

(Q4) If $F \in M_{\chi}(N; \Lambda)$ is a common Hecke eigenform with $a(1, F) = 1$ with $\Lambda$-unit eigenvalue for $U(p)$, writing $d\varphi_k$ for the $p$-adic measure constructed for $F(\gamma^k-1)$ in [LFE] in Section 6.5 and in [LEC] Theorem 2.36, what is the relation among $d\varphi_k$ for $k \gg 0$?

We try to answer some of these questions.

1.5. Modular forms of level $N$. We generalize a bit the notion of modular forms. Let

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) | a \equiv d \equiv 1 \mod N \right\}. $$

A modular form $f \in M_k(\Gamma_1(N))$ is a holomorphic function on $\mathfrak{H}$ satisfying the conditions (M1–2) in 1.2 for $\Gamma_1(N)$ in place of $\Gamma_0(N)$. Since $d \equiv 1 \mod N$ and $\chi$ is a character modulo $N$ in (M1), this space is independent of the choice of $\chi$, and hence the subscript $\chi$ is dropped from the notation. Similarly we define the subspace of cusp forms $S_k(\Gamma_1(N))$ inside $M_k(\Gamma_1(N))$ by imposing (S) in addition to (M1–2). Then we define first for a ring $A \subset \mathbb{C}$

$$M_k(\Gamma_1(N); A) = \left\{ f \in M_k(\Gamma_1(N)) | a(n, f) \in A \text{ for all } n \geq 0 \right\}$$
We have Lemma 1.27.

\[
M_k(\Gamma_1(N); \mathbb{Z}) \otimes_\mathbb{Z} A = M_k(\Gamma_1(N); A) \quad \text{and} \quad M_k(\Gamma_1(N); A) \otimes_A \mathbb{C} = M_k(\Gamma_1(N))
\]

Thus, for an algebra \( X \) with \( W \subset X \subset \overline{\mathbb{Q}}_p \), taking \( A = X \cap \overline{\mathbb{Q}} \), we may define \( M_k(\Gamma_1(N); X) = M_k(\Gamma_1(N); A) \otimes_A X \) and \( S_k(\Gamma_1(N); X) = M_k(\Gamma_1(N); A) \otimes_A X \). These spaces can be embedded into \( X[[q]] \) by \( q \)-expansion.

Since \( \Gamma_0(N)/\Gamma_1(N) \cong (\mathbb{Z}/N\mathbb{Z})^\times \) by \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (d \mod N) \), the finite group \( (\mathbb{Z}/N\mathbb{Z})^\times \) acts on \( M_k(\Gamma_1(N)) \). Then by definition, the \( \chi \)-eigenspace of \( M_k(\Gamma_1(N)) \) is the space \( M_{k,\chi}(\Gamma_0(N)) \):

\[
M_{k,\chi}(\Gamma_0(N)) = \left\{ f \in M_k(\Gamma_1(N)) \mid f|\langle d \rangle = \chi(d)f \text{ for all } d \in (\mathbb{Z}/N\mathbb{Z})^\times \right\},
\]

where \( f|\langle a \rangle = f|k \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) for \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) \). Thus we get

**Lemma 1.27.** We have

\[
M_k(\Gamma_1(N)) = \bigoplus_{\chi} M_{k,\chi}(\Gamma_0(N))
\]

\[
S_k(\Gamma_1(N)) = \bigoplus_{\chi} S_{k,\chi}(\Gamma_0(N)),
\]

where \( \chi \) runs over all Dirichlet characters modulo \( N \).

The Hecke operator \( T(n) \) on each \( M_{k,\chi}(\Gamma_0(N)) \) gives rise to a Hecke operator on the sum \( M_k(\Gamma_1(N)) \) over \( \chi \). In other words, writing \( f \in M_k(\Gamma_1(N)) \) as \( f = \bigoplus \chi f_\chi \) with \( f_\chi \in M_{k,\chi}(\Gamma_0(N)) \), we have \( f|T(n) = \bigoplus \chi (f_\chi|T(n)) \).

**Exercise 1.28.** Prove that for \( f \in M_k(\Gamma_1(N)) \)

\[
a(m, f|T(n)) = \sum_{0<d|(m,n), l|d,N=1} d^{k-1}a\left(\frac{mn}{d^2}, f|\langle d \rangle\right)
\]

and \( T(\ell)^2 - T(\ell^2) = \ell^{k-1}|\ell \) if \( \ell \) is a prime outside \( N \).

