COHOMOLOGICAL MODULAR FORMS AND p-ADIC L-FUNCTIONS

HARUZO HIDA

Contents

1. Introduction 2
1.1. Cohomology groups 2
1.2. Cohomology of $\mathbb{G}_m(\mathbb{C})$ and Dirichlet L-values 5
1.3. Relative cohomology 7
1.4. Hecke operators 8
1.5. p-Adic measure 9
1.6. p-Adic measure and Hecke operators 11
2. Modular p-adic L-functions 15
2.1. Elliptic modular forms 15
2.2. Modular cohomology group 16
2.3. Hecke operators 17
2.4. Duality 19
2.5. Modular Hecke L-functions 22
2.6. Rationality of Hecke L-values 23
2.7. p-Old and p-new forms 24
2.8. Elliptic modular p-adic measure 25
References 28

In this course, assuming basic knowledge of algebraic number theory, elliptic modular forms, commutative algebra and topology, we will make p-adic study of cohomological modular forms on $GL(1)$ and $GL(2)$. We plan to discuss the following four topics:

(1) Isomorphism of Eichler-Shimura type connecting modular forms and cohomology groups,
(2) Rationality and integrality of L-values,
(3) p-adic measure theory,
(4) Construction of analytic p-adic L-functions.

Along with these main topics, we will give a brief description of different cohomology theory we will use. In this note, all rings are supposed to have the identity.

Date: May 22, 2007.
The author is partially supported by the NSF grant: DMS 0244401 and DMS 0456252.

1. Introduction

In this introduction, without going into technical details, we describe a prototypical example of the cohomology groups we deal with and construction of p-adic L-functions. Starting with the third week, we start to justify our construction and give a brief description of cohomology theory. The example we describe is from [LFE] Chapter 4.

1.1. **Cohomology groups.** We consider the multiplicative group $\mathbb{G}_m$ as an algebraic group. Thus as a scheme, $\mathbb{G}_m = \mathrm{Spec}(\mathbb{Z}[t, t^{-1}])$ whose A-points $\mathbb{G}_m(A)$ is given by $\mathrm{Hom}_{\mathrm{alg}}(\mathbb{Z}[t, t^{-1}], A) \cong A^\times$ by $\phi \mapsto \phi(t)$ for each commutative ring A.

Exercise 1.1. *Prove the following assertion:*

(1) $\mathrm{Hom}_{alg}(\mathbb{Z}[t, t^{-1}], A) \cong A^\times$ *as sets for a commutative algebras* A*, where* Hom_{alg} *denotes the set of algebra homomorphisms.*
(2) *The "map" assigning each commutative algebra* A *the group* $\mathbb{G}_m(A)$ *gives rise to a covariant functor from the category of commutative algebras with identity into the category of abelian groups.*

The space $\mathbb{G}_m(\mathbb{C})$ has nontrivial homology group of positive degree. Indeed, we have $H_1(\mathbb{G}_m(\mathbb{C}), A) \cong A$. Here intuitively, for any given commutative ring A, a C^∞ n-chain in a C^∞ manifold X is a formal A-linear combination of C^∞ maps from $\Delta^n = [0,1]^n$ for the closed interval $[0,1]$ into X. Then the totality of n-chains form an A-free module $C_n(X; A)$ generated by $\phi : \Delta^n \to X$. Since Δ^n has natural boundaries

$$[0,1] \times \cdots \times [0,1] \times \overset{i}{x} \times [0,1] \times \cdots \times [0,1] = \Delta^{n-1}_x$$

with orientation, identifying $\Delta^{n-1}_{i,x}$ for $x = 0, 1$, we can think of the boundary $\partial\phi = \sum_{i,x} (-1)^{i+1+x} \phi|_{\Delta^{n-1}_{i,x}}$ which is a $n-1$ chain, identifying $\Delta^{n-1}_{i,x}$ with Δ^n (removing the i-th coordinate). We set Δ^0 to be the origin 0 (one point). By linearity, we can extend ∂ to an A-linear map $\partial : C_n(X; A) \to C_{n-1}(X; A)$ if $n \geq 1$. Thus we have a chain complex

$$\cdots \xrightarrow{\partial} C_n(X; A) \xrightarrow{\partial} C_{n-1}(X; A) \xrightarrow{\partial} \cdots \xrightarrow{\partial} C_0(X; A) \xrightarrow{\deg} A \to 0,$$

where $\deg(\sum_\phi a_\phi \phi) = \sum_\phi a_\phi$. An n-chain ϕ with $\partial\phi = 0$ is called an n cycle.

