1. INTRODUCTION

In this introduction, first, without going into technical details, we describe a prototypical example of a p-adic analytic families of modular forms. Starting with the third week (or slightly earlier), we start to justify our construction cohomologically. The examples we describe are from [LFE] Chapter 7.

1.1. p-Adic L-functions as a power series.

We start with a general fact on the Kubota-Leopoldt p-adic L-functions. We consider the binomial formula:

$$(1+T)^s = \sum_{n=0}^{\infty} \binom{s}{n} T^n. \tag{1.1}$$

Since $s \mapsto \binom{s}{n} = \frac{s(s-1)\cdots(s-(n-1)}{n!}$ is a polynomial in s and has integer value over natural numbers, it is a polynomial on $\mathbb{Z}_p$ with values in $\mathbb{Z}_p$. Thus if $\gamma \equiv 1 \mod p$, we have the p-adic power $\gamma^s = \sum_{n=0}^{\infty} \binom{s}{n}(\gamma-1)^n \in \mathbb{Z}_p$ (convergent p-adically) for all $s \in \mathbb{Z}_p$.

Let K be a finite extension of $\mathbb{Q}_p$ with the p-adic integer ring $W = \{w \in K \big| |w|_p \leq 1\}$. Let φ be a p-adic measure on $\mathbb{Z}_p$ with values in W (so it is a bounded measure). Since the power series ring $W[[T]]$ is a Banach algebra under the norm $|| \sum_{n=1}^{\infty} a_n T^n|| = \mathrm{Sup}_n |a_n|_p$, we can integrate any continuous function $\phi : \mathbb{Z}_p \to W[[T]]$ under $d\varphi$. In other words, we approximate ϕ by step functions $\phi_n : \mathbb{Z}_p \to W[[T]]$ factoring through $(\mathbb{Z}/p^n\mathbb{Z})$ so that $\lim_{n\to\infty} \phi_n = \phi$ under the norm $|| \cdot ||$ and define

$$\int_{\mathbb{Z}_p^\times} \phi d\varphi = \lim_{n\to\infty} \int_{\mathbb{Z}_p^\times} \phi_n d\varphi \in W[[T]].$$

Exercise 1.1. *Prove that*

$$\int_{\mathbb{Z}_p^\times} (1+T)^s d\varphi(s) = \sum_{\mathbb{Z}_p^\times} \binom{s}{n} d\varphi(s) T^n =: \Phi_\varphi(T).$$

Let $\Gamma = 1 + p\mathbb{Z}_p$ and $z \mapsto \langle z \rangle = \omega(z)^{-1} z$ be the projection of $\mathbb{Z}_p^\times$ onto Γ, where ω is the Teichmüller character defined in [LEC] Theorem 1.33. By the existence of a primitive root, for an odd prime p, the multiplicative group $(\mathbb{Z}/p^n\mathbb{Z})^\times$ is a cyclic group, and hence its subgroup $\{x \in (\mathbb{Z}/p^n\mathbb{Z})^\times | x \equiv 1 \mod p\}$ is cyclic generated by $\gamma = 1 + p$.

Exercise 1.2. *Let $\Gamma^n = \{u^n | u \in \Gamma\} \subset \Gamma$ and p be an odd prime. Prove the following facts*

1. $\Gamma^n = \Gamma$ *if* $p \nmid n$;
2. $\Gamma/\Gamma^{p^{n-1}} \cong \{x \in (\mathbb{Z}/p^n\mathbb{Z})^\times | x \equiv 1 \mod p\}$ *by sending* $u\Gamma^{p^{n-1}}$ *to* $u \mod p^n$. *In particular,* $[\Gamma : \Gamma^{p^n}] = p^n$;
3. $\Gamma \cong \mathbb{Z}_p$ *by* $\gamma^s \mapsto s \in \mathbb{Z}_p$ *for* $\gamma = 1 + p$;
4. $1 + 4\mathbb{Z}_2 \cong \mathbb{Z}_2$ *by* $\gamma^s \mapsto s \in \mathbb{Z}_2$ *for* $\gamma = 5$.

We have a projection $\langle \cdot \rangle : \mathbb{Z}_p^\times \to \Gamma$. Thus we can define a bounded measure $\langle \varphi \rangle$ on Γ by $\int_\Gamma \phi d\langle\varphi\rangle = \int_{\mathbb{Z}_p^\times} \phi(\langle z \rangle) d\varphi$. Identifying Γ with $\mathbb{Z}_p$ by $\gamma^s \leftrightarrow s \in \mathbb{Z}_p$, consider $\Phi_{\langle\varphi\rangle}(T) \in W[[T]]$.

Lemma 1.3. *We have* $\int_\Gamma u^s d\langle\varphi\rangle(u) = \Phi_{\langle\varphi\rangle}(\gamma^s - 1)$.

Proof. For the isomorphism $\iota : \Gamma \cong \mathbb{Z}_p$ with $\iota(\gamma^z) = z$, we can define a measure φ_+ on $\mathbb{Z}_p$ by $\int_{\mathbb{Z}_p} \phi d\varphi_+ = \int_\Gamma \phi \circ \iota d\langle\varphi\rangle$. Then we have $\Phi_{\langle\varphi\rangle} = \Phi_{\varphi_+}$, and $\Phi_{\varphi_+}(T) = \int_{\mathbb{Z}_p}(1+T)^z d\varphi_+(z)$. Replacing T by $\gamma^s - 1$ and writing $u = \gamma^z$, we get

$$\Phi_{\varphi_+}(\gamma^s - 1) = \int_{\mathbb{Z}_p} \gamma^{sz} d\varphi_+(z) = \int_\Gamma u^s d\langle\varphi\rangle(u),$$

which shows the assertion. □

Exercise 1.4. *Define a Dirac measure* δ_z *for* $z \in \mathbb{Z}_p$ *by* $\int_{\mathbb{Z}_p} \phi d\delta_z = \phi(z)$. *Prove that* $\Phi_{\delta_z}(T) = (1+T)^z$.