Let

\[
\Delta_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}) \mid c \equiv 0 \mod N, \ a \equiv 1 \mod N, \ ad - bc > 0 \right\}.
\]

**Exercise 1.29.** Splitting \( T_n = \{ \alpha \in \Delta_1(N) \mid \det(\alpha) = n \} \) into a disjoint union \( T_n = \bigsqcup_\alpha \Gamma_0(pN)\alpha \), prove that \( f|T(n) = \sum_\alpha f|k,\alpha \).

The modular curve \( X_1(N)(\mathbb{C}) = \Gamma_1(N)\backslash(\mathfrak{H} \cup \mathbb{P}^1(\mathbb{Q})) \) has a regular model \( X_1(N) \) over \( \mathbb{Z} \). We admit the following nontrivial facts which could be proven using algebraic geometry on the regular scheme \( X_1(N)/\mathbb{Z} \):

**Theorem 1.30.** The space \( M_k(\Gamma_1(N); A) \) and \( S_k(\Gamma_1(N); A) \) are stable under the action of \((\mathbb{Z}/N\mathbb{Z})^\times\) and hence under \( T(n) \) for all \( n \) for any algebra \( A \).
1.6. **Slope of modular forms.** A modular form \( f \in M_k(\Gamma_1(pN); K) \) \((r > 0)\) has slope \( \alpha \) if \( f((U(p) - a)^M = 0 \) (for a sufficiently large integer \( M \gg 0 \)) and \( |a|_p = p^{-\alpha} \). If \( f|U(p)^M = 0 \) (for \( M \gg 0 \)), we call \( f \) to have infinity slope.

**Lemma 1.31.** Let \( X \) be a \( W \)-module of finite type and \( T : X \to X \) be a \( W \)-linear operator. Then the \( p \)-adic limit \( e = \lim_{n \to \infty} T^n \) exists in \( \text{End}_W(X) \) and gives an idempotent of \( \text{End}_W(X) \).

**Proof.** Since \( \text{End}_W(X) \) is a \( W \)-module of finite type, the \( W \)-subalgebra \( A \) generated by \( T \) is a \( W \)-algebra which is of finite type as \( W \)-modules. Thus we need to prove the existence of \( \lim_{n \to \infty} T^n \) in a \( W \)-algebra \( X \) which is a \( W \)-module of finite type. In particular, we have a finitely many generators \( a_1, \ldots, a_r \) of \( A \) over \( W \) and a \( W \)-linear surjection: \( W^r \to A \) sending \((x_j)_{j=1,\ldots,r}\) to \( \sum_j x_ja_j \). In particular \( A/pA \) is finite; so, writing \( \Omega \) the set of all maximal ideals of \( A \), \( \Omega \) is a finite set. Also \( A \) is \( p \)-adically complete; so, \( A = \lim_{n \to \infty} A/p^nA \). We have \( \bigcap_n \bigcap_{m \in \Omega} m^n = 0 \); so, \( A = \lim_{n \to \infty} A/\bigcap_{m \in \Omega} m^n \). By Chinese remainder theorem applied to \( A/\bigcap_{m \in \Omega} m^n \), we find \( A/\bigcap_{m \in \Omega} m^n = \bigoplus_m A/m^nA \). Thus we find that \( A = \bigoplus_{m \in \Omega} A_m \) for \( A_m = \lim_{n \to \infty} A/m^nA \). Then \( A \) is a direct product of local rings \( A_m \). Thus we may assume that \( A \) is local. Then \( \lim_{n \to \infty} T^n = 0 \) if \( T \in m \).

Since \( A/mA \) is a finite field of characteristic \( p \), its order is \( p^j \). Then \( a_m = |(A/m^nA)^x| = p^{m(a)}(p^j - 1) \) for an increasing sequence \( m_n \). In particular, \( T^{m_n} \equiv 1 \mod m^n \), and hence \( 1 = \lim_{n \to \infty} T^{m_n} = \lim_{n \to \infty} T^n \). \( \square \)

**Exercise 1.32.** Under the notation of the above proof, give a detailed proof of the following facts:

1. \( p \in \bigcap_{m \in \Omega} m \);
2. \( pA \supset \left( \bigcap_{m \in \Omega} m \right)^n \) for sufficiently large \( n > 0 \);
3. \( \bigcap_{n=1}^\infty \bigcap_{m \in \Omega} m^n = \bigcap_{n=1}^\infty p^nA = 0 \);
4. \( \lim_{n \to \infty} T^{a_n} = \lim_{n \to \infty} T^n \).