Exercise 1.2. *Prove* $\partial \circ \partial = 0$.

By taking A-dual, we put $C^n(X; A) = \mathrm{Hom}(C_n(X; A), A)$. Then we have a reversed chain complex

$$0 \to A \xrightarrow{\deg^*} C^0(X; A) \xrightarrow{\partial^*} \cdots \xrightarrow{\partial^*} C^n(X; A) \xrightarrow{\partial^*} \cdots,$$

where $\partial^*\phi = \phi \circ \partial$. An n-cochain $\phi \in C^n(X; A)$ with $\partial^*\phi = 0$ is called an n-cocycle. Similarly, an n-cocycle (resp. an n-cycle) which is in the image of ∂^* (resp. ∂) is called an n-coboundary (resp. an n-boundary). Here is a definition of the singular (or Betti) cohomology group of degree $n \geq 1$:

$$H_n(X,A) = \frac{\mathrm{Ker}(\partial : C_n(X;A) \to C_{n-1}(X;A))}{\mathrm{Im}(\partial : C_{n+1}(X;A) \to C_n(X;A))}$$
(1.1)
$$H^n(X,A) = \frac{\mathrm{Ker}(\partial^* : C^n(X;A) \to C^{n+1}(X;A))}{\mathrm{Im}(\partial^* : C^{n-1}(X;A) \to C^n(X;A))}$$

When $n = 0$, we just define $H_0(X,A) = C_0(X;A)/\mathrm{Im}(\partial)$ and $H^0(X,A) = \mathrm{Ker}(\partial : C^0(X;A) \to C^1(X;A))$.

Exercise 1.3. *Prove that $H_0(X,A) \cong H^0(X,A) \cong A$ if X is connected.*

If X is a C^∞ real manifold of dimension d, $H^n(X,A) = H_n(X,A) = 0$ for $n > d$.

By definition, taking a cohomology class $[c]$ represented by an n-cocycle c and a homology class $[\gamma]$ represented by a cycle γ. Then $c \in \mathrm{Hom}(C_n(X;A), A)$; so, we have the value $c(\gamma)$. If we replace γ by $\gamma + \partial b$, we have $c(\gamma) = c(\gamma) + c(\partial b) = c(\gamma)$, because $c(\partial b) = (\partial^* c)(b) = 0$. Similarly, $(c + \partial^*\beta)(\gamma) = c(\gamma)$, and hence $c(\gamma) \in A$ only depends on the classes $[c]$ and $[\gamma]$. Thus we have the Poincaré duality pairing:

$$\langle \cdot, \cdot \rangle : H^n(X,A) \times H_n(X,A) \to A \tag{1.2}$$

given by $\langle [c], [\gamma] \rangle = c(\gamma)$.

Here is an example.

Example 1.1. Let S^1 be the unit circle. Then 1-chain $\phi : [0,1] \to S^1$ has $\partial\phi = \phi(0) - \phi(1)$. Thus $\phi \in \mathrm{Ker}(\partial) \Leftrightarrow \phi(0) = \phi(1)$, and therefore, if one moves x from 0 to 1, $\phi(x)$ circles m times forward in counterclockwise or backward. Assigning ϕ to this number $n = [\phi]$ when $\phi(x)$ moves forward and $-n = [\phi]$ when $\phi(x)$ moves backwards, we have a linear map $H_1(S^1, \mathbb{Z}) \to \mathbb{Z}$, and actually, we have $H_1(S^1, \mathbb{Z}) \cong \mathbb{Z}$.