Let N be a positive integer prime to p. We defined in [LEC] Theorem 1.33 the p-adic Dirichlet L-function for each primitive odd character χ modulo Np^r (with values in K). Reformulating the result there (by making a variable change $\chi \mapsto \chi\omega^{-1}$; so now χ is even), by the above lemma, we thus get

Theorem 1.5. *Let* N *be a positive integer prime to* p *and* χ *with* $\chi(-1) = 1$ *be a Dirichlet character modulo* Np. *Suppose that* χ_N *is primitive modulo* N. *Then there exists a power series* $\Phi_\chi(T) \in W[[T]]$ *such that* $L_p(1-s, \chi) = \Phi_\chi(\gamma^s - 1)$ *if* $\chi_N \neq \mathbf{1}$ *and* $L_p(1-s, \mathbf{1}) = \frac{\Phi_{\mathbf{1}}(\gamma^s-1)}{\gamma^s-1}$.

Exercise 1.6. *Give a detailed proof of the above theorem.*

A p-adic analytic function on $\mathbb{Z}_p$ of the form $s \mapsto \Phi(\gamma^s - 1)$ for a power series $\Phi(T) \in W[[T]]$ is called an Iwasawa function. Iwasawa functions form a special subclass of p-adic analytic functions on $\mathbb{Z}_p$.

1.2. **Eisenstein series.** Let $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \overline{\mathbb{Q}}^\times$ be a primitive Dirichlet character. We consider the Eisenstein series of weight $0 < k \in \mathbb{Z}$

$$E'_k(z,s) = \sum_{(m,n)\in\mathbb{Z}^2-\{(0,0)\}} \chi^{-1}(n)(mNz+n)^{-k}|mNz+n|^{-2s},$$

where $z \in \mathfrak{H}$ and $s \in \mathbb{C}$. When $N = 1$, χ is the trivial character $\mathbf{1}$. For the following exercise, see [MFM] Section 2.6 and Chapter 7.

Exercise 1.7. *Prove*

1. $E'_{k,\chi}(z,s)$ *converges absolutely and locally uniformly with respect to* $(z,s) \in \mathfrak{H}\times\mathbb{C}$ *if* $\mathrm{Re}(2s+k) > 2$;
2. $E'_{k,\chi}(z,s) = 0$ *if* $\chi(-1) \neq (-1)^k$ *(assuming convergence);*
3. $E'_{k,\chi}(z) = E'_{k,\chi}(z,0)$ *is a holomorphic function of* z *if* $k > 2$ *(this fact is actually true if* $k = 2$ *and* $\chi \neq \mathbf{1}$ *for the limit* $E'_{k,\chi}(z) = \lim_{s\to+0} E'_{k,\chi}(z,s)$*);*
4. $E'_{k,\chi}(\gamma(z)) = \chi(d)(cz+d)^k E'_{k,\chi}(z)$ *for* $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$.

A holomorphic function $f : \mathfrak{H} \to \mathbb{C}$ is called a modular form on $\Gamma_0(N)$ of weight k with character χ if f satisfies the following conditions:

(M1) $f(\frac{az+b}{cz+d}) = \chi(d)f(z)(cz+d)^k$ for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$;

(M2) f is finite at all cusps of $X_0(N)$; in other words, for all $\alpha = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$, $f|_k\alpha(z) = f(\alpha(z))(cz+d)^{-k}$ has Fourier expansion of the form

$$\sum_{0 \le n \in N^{-1}\mathbb{Z}} a(n, f|_k\alpha)\exp(2\pi i n z) \quad (\text{with } a(n, f|_k\alpha) \in \mathbb{C}).$$

The above condition means that the function f is finite at the cusp $\alpha(\infty)$ of $X_0(N)$ (whose value at the cusp is $a(0, f|_k\alpha)$. We write $M_{k,\chi}(\Gamma_0(N))$ for the space of functions satisfying (M1–2). Replace (M2) by

(S) f is vanishing at all cusps of $X_0(N)$ (that is, $a(n, f|_k\alpha) = 0$ for all $\alpha \in SL_2(\mathbb{Z})$ and $n \le 0$),

we define subspace $S_{k,\chi}(\Gamma_0(N)) \subset M_{k,\chi}(\Gamma_0(N))$ by imposing (S). Functions in the space $S_{k,\chi}(\Gamma_0(N))$ are called holomorphic cusp forms on $\Gamma_0(N)$ of weight k with character χ.

Exercise 1.8. *Prove that* $M_{0,\chi}(\Gamma_0(N))$ *is either* $\mathbb{C}$ *(constants) or* 0 *according as* $\chi = \mathbf{1}$ *or not.*

Exercise 1.9. *Prove that* $M_{k,\chi}(\Gamma_0(N)) = 0$ *if* $\chi(-1) \ne (-1)^k$.

Proposition 1.10. *Let* χ *be a primitive Dirichlet character modulo* N. *The Eisenstein series* $E'_{k,\chi}(z,s)$ *for* $0 < k \in \mathbb{Z}$ *can be meromorphically continued as a function of* s *for a fixed* z *giving a real analytic function of* z *if* $E'_{k,\chi}(z,s)$ *is finite at* $s \in \mathbb{C}$. *If* $\chi \ne \mathbf{1}$ *or* $k \ne 2$, $E'_{k,\chi}(z) = E'_{k,\chi}(z,0)$ *is an element in* $M_{k,\chi}(\Gamma_0(N))$.

We only prove the last assertion for $k > 2$, since the proof of the other assertions require more preparation from real analysis. See [LFE] Chapter 9 (or [MFM] Chapter 7) for a proof of these assertions not proven here.