Let \( e = \lim_{n \to \infty} U(p)^n \) in \( \text{End}_W(M_k(\Gamma_1(Np^r); A)) \) for \( A = W \) or \( K \), and define

\[
M^\text{ord}_k(\Gamma_1(Np^r); A) = e(M_k(\Gamma_1(Np^r); A)).
\]

The following lemma is easy:

**Lemma 1.33.** \( f \in M_k(\Gamma_1(Np^r); W) \) is of slope zero if and only if \( f \in M^\text{ord}_k(\Gamma_1(Np^r); W) \) and \( f \) is an eigenform for \( U(p) \).

**Exercise 1.34.** Prove the above lemma.

**Definition 1.35.** Define a Hecke algebra \( \mathbb{H}_k(Np^r; A) \) (resp. \( \mathfrak{h}_k(Np^r; A) \)) by the \( \mathcal{A} \)-subalgebra of \( \text{End}_A(M_k(\Gamma_1(Np^r); A)) \) (resp. \( \text{End}_A(S_k(\Gamma_1(Np^r); A)) \)) generated by Hecke operators \( T(n) \) for all \( n \).

We can define the corresponding spaces \( M^\text{ord}_\chi(N; A) \) of \( p \)-ordinary analytic families as follows:
**Definition 1.36.**

\[ M^\text{ord}_X(N; \Lambda) = \left\{ F \in M_X(N; \Lambda) \mid F(\gamma^k - 1) \in M^\text{ord}_{k,\chi^\omega - k}(\Gamma_0(Np)) \text{ for all } k \gg 0 \} \]

and \( S^\text{ord}_X(N; \Lambda) = S_X(N; \Lambda) \cap M^\text{ord}_X(N; \Lambda) \).

The following theorem is proven in 1986 in my papers [H86a], [H86b] and [LFE] Chapter 7 (except for the case for \( M^{2,1}_2(\Gamma_0(p); W) \)):

**Theorem 1.37.** \( M^\text{ord}_X(N; \Lambda) \) and \( S^\text{ord}_X(N; \Lambda) \) are free of finite rank over \( \Lambda \), and the specialization map induces isomorphisms

\[
M^\text{ord}_X(N; \Lambda) \otimes_\Lambda \Lambda/(T - (\gamma^k - 1)) \cong M^\text{ord}_{k,\chi^\omega - k}(\Gamma_0(pN); W),
\]

\[
S^\text{ord}_X(N; \Lambda) \otimes_\Lambda \Lambda/(T - (\gamma^k - 1)) \cong S^\text{ord}_{k,\chi^\omega - k}(\Gamma_0(pN); W)
\]

for all \( k \geq 2 \).

**Corollary 1.38.** Any element in \( M^\text{ord}_{k,\chi}(\Gamma_0(pN); W) \) for \( k \geq 2 \) can be lifted to a \( p \)-adic analytic family. Moreover if \( k \geq 2 \), we have

\[
\text{rank}_W M^\text{ord}_{k,\chi^\omega - k}(\Gamma_0(pN); W) = \text{rank}_\Lambda M^\text{ord}_X(N; \Lambda),
\]

\[
\text{rank}_W S^\text{ord}_{k,\chi^\omega - k}(\Gamma_0(pN); W) = \text{rank}_\Lambda S^\text{ord}_X(N; \Lambda)
\]

which are independent of \( k \geq 2 \).

**Definition 1.39.** Let \( 0 < k \in \mathbb{Z} \) be an integer with divisible by \( p - 1 \) (so, \( \omega^k = 1 \)). A weak \( p \)-adic analytic family of modular forms (centered at \( 0 < k \in \mathbb{Z} \)) is a formal power series \( F = \sum_{n=0}^{\infty} a(n, F)(T)q^n \) with \( a(n, F)(T) \in K[[T]] \) convergent at \( \gamma^{k'} - 1 \) for all \( k' \) in a small \( p \)-adic neighborhood \( U \) in \( k \cdot \Gamma \subset \mathbb{Z}_p^* \) of \( k \) such that \( F(\gamma^{k'} - 1) \in M_{k',\chi}(\Gamma_0(pN); K) \) for all \( k' \gg 0 \) in \( U \).