Exercise 1.4. *Prove that $H_1(S^1, \mathbb{Z}) \cong \mathbb{Z}$. (Hint: if $[\phi] = [\varphi] = 1$ and $\phi(0) = \varphi(0) = 0 \in S^1 = \mathbb{R}/\mathbb{Z}$, then defining $\Delta^2 \xrightarrow{\Phi} S^1$ by $\Phi(x,y) = \phi(x)y + \varphi(x)(1-y)$ identifying $S^1 - \{0\}$ with $(0,1)$ naturally, we have $\partial\Phi = \phi - \varphi$; so, the class of ϕ and φ are equal in $H_1(S^1, \mathbb{Z})$.)*

Example 1.2. We now consider $X = \mathbb{R}$. Then it is well known that $H_1(\mathbb{R}, A) = H^1(\mathbb{R}, A) = 0$. By the Künneth formula (see [CGP] 0.8),

$$H^n(X \times Y, A) \cong \bigoplus_{i+j=n} H^i(X,A) \otimes_A H^j(Y,A)$$

as long as A is PID (or Dedekind domain) and $H^j(X,A)$ and $H^j(Y,A)$ are A-free for all $j = 0, 1, \ldots, n$. In particular,

$$H^1(X \times Y, A) \cong (H^1(X,A) \otimes_A H^0(Y,A)) \oplus (H^0(X,A) \otimes_A H^1(Y,A))$$

if $H^1(X,A)$ and $H^1(Y,A)$ is A-free. By polar coordinates, we have $\mathbb{C}^\times \cong (0,\infty) \times S^1 \cong \mathbb{R} \times S^1$. Thus $H^1(\mathbb{C}^\times, \mathbb{Z}) \cong \mathbb{Z}$ by regarding a generator $\phi \in H_1(S^1, \mathbb{Z})$ as a cycle having values in $\mathbb{C}^\times$. We may identify $\mathbb{R} \times S^1$ with $T = \mathbb{C}/\mathbb{Z}$, and the isomorphism $T \cong \mathbb{C}^\times$ can be given by $z \mapsto \mathbf{e}(z) = \exp(2\pi i z)$. Then $H_1(T, \mathbb{Z})$ is generated by $\phi : [0,1] \hookrightarrow \mathbb{R}/\mathbb{Z} \subset T$, which is the real circle in T. In this case, we have $H_1(T,A) = H_1(T,\mathbb{Z}) \otimes_{\mathbb{Z}} A$, and the above fact holds for all commutative rings A.

There is another way of computing a cohomology group over $\mathbb{C}$. We consider a C^∞ class differential form ω on X of degree n. Pick a point $x \in X$ and a coordinate neighborhood U of x with coordinate $t_1, \dots, t_d$ with x giving the origin of U, ω has the following form $\sum_{i_1<\dots<i_n} f_{i_1 i_2 \dots i_n}(t) dt_{i_1} \wedge \dots \wedge dt_{i_n}$ with C^∞ functions $f_{i_1 i_2 \dots i_n}(t)$. If one change coordinates system, the expression of ω changes according to the chain-rule. For example, if $n = 1$, and $z_1, \dots, z_d$ is another coordinates, $dt_i = \sum_j \frac{\partial t_i}{\partial z_j} dz_j$ and the expression of $\omega = \sum_i f_i dt_i$ with respect to z_j is given by

$$\omega = \sum_i f_i \sum_j \frac{\partial t_i}{\partial z_j} dz_j = \sum_j \left(\sum_i f_i \frac{\partial t_i}{\partial z_j} \right) dz_j.$$

If (U, t_i) and (V, z_j) are coordinate neighborhoods with $U \cap V \neq \emptyset$, the expression of ω on U and V is related by the above chain-rule. Write $\Omega^n(X,\mathbb{C})$ for the $\mathbb{C}$-vector space of differential forms as above of degree n. When $n = 0$, $\Omega^0(X,\mathbb{C})$ is just the vector space of C^∞-class functions on X. We can define the exterior derivative $d\omega \in \Omega^{n+1}(X,\mathbb{C})$ as follows. For $f \in \Omega^0(X,\mathbb{C})$, we just put $df = \sum_i \frac{\partial f}{\partial t_i} dt_i$. For $\omega = \sum_{i_1<\dots<i_n} f_{i_1 i_2 \dots i_n}(t) dt_{i_1} \wedge \dots \wedge dt_{i_n}$, we define

$$d\omega = \sum_{i_1<\dots<i_n} df_{i_1 i_2 \dots i_n}(t) \wedge dt_{i_1} \wedge \dots \wedge dt_{i_n}.$$