Proof. Suppose $k > 2$. Then $E'_{k,\chi}$ is absolutely and locally uniformly convergent by the exercise above, and hence $E'_{k,\chi}$ is a holomorphic functions in $z \in \mathfrak{H}$. Thus we need to compute its Fourier expansion. Since the computation is basically the same for all cusps, we only do the computation at the cusp ∞. We use the following partial fraction expansion of cotangent function (can be found any advanced Calculus text or [LFE] (2.1.5-6) in page 28):

$$\begin{aligned} \pi\cot(\pi z) &= \pi i\frac{\exp(2\pi i z)+1}{\exp(2\pi i z)-1} = \frac{1}{z} + \sum_{n=1}^{\infty}\left(\frac{1}{z+n} + \frac{1}{z-n}\right) \\ \pi\cot(\pi z) &= \pi i\frac{\exp(2\pi i z)+1}{\exp(2\pi i z)-1} = \pi i\left(-1 - 2\sum_{n=1}^{\infty} q^n\right), \quad q = \exp(2\pi i z). \end{aligned} \tag{1.2}$$

The two series converge locally uniformly on $\mathfrak{H}$ and periodic on $\mathbb{C}$ by definition. Applying the differential operator $(2\pi i)^{-1}\frac{\partial}{\partial z}$ to the formulas in (1.2) term by term, we get

$$S_k(z) = \sum_{n\in\mathbb{Z}} \frac{1}{(z+n)^k} = \frac{(-2\pi i)^k}{(k-1)!}\sum_{n=1}^{\infty} n^{k-1}q^n. \tag{1.3}$$

Form this, assuming $\chi(-1)=(-1)^k$, we have

$$\begin{aligned} E'_{k,\chi}(z) &= 2\sum_{n=1}^{\infty}\chi(n)^{-1}n^{-k} + 2\sum_{r=1}^{N}\chi^{-1}(r)\sum_{m=1}^{\infty}\sum_{n\in\mathbb{Z}}N^{-k}(mz+\frac{r}{N}+n)^{-k} \\ &= 2L(k,\chi^{-1}) + 2\sum_{r=1}^{N}\chi^{-1}(r)\sum_{m=1}^{\infty}N^{-k}S_k(mz+\frac{r}{N}) \qquad (1.4) \\ &\overset{(1.3)}{=} 2L(k,\chi^{-1}) + 2N^{-k}\frac{(-2\pi i)^k}{(k-1)!}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}n^{k-1}q^m\sum_{r=1}^{N}\chi^{-1}(r)\exp(2\pi i\frac{nr}{N}). \end{aligned}$$

By the functional equation (see [LFE] Theorem 2.3.2), we have, if $\chi(-1)=(-1)^k$,

$$L(k,\chi^{-1}) = G(\chi^{-1})\frac{(-2\pi i)^k}{N^k(k-1)!}L(1-k,\chi), \tag{1.5}$$

where $G(\psi)$ for a primitive character ψ modulo C is the Gauss sum $\sum_{r=1}^{C}\psi(r)\exp(2\pi i\frac{u}{C})$. We have $\sum_{r=1}^{N}\chi^{-1}(r)\exp(2\pi i\frac{nr}{N}) = \begin{cases}\chi(n)G(\chi^{-1}) & \text{if } n \text{ is prime to } N,\\ 0 & \text{otherwise,}\end{cases}$ and we get the formula

$$E'_{k,\chi}(z) = G(\chi^{-1})\frac{2(-2\pi i)^k}{N^k(k-1)!}E_{k,\chi}(z) \tag{1.6}$$

for

$$E_{k,\chi}(z) = 2^{-1}L(1-k,\chi) + \sum_{n=1}^{\infty}\sigma_{k-1,\chi}(n)q^n$$

for $\sigma_{k-1,\chi}(n) = \sum_{0<d|n}\chi(d)d^{k-1}$. Here we used the convention that $E_{k,\chi}(z)=0$ if $\chi(-1)\neq(-1)^k$. □

Exercise 1.11. *Give a proof of*

$$\sum_{r=1}^{N}\chi^{-1}(r)\exp(2\pi i\frac{nr}{N}) = \begin{cases}\chi(n)G(\chi^{-1}) & \textit{if } n \textit{ is prime to } N,\\ 0 & \textit{otherwise.}\end{cases}$$

Exercise 1.12. *Let* p *be a prime, and write* $\mathbf{1}_p$ *for the imprimitive identity character of* $(\mathbb{Z}/p\mathbb{Z})^\times$. *Prove that*

$$E_{k,\mathbf{1}}(z) - p^{k-1}E_{k,\mathbf{1}}(pz) = 2^{-1}(1-p^{k-1})\zeta(1-k) + \sum_{n=1}^{\infty}\sigma^{(p)}_{k-1,\mathbf{1}}(n)q^n$$

for $\sigma^{(p)}_{k-1,\mathbf{1}}(n) = \sum_{0<d|n,p\nmid n}d^{k-1}$. *More generally, if* N *is prime to* p, *prove that*

$$E_{k,\chi}(z) - \chi(p)p^{k-1}E_{k,\chi}(pz) = 2^{-1}(1-\chi(p)p^{k-1})L(1-k,\chi) + \sum_{n=1}^{\infty}\sigma^{(p)}_{k-1,\chi}(n)q^n$$

for $\sigma^{(p)}_{k-1,\chi}(n) = \sum_{0<d|n,p\nmid n}\chi(d)d^{k-1}$.

1.3. **Eisenstein family.** We continue to fix a positive integer N prime to p and a Dirichlet character χ modulo Np with $\chi(-1) = (-1)^k$. We know by a work of Shimura (recalled in [LEC] Proposition 2.18 when $k = 2$ and $\chi = \mathbf{1}$) that $M_{k,\chi}(\Gamma_0(Np^r); A) \otimes_A \mathbb{C} = M_{k,\chi}(\Gamma_0(Np^r))$ for any algebra $A \subset \mathbb{C}$ containing the values of χ, where

$$M_{k,\chi}(\Gamma_0(Np^r); A) = \left\{ f \in M_{k,\chi}(\Gamma_0(Np^r)) \middle| a(n, f) \in A \text{ for all } n \geq 0 \right\}.$$

Here we write the q-expansion of f as $f = \sum_{n=0}^{\infty} a(n, f)q^n$. Then we take $A = W \cap \overline{\mathbb{Q}}$ and define $M_{k,\chi}(\Gamma_0(Np^r); W) = M_{k,\chi}(\Gamma_0(Np^r); A) \otimes_A W$ and

$$M_{k,\chi}(\Gamma_0(Np^r); K) = M_{k,\chi}(\Gamma_0(Np^r); W) \otimes_W K = M_{k,\chi}(\Gamma_0(Np^r); A) \otimes_A K.$$

By definition, $M_{k,\chi}(\Gamma_0(Np^r); A) \hookrightarrow A[[q]]$ via q-expansion.