This type of weak families was introduced in [GM] by Mazur and Gouvêa in 1992. For a given slope \( \alpha \in \mathbb{Q} \), we define \( M^{(\alpha)}_{k,\chi}(\Gamma_0(pN); K) \) be the space spanned by slope \( \alpha \) modular forms in \( M_{k,\chi}(\Gamma_0(pN); K) \) and put \( S^{(\alpha)}_{k,\chi}(\Gamma_0(pN); K) = M^{(\alpha)}_{k,\chi}(\Gamma_0(pN); K) \cap S_{k,\chi}(\Gamma_0(pN); K) \). By definition, we have

\[
M^{(0)}_{k,\chi}(\Gamma_0(pN); K) = M^\text{ord}_{k,\chi}(\Gamma_0(pN); K).
\]

Moreover, we have

\[
M_{k,\chi}(\Gamma_0(pN); K) = \bigoplus_{\alpha} M^{(\alpha)}_{k,\chi}(\Gamma_0(pN); K).
\]

**Exercise 1.40.** Prove the above decomposition.

Gouvêa and Mazur made the following conjecture

**Conjecture 1.41** (Gouvêa and Mazur, 1992).

1. If \( k, k' \geq 2\alpha + 2 \) and \( k \equiv k' \pmod{p^n(p - 1)} \) for \( n \geq \alpha \), then

\[
\dim_K S^{(\alpha)}_{k,\chi}(\Gamma_0(pN); K) = \dim_K S^{(\alpha)}_{k',\chi}(\Gamma_0(pN); K),
\]
(2) If $k \geq 2\alpha + 2$, any $f \in S^{(\alpha)}_{k, \chi}(\Gamma_0(pN); K)$ can be lifted to a weak analytic family of slope $\alpha$ (centered at $k$).

Their conjecture is actually slightly stronger than what is stated here. In this conjecture, they predict the neighborhood $U = U_k$ of a given $k \geq 2\alpha + 2$ (appearing in the definition of the weak families) is specified as

$$U_k = \{ k' \in k \cdot \Gamma \mid |k' - k|_p \leq p^{-\lceil \alpha \rceil} \}.$$

Though K. Buzzard found a counter example against the lower bound $k \geq 2\alpha + 2$ of (1) when $p = 2$, the conjecture would be true for $p > 3$ (as Gouvêa and Mazur actually assumed). A slightly different version of the conjecture (2) valid for $k \geq \alpha + 1$ (for a neighborhood $U \subset U_k$) was proven by Coleman [C] in 1998 (a lower bound for $k$ for the validity of (1) quadratic in $\alpha$ was proven by Wan [W] soon after [C]). This result implies that actually that Hecke eigenforms in $M_{k', \chi' \cdot \omega - \kappa'}(\Gamma_0(pN); \overline{\mathbb{Q}}_p)$ is parameterized by $k' \in U \subset U_k$ (as we will see later for $p$-ordinary forms). Then Coleman and Mazur further went on to globalize the (local) parameter space $U_k$ of modular Hecke eigenforms to a rigid analytic curve (the so called eigencurve) in [CM].

Theorem 1.37 (proven earlier than the conjecture) gives a finer result than the conjecture for slope 0 forms and was a main supporting evidence for the conjecture. Indeed, in this case, the eigencurve is actually a formal scheme finite flat over $\text{Spec}(\Lambda)$ (not just a rigid analytic space) and is given by $\text{Spec}(\mathbb{H}_\chi)$ for the Hecke algebra $\mathbb{H}_\chi \subset M_\chi(N; \Lambda)$ generated by Hecke operators $T(n)$ over $\Lambda$. Note here the rigid analytic space $\text{Spec}(\mathbb{H}_\chi)(\mathbb{C}_p)$ is a $p$-adic open unit disk. We would prove Theorem 1.37 to some extent in this course by a cohomological means in [H86b] (two other methods are discussed in [H86a] and [LFE] Chapter 7, respectively).

**Exercise 1.42.** Prove that $\text{Spec}(\Lambda)(\overline{\mathbb{Q}}_p) = \text{Hom}_{W\text{-alg}}(\Lambda, \overline{\mathbb{Q}}_p)$ is isomorphic to the open unit disk $D$ in $\overline{\mathbb{Q}}_p$ (centered at the origin 0) by sending a $W$-algebra homomorphism $\phi : \Lambda \to \overline{\mathbb{Q}}_p$ to $\phi(T)$. 