Definition 1.13. *A p-adic analytic family of modular forms of character χ (modulo Np) with coefficients in $\Lambda = W[[T]]$ is a formal q-expansion $F(T) = \sum_{n=0}^{\infty} a(n, F)(T)q^n \in \Lambda[[q]]$ such that for all sufficiently large integers $k \gg 0$, $F(\gamma^k - 1)$ is the q-expansion of an element in $M_{k,\chi\omega^{-k}}(\Gamma_0(Np); W)$ for the Teichmüller character $\omega(z) = \lim_{n\to\infty} z^{p^n}$ for $z \in \mathbb{Z}_p$ (which factors through $\mathbb{Z}/p\mathbb{Z}$).*

Exercise 1.14. *Prove that the limit $\omega(z) = \lim_{n\to\infty} z^{p^n}$ exists in $\mathbb{Z}_p$ and that it gives rise to a Dirichlet character modulo p.*

Exercise 1.15. *Prove that* $\log_p(z) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{(z-1)^n}{n}$ *converges p-adically for $z \in \Gamma$ to an element in $p\mathbb{Z}_p$ and satisfies* $\log_p(zw) = \log_p(z) + \log_p(w)$ *and* $\gamma^{\log_p(\langle n\rangle)/\log_p(\gamma)} = \langle n\rangle$ *for all integer n prime to p (cf.* [LFE] *Section 1.3). Similarly, prove that* $\exp_p(z) = \sum_{n=0}^{\infty}\frac{z^n}{n!}$ *converges to an element in Γ p-adically over $p\mathbb{Z}_p$ and show that* $\exp_p \circ \log_p$ *and* $\exp_p \circ \log_p$ *are the identity maps.*

Define $\Phi_\chi(T) \in W[[T]]$ by

$$\Phi_\chi(\gamma^s - 1) = \begin{cases} 2^{-1}L_p(1-s,\chi) & \text{if } \chi \neq \mathbf{1} \\ 2^{-1}(\gamma^s - 1)L_p(1-s,\chi) & \text{otherwise} \end{cases}$$

for $s \in \mathbb{Z}_p$ and

$$a(n, \mathcal{E}_\chi)(T) = \begin{cases} \sum_{0<d|n, p\nmid d} \chi(d)(1+T)^{\log_p(\langle d\rangle)/\log_p(\gamma)} & \text{if } \chi \neq \mathbf{1}, \\ T\sum_{0<d|n, p\nmid d}(1+T)^{\log_p(\langle d\rangle)/\log_p(\gamma)} & \text{if } \chi = \mathbf{1}. \end{cases}$$

Exercise 1.16. *Using* Theorem 1.5, *prove the existence and uniqueness of $\Phi_\chi(T) \in W[[T]]$ if χ_N is primitive modulo N.*

Theorem 1.17. *Let χ be an even Dirichlet character modulo Np with primitive χ_N. Then the q-expansion $\mathcal{E}_\chi = \Phi_\chi(T) + \sum_{n=1}^{\infty} a(n, \mathcal{E}_\chi)q^n$ gives a p-adic analytic family of modular form with character χ. Moreover $E_\chi(\gamma^k - 1) \in M_{k,\chi\omega^{-k}}(\Gamma_0(Np); W)$ for $k \geq 2$ except for the case where $\chi = \mathbf{1}$. When $N = 1$ and $\chi = \mathbf{1}$, $E_{\mathbf{1}}(\gamma^k - 1) \in M_{k,\omega^{-k}}(\Gamma_0(Np); W)$ if $k = 0$ or $k > 3$.*

Proof. We prove the result assuming $\chi \neq \mathbf{1}$, since the case of $\chi = \mathbf{1}$ and $N = 1$ is similar. By computation, we have

$$\begin{aligned} a(n, \mathcal{E}_\chi)(\gamma^k - 1) &= \sum_{0<d|n, p\nmid n} \chi(d) \gamma^{k \log_p(\langle d \rangle)/\log_p(\gamma)} \\ &= \sum_{0<d|n, p\nmid n} \chi(d) \exp_p(\log_p(\gamma))^{k \log_p(\langle d \rangle)/\log_p(\gamma)} = \sum_{0<d|n, p\nmid n} \chi(d) \langle n \rangle^k \\ &= \sum_{0<d|n, p\nmid n} \chi\omega^{-k}(d) d^k = \sigma^{(p)}_{k,\chi}(n). \end{aligned}$$

Similarly by definition, $\Phi_\chi(\gamma^k - 1) = 2^{-1} L_p(1-k, \chi\omega^{-k})$. Thus we have from Exercise 1.12

$$\mathcal{E}_\chi(\gamma^k - 1) = \begin{cases} E_{k,\chi\omega^{-k}} & \text{if } \chi\omega^{-k} \text{ is primitive modulo } Np, \\ E_{k,\chi_N}(z) - \chi_N(p) p^{k-1} E_{k,\chi_N}(pz) & \text{otherwise.} \end{cases}$$

This finishes the proof. □

Exercise 1.18. *Give a detailed proof of the above theorem when* $\chi = \mathbf{1}$ *and* $N = 1$.

The collection of all p-adic analytic families of modular forms with character χ form a Λ-module $M_\chi(N; \Lambda)$. If $F \in M_\chi(N; \Lambda)$ specializes to a cusp form $F(\gamma^k - 1) \in S_{k,\chi\omega^{-k}}(\Gamma_0(N); W)$ for all sufficiently large $k \gg 0$, F is called a p-adic analytic cuspidal family. The correction of all cuspidal families is written as $S_\chi(N; \Lambda)$. For a given modular form $f \in M_{\ell,\psi}(\Gamma_0(pN); W)$, we can define a convoluted product $f * \mathcal{E}_\chi$ by

$$f * \mathcal{E}_\chi(T) = f \mathcal{E}_\chi(\gamma^{-\ell}(1+T) - 1)).$$

Then $f * \mathcal{E}_k \in \Lambda[[q]]$ and by computation, we have $f * \mathcal{E}_\chi(\gamma^k - 1) = f \cdot \mathcal{E}(\gamma^{k-\ell} - 1)$. Since $\mathcal{E}(\gamma^{k-\ell} - 1) \in M_{k-\ell,\chi\omega^{\ell-k}}(\Gamma_0(N); W)$, we find $f \cdot \mathcal{E}(\gamma^{k-\ell} - 1) \in M_{k,\chi\psi\omega^{\ell-k}}(\Gamma_0(N); W)$ if $k \geq \ell + 2$. This shows

Corollary 1.19. *We have* $f * \mathcal{E}_\chi \in M_{\chi\psi\omega^\ell}(\Gamma_0(N); \Lambda)$ *if* $f \in M_{\ell,\psi}(\Gamma_0(pN); W)$. *If* $f \in S_{\ell,\psi}(\Gamma_0(pN); W)$, *we have* $f * \mathcal{E}_\chi \in S_{\chi\psi\omega^\ell}(\Gamma_0(N); \Lambda)$.

In this way, we can produce a lot of p-adic analytic families.

Exercise 1.20. *Prove that* $f * \mathcal{E}_\chi \in S_{\chi\psi\omega^\ell}(\Gamma_0(N); \Lambda)$ *if* $f \in S_{\ell,\psi}(\Gamma_0(pN); W)$.

1.4. **Hecke operators.** Recall

$$\Delta_0(pN) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in M_2(\mathbb{Z}) \middle| c \equiv 0 \mod pN, a\mathbb{Z} + Np\mathbb{Z} = \mathbb{Z}, ad - bc > 0 \right\}.$$

We define a character χ_Δ of $\Delta_0(pN)$ by $\chi_\Delta \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \chi^{-1}(a)$.

Exercise 1.21. *Prove that* $\chi_\Delta \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \chi(d)$ *if* $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(pN)$.

Define for $\alpha = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Delta_0(pN)$ and a function $f : \mathfrak{H} \to \mathbb{C}$, a new function $f|_{k,\chi}\alpha(z)$ by $f|_{k,\chi}\alpha(z) = \det(\alpha)^{k-1} f(\alpha(z)) \chi_\Delta^{-1}(\alpha)(cz+d)^{-k}$. Splitting $T_n = \{\alpha \in \Delta_0(pN) | \det(\alpha) = n\}$ into a disjoint union $T_n = \bigsqcup_\alpha \Gamma_0(pN)\alpha$, we define $f|T(n) = \sum_\alpha f|_{k,\chi}\alpha$. Then the same proof of Lemma 2.14 in [LEC] gives

Lemma 1.22. (1) *Write* $T(n)$ *for the operator corresponding to* T_n. *Then* $T(n)$ *gives a linear endomorphism of* $M_{k,\chi}(\Gamma_0(pN))$.

(2) *We get the following identity of Hecke operators for* $f \in M_{k,\chi}(\Gamma_0(pN))$:

$$a(m, f|T(n)) = \sum_{0<d|(m,n),(d,pN)=1} \chi(d)d^{k-1} \cdot a(\frac{mn}{d^2}, f).$$

(3) $T(m)T(n) = T(n)T(m)$ *for all integers* m *and* n.

When $m|pN$, we often write $U(m)$ for $T(m)$.

Exercise 1.23. *Give a detailed proof of the above Lemma.*

Corollary 1.24. *If* $k \geq 1$ *and* A *contains the values of* χ, *the Hecke operators* $T(n)$ *preserves* $M_{k,\chi}(\Gamma_0(pN), \chi)$.

Definition 1.25. *We consider the operator* $T_\Lambda(n)$ *on* $F = \sum_{n=0}^{\infty} a(n, F)(T)q^n\Lambda[[q]]$ *defined by* $a(m, F|T_\Lambda(n)) = \sum_{0<d|(m,n),(d,pN)=1} \chi(d)d^{-1}(1+T)^{\log_p(\langle d\rangle)/\log_p(\gamma)} \cdot a(\frac{mn}{d^2}, F)$.

Since $(1+T)^{\log_p(\langle d\rangle)/\log_p(\gamma)}|_{T=\gamma^k-1} = \langle d\rangle^k = \omega^{-k}(d)d^k$, after specializing $T = \gamma^k - 1$, we find $(F|T_\Lambda(n))(\gamma^k-1) = (F(\gamma^k-1)|T(n)$. Thus $T_\Lambda(n)$ preserves $M_\chi(N;\Lambda)$ and $S_\chi(N;\Lambda)$.

Proposition 1.26. *We have a linear operator* $T_\Lambda(n)$ *defined by* Definition 1.25 *acting on* $M_\chi(N;\Lambda)$ *which preserves* $S_\chi(N;\Lambda)$. *In particular,* $(F|T_\Lambda(n))(\gamma^k-1) = (F(\gamma^k-1)|T(n)$ *for all* $F \in M_k(N;\Lambda)$ *and all* $k \gg 0$ *and* $T_\Lambda(m)T_\Lambda(n) = T_\Lambda(n)T_\Lambda(m)$ *and* $T_\Lambda(m)T_\Lambda(n) = T_\Lambda(mn)$ *if* $m\mathbb{Z} + n\mathbb{Z} = \mathbb{Z}$.

There are a lot of questions we can ask for p-adic analytic families; for example,

(Q1) Is the Λ-module $M_\chi(N;\Lambda)$ finitely generated?
(Q2) Is the module $M_\chi(N;\Lambda)$ spanned by Hecke eigenforms (at least topologically if it is infinite rank)?
(Q3) What is $F(\zeta - 1)$ for a general $\zeta \in \overline{\mathbb{Q}}_p$ with $|\zeta - 1|_p < 1$?
(Q4) If $F \in M_\chi(N;\Lambda)$ is a common Hecke eigenform with $a(1, F) = 1$ with Λ-unit eigenvalue for $U(p)$, writing $d\varphi_k$ for the p-adic measure constructed for $F(\gamma^k - 1)$ in [LFE] in Section 6.5 and in [LEC] Theorem 2.36, what is the relation among $d\varphi_k$ for $k \gg 0$?

We try to answer some of these questions.

1.5. Modular forms of level N.

We generalize a bit the notion of modular forms. Let

$$\Gamma_1(N) = \left\{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N) \middle| a \equiv d \equiv 1 \mod N\right\}.$$

A modular form $f \in M_k(\Gamma_1(N))$ is a holomorphic function on $\mathfrak{H}$ satisfying the conditions (M1–2) in 1.2 for $\Gamma_1(N)$ in place of $\Gamma_0(N)$. Since $d \equiv 1 \mod N$ and χ is a character modulo N in (M1), this space is independent of the choice of χ, and hence the subscript χ is dropped from the notation. Similarly we define the subspace of cusp forms $S_k(\Gamma_1(N))$ inside $M_k(\Gamma_1(N))$ by imposing (S) in addition to (M1–2). Then we define first for a ring $A \subset \mathbb{C}$

$$M_k(\Gamma_1(N); A) = \left\{f \in M_k(\Gamma_1(N)) \middle| a(n, f) \in A \text{ for all } n \geq 0\right\}$$

and put $S_k(\Gamma_1(N);A) = M_k(\Gamma_1(N);A) \cap S_k(\Gamma_1(N))$. Then again it is known by Shimura (cf. [IAT] Theorem 3.52) that

$$\begin{aligned} &M_k(\Gamma_1(N);\mathbb{Z}) \otimes_{\mathbb{Z}} A = M_k(\Gamma_1(N);A) \text{ and } M_k(\Gamma_1(N);A) \otimes_A \mathbb{C} = M_k(\Gamma_1(N)) \\ &S_k(\Gamma_1(N);\mathbb{Z}) \otimes_{\mathbb{Z}} A = S_k(\Gamma_1(N);A) \text{ and } S_k(\Gamma_1(N);A) \otimes_A \mathbb{C} = S_k(\Gamma_1(N)) \end{aligned} \tag{1.7}$$

Thus, for an algebra X with $W \subset X \subset \overline{\mathbb{Q}}_p$, taking $A = X \cap \overline{\mathbb{Q}}$, we may define $M_k(\Gamma_1(N);X) = M_k(\Gamma_1(N);A) \otimes_A X$ and $S_k(\Gamma_1(N);X) = M_k(\Gamma_1(N);A) \otimes_A X$. These spaces can be embedded into $X[[q]]$ by q-expansion.

Since $\Gamma_0(N)/\Gamma_1(N) \cong (\mathbb{Z}/N\mathbb{Z})^\times$ by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto (d \mod N)$, the finite group $(\mathbb{Z}/N\mathbb{Z})^\times$ acts on $M_k(\Gamma_1(N))$. Then by definition, the χ-eigenspace of $M_k(\Gamma_1(N))$ is the space $M_{k,\chi}(\Gamma_0(N))$:

$$M_{k,\chi}(\Gamma_0(N)) = \left\{ f \in M_k(\Gamma_1(N)) \middle| f|\langle d \rangle = \chi(d) f \text{ for all } d \in (\mathbb{Z}/N\mathbb{Z})^\times \right\},$$

where $f|\langle a \rangle = f|_k \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$. Thus we get

Lemma 1.27. *We have*

$$M_k(\Gamma_1(N)) = \bigoplus_{\chi} M_{k,\chi}(\Gamma_0(N))$$
$$S_k(\Gamma_1(N)) = \bigoplus_{\chi} S_{k,\chi}(\Gamma_0(N)),$$

where χ runs over all Dirichlet characters modulo N.

The Hecke operator $T(n)$ on each $M_{k,\chi}(\Gamma_0(N))$ gives rise to a Hecke operator on the sum $M_k(\Gamma_1(N))$ over χ. In other words, writing $f \in M_k(\Gamma_1(N)$ as $f = \oplus_\chi f_\chi$ with $f_\chi \in M_{k,\chi}(\Gamma_0(N))$, we have $f|T(n) = \oplus_\chi (f_\chi|T(n))$.

Exercise 1.28. *Prove that for $f \in M_k(\Gamma_1(N))$*

$$a(m, f|T(n)) = \sum_{0<d|(m,n),(d,N)=1} d^{k-1} a(\frac{mn}{d^2}, f|\langle d \rangle)$$

and $T(\ell)^2 - T(\ell^2) = \ell^{k-1}\langle \ell \rangle$ if ℓ is a prime outside N.

Let

$$\Delta_1(N) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in M_2(\mathbb{Z}) \middle| c \equiv 0 \mod N,\ a \equiv 1 \mod N,\ ad - bc > 0 \right\}.$$

Exercise 1.29. *Splitting $T_n = \{\alpha \in \Delta_1(N) | \det(\alpha) = n\}$ into a disjoint union $T_n = \bigsqcup_\alpha \Gamma_0(pN)\alpha$, prove that $f|T(n) = \sum_\alpha f|_{k,\mathbf{1}}\alpha$.*

The modular curve $X_1(N)(\mathbb{C}) = \Gamma_1(N)\backslash(\mathfrak{H} \cup \mathbf{P}^1(\mathbb{Q}))$ has a regular model $X_1(N)$ over $\mathbb{Z}$. We admit the following nontrivial facts which could be proven using algebraic geometry on the regular scheme $X_1(N)_{/\mathbb{Z}}$:

Theorem 1.30. *The space $M_k(\Gamma_1(N);A)$ and $S_k(\Gamma_1(N);A)$ are stable under the action of $(\mathbb{Z}/N\mathbb{Z})^\times$ and hence under $T(n)$ for all n for any algebra A.*

1.6. **Slope of modular forms.** A modular form $f \in M_k(\Gamma_1(p^rN);K)$ $(r > 0)$ has slope α if $f|(U(p)-a)^M = 0$ (for a sufficiently large integer $M \gg 0$) and $|a|_p = p^{-\alpha}$. If $f|U(p)^M = 0$ (for $M \gg 0$), we call f to have infinity slope.

Lemma 1.31. *Let X be a W-module of finite type and $T : X \to X$ be a W-linear operator. Then the p-adic limit $e = \lim_{n\to\infty} T^{n!}$ exists in $\mathrm{End}_W(X)$ and gives an idempotent of $\mathrm{End}_W(X)$.*

Proof. Since $\mathrm{End}_W(X)$ is a W-module of finite type, the W-subalgebra A generated by T is a W-algebra which is of finite type as W-modules. Thus we need to prove the existence of $\lim_{n\to\infty} T^{n!}$ in a W-algebra X which is a W-module of finite type. In particular, we have a finitely many generators $a_1, \dots, a_r$ of A over W and a W-linear surjection: $W^r \twoheadrightarrow A$ sending $(x_j)_{j=1,\dots,r}$ to $\sum_j x_j a_j$. In particular A/pA is finite; so, Writing Ω the set of all maximal ideals of A, Ω is a finite set. Also A is p-adically complete; so, $A = \varprojlim_n A/p^rA$. We have $\bigcap_n \bigcap_{\mathfrak{m}\in\Omega} \mathfrak{m}^n = 0$; so, $A = \varprojlim_n A/\bigcap_{\mathfrak{m}\in\Omega} \mathfrak{m}^n$. By Chinese remainder theorem applied to $A/\bigcap_{\mathfrak{m}\in\Omega} \mathfrak{m}^n$, we find $A/\bigcap_{\mathfrak{m}\in\Omega} \mathfrak{m}^n = \bigoplus_{\mathfrak{m}} A/\mathfrak{m}^n$. Thus we find that $A = \bigoplus_{\mathfrak{m}\in\Omega} A_{\mathfrak{m}}$ for $A_{\mathfrak{m}} = \varprojlim_n A/\mathfrak{m}^nA$. Then A is a direct product of local rings $A_{\mathfrak{m}}$. Thus we may assume that A is local. Then $\lim_n T^{n!} = 0$ if $T \in \mathfrak{m}$. Since $A/\mathfrak{m}A$ is a finite field of characteristic p, its order is p^f. Then $a_n = |(A/\mathfrak{m}^nA)^\times| = p^{m_n}(p^f - 1)$ for an increasing sequence m_n. In particular, $T^{a_n} \equiv 1 \mod \mathfrak{m}^n$, and hence $1 = \lim_{n\to\infty} T^{a_n} = \lim_{n\to\infty} T^{n!}$. □

Exercise 1.32. *Under the notation of the above proof, give a detailed proof of the following facts:*

1. $p \in \bigcap_{\mathfrak{m}\in\Omega} \mathfrak{m}$;
2. $pA \supset \left(\bigcap_{\mathfrak{m}\in\Omega} \mathfrak{m}\right)^n$ *for sufficiently large* $n > 0$;
3. $\bigcap_{n=1}^{\infty} \bigcap_{\mathfrak{m}\in\Omega} \mathfrak{m}^n = \bigcap_{n=1}^{\infty} p^nA = 0$;
4. $\lim_{n\to\infty} T^{a_n} = \lim_{n\to\infty} T^{n!}$.

Let $e = \lim_{n\to\infty} U(p)^{n!}$ in $\mathrm{End}_W(M_k(\Gamma_1(Np^r);A)$ for $A = W$ or K, and define

$$M_k^{ord}(\Gamma_1(Np^r);A) = e(M_k(\Gamma_1(Np^r);A)).$$

The following lemma is easy:

Lemma 1.33. *$f \in M_k(\Gamma_1(Np^r);W))$ is of slope zero if and only if $f \in M_k^{ord}(\Gamma_1(Np^r);W)$ and f is an eigenform for $U(p)$.*

Exercise 1.34. *Prove the above lemma.*

Definition 1.35. *Define a Hecke algebra $\mathbb{H}_k(Np^r;A)$ (resp. $\mathbf{h}_k(Np^r;A)$) by the A-subalgebra of $\mathrm{End}_A(M_k(\Gamma_1(Np^r);A))$ (resp. $\mathrm{End}_A(S_k(\Gamma_1(Np^r);A))$) generated by Hecke operators $T(n)$ for all n.*

We can define the corresponding spaces $M_\chi^{ord}(N;\Lambda)$ of p-ordinary analytic families as follows:

Definition 1.36.

$$M^{ord}_{\chi}(N;\Lambda) = \left\{F \in M_{\chi}(N;\Lambda)\middle| F(\gamma^k-1) \in M^{ord}_{k,\chi\omega^{-k}}(\Gamma_0(Np)) \quad \textit{for all } k \gg 0\right\}$$

and $S^{ord}_{\chi}(N;\Lambda) = S_{\chi}(N;\Lambda) \cap M^{ord}_{\chi}(N;\Lambda)$.

The following theorem is proven in 1986 in my papers [H86a], [H86b] and [LFE] Chapter 7 (except for the case for $M^{ord}_{2,\mathbf{1}}(\Gamma_0(p);W)$):

Theorem 1.37. *$M^{ord}_{\chi}(N;\Lambda)$ and $S^{ord}_{\chi}(N;\Lambda)$ are free of finite rank over Λ, and the specialization map induces isomorphisms*

$$M^{ord}_{\chi}(N;\Lambda) \otimes_{\Lambda} \Lambda/(T-(\gamma^k-1)) \cong M^{ord}_{k,\chi\omega^{-k}}(\Gamma_0(pN);W),$$
$$S^{ord}_{\chi}(N;\Lambda) \otimes_{\Lambda} \Lambda/(T-(\gamma^k-1)) \cong S^{ord}_{k,\chi\omega^{-k}}(\Gamma_0(pN);W)$$

for all $k \geq 2$.

Corollary 1.38. *Any element in $M^{ord}_{k,\chi}(\Gamma_0(pN);W)$ for $k \geq 2$ can be lifted to a p-adic analytic family. More over if $k \geq 2$, we have*

$$\mathrm{rank}_W\, M^{ord}_{k,\chi\omega^{-k}}(\Gamma_0(pN);W) = \mathrm{rank}_{\Lambda}\, M^{ord}_{\chi}(N;\Lambda),$$
$$\mathrm{rank}_W\, S^{ord}_{k,\chi\omega^{-k}}(\Gamma_0(pN);W) = \mathrm{rank}_{\Lambda}\, S^{ord}_{\chi}(N;\Lambda)$$

which are independent of $k \geq 2$.

Definition 1.39. *Let $0 < k \in \mathbb{Z}$ be an integer with divisible by $p-1$ (so, $\omega^k = 1$). A weak p-adic analytic family of modular forms (centered at $0 < k \in \mathbb{Z}$) is a formal power series $F = \sum_{n=0}^{\infty} a(n,F)(T)q^n$ with $a(n,F)(T) \in K[[T]]$ convergent at $\gamma^{k'}-1$ for all k' in a small p-adic neighborhood U in $k\cdot\Gamma \subset \mathbb{Z}_p^{\times}$ of k such that $F(\gamma^{k'}-1) \in M_{k',\chi}(\Gamma_0(pN);K)$ for all $k' \gg 0$ in U.*

This type of weak families was introduced in [GM] by Mazur and Gouvêa in 1992. For a given slope $\alpha \in \mathbb{Q}$, we define $M^{(\alpha)}_{k,\chi}(\Gamma_0(pN);K)$ be the space spanned by slope α modular forms in $M_{k,\chi}(\Gamma_0(pN);K)$ and put $S^{(\alpha)}_{k,\chi}(\Gamma_0(pN);K) = M^{(\alpha)}_{k,\chi}(\Gamma_0(pN);K) \cap S_{k,\chi}(\Gamma_0(pN);K)$. By definition, we have

$$M^{(0)}_{k,\chi}(\Gamma_0(pN);K) = M^{ord}_{k,\chi}(\Gamma_0(pN);K).$$

Moreover, we have

$$M_{k,\chi}(\Gamma_0(pN);K) = \bigoplus_{\alpha} M^{(\alpha)}_{k,\chi}(\Gamma_0(pN);K).$$

Exercise 1.40. *Prove the above decomposition.*

Gouvêa and Mazur made the following conjecture

Conjecture 1.41 (Gouvêa and Mazur, 1992)**.**

(1) *If $k, k' \geq 2\alpha+2$ and $k \equiv k' \mod p^n(p-1)$ for $n \geq \alpha$, then*

$$\dim_K S^{(\alpha)}_{k,\chi}(\Gamma_0(pN);K) = \dim_K S^{(\alpha)}_{k',\chi}(\Gamma_0(pN);K),$$

(2) *If $k \geq 2\alpha + 2$, any $f \in S^{(\alpha)}_{k,\chi}(\Gamma_0(pN); K)$ can be lifted to a weak analytic family of slope α (centered at k).*

Their conjecture is actually slightly stronger than what is stated here. In this conjecture, they predict the neighborhood $U = U_k$ of a given $k \geq 2\alpha + 2$ (appearing in the definition of the weak families) is specified as

$$U_k = \left\{ k' \in k \cdot \Gamma \middle| |k' - k|_p \leq p^{-\lceil \alpha \rceil} \right\}.$$

Though K. Buzzard found a counter example against the lower bound $k \geq 2\alpha + 2$ of (1) when $p = 2$, the conjecture would be true for $p > 3$ (as Gouvêa and Mazur actually assumed). A slightly different version of the conjecture (2) valid for $k \geq \alpha + 1$ (for a neighborhood $U \subset U_k$) was proven by Coleman [C] in 1998 (a lower bound for k for the validity of (1) quadratic in α was proven by Wan [W] soon after [C]). This result implies that actually that Hecke eigenforms in $M_{k',\chi\omega^{-k'}}(\Gamma_0(pN); \overline{\mathbb{Q}}_p)$ is parameterized by $k' \in U \subset U_k$ (as we will see later for p-ordinary forms). Then Coleman and Mazur further went on to globalize the (local) parameter space U_k of modular Hecke eigenforms to a rigid analytic curve (the so called eigencurve) in [CM].

Theorem 1.37 (proven earlier than the conjecture) gives a finer result than the conjecture for slope 0 forms and was a main supporting evidence for the conjecture. Indeed, in this case, the eigencurve is actually a formal scheme finite flat over $\mathrm{Spec}(\Lambda)$ (not just a rigid analytic space) and is given by $\mathrm{Spec}(\mathbb{H}_\chi)$ for the Hecke algebra $\mathbb{H}_\chi \subset M_\chi(N; \Lambda)$ generated by Hecke operators $T(n)$ over Λ. Note here the rigid analytic space $\mathrm{Spec}(\Lambda)(\mathbb{C}_p)$ is a p-adic open unit disk. We would prove Theorem 1.37 to some extent in this course by a cohomological means in [H86b] (two other methods are discussed in [H86a] and [LFE] Chapter 7, respectively).

Exercise 1.42. *Prove that* $\mathrm{Spec}(\Lambda)(\overline{\mathbb{Q}}_p) = \mathrm{Hom}_{W\text{-}alg}(\Lambda, \overline{\mathbb{Q}}_p)$ *is isomorphic to the open unit disk D in $\overline{\mathbb{Q}}_p$ (centered at the origin 0) by sending a W-algebra homomorphism $\phi : \Lambda \to \overline{\mathbb{Q}}_p$ to $\phi(T)$.*