Exercise 1.5. (1) *Check that $d\omega$ is a well defined differential form of degree $n+1$ if ω is of degree n. Here you need to verify that the chain-rule is satisfied by $d\omega$ if one changes coordinates.*
(2) *Prove $d \circ d = 0$.*

We simply put $\Omega^n(X,\mathbb{C}) = 0$ if $n > d = \dim X$. By the above exercise, we have a complex:

$$0 \to \mathbb{C} \to \Omega^0(X,\mathbb{C}) \xrightarrow{d} \Omega^1(X,\mathbb{C}) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^d(X,\mathbb{C}) \to 0.$$

We define the de Rham cohomology group $H^n_{DR}(X,\mathbb{C})$ (in differential geometry) by

$$H^n_{DR}(X,\mathbb{C}) = \frac{\mathrm{Ker}(d : \Omega^n(X,\mathbb{C}) \to \Omega^{n+1}(X,\mathbb{C}))}{\mathrm{Im}(d : \Omega^{n-1}(X,\mathbb{C}) \to \Omega^n(X,\mathbb{C}))}$$

if $n \geq 1$. When $n = 0$, we simply put $H^0_{DR}(X,\mathbb{C}) = \mathrm{Ker}(d : \Omega^0(X,\mathbb{C}) \to \Omega^1(X,\mathbb{C}))$, which is isomorphic to the space of constant functions in $\Omega^0(X,\mathbb{C})$ (and is isomorphic to $\mathbb{C}$ if X is connected) by the fundamental theorem of Calculus.

Theorem 1.6. *If X is a compact differentiable manifold of dimension d, we have a canonical isomorphism $H^n(X,\mathbb{C}) \cong H^n_{DR}(X,\mathbb{C})$.*

Here is a brief sketch of the proof. By the Poincaré duality, we have $H^n(X,\mathbb{C}) \cong \mathrm{Hom}_{\mathbb{C}}(H_n(X,\mathbb{C}),\mathbb{C})$. We have a pairing $H^n_{DR}(X,\mathbb{C}) \times H_n(X,\mathbb{C}) \to \mathbb{C}$ given by $(\omega,\gamma) = \int_\gamma \omega$. By Stokes' theorem, $\int_{\partial\gamma} \omega = \int_\gamma d\omega$, and hence the pairing is well defined. de Rham proved the non-degeneracy of the above pairing, which implies the theorem. □

1.2. **Cohomology of $\mathbb{G}_m(\mathbb{C})$ and Dirichlet L-values.** Consider a Dirichlet character $\chi : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}^\times$; so, $\chi(m)\chi(n) = \chi(mn)$ and $\chi(m) = 0$ if m is not prime to N. The Dirichlet L-function $L(s,\chi)$ is defined by $L(s,\chi) = \sum_{n=1}^{\infty} \chi(n)n^{-s}$ which is absolutely and locally uniformly convergent if $\mathrm{Re}(s) > 1$. The holomorphic function $L(s,\chi)$ can be continued analytically to the whole complex plane if χ is non-trivial (see [LFE] Chapter 2). When $N = 1$ and $\chi = 1$, we define $L(s,\chi)$ by the Riemann zeta function $\zeta(s)$. For simplicity, we suppose that χ is nontrivial.

Identify $\mathbb{G}_m(\mathbb{C})$ with $T = \mathbb{C}/\mathbb{Z}$. Let z be the complex coordinate of T, and put $q = \exp(2\pi i z)$ which is the coordinate of $\mathbb{G}_m(\mathbb{C})$. We consider $\theta_\chi(z) = \sum_{n=1}^{\infty} \chi(n)q^n$. Then we see

$$\theta_\chi(z) = \sum_{a=1}^{N} \chi(a) \sum_{n=0}^{\infty} q^{a+nN} = \sum_{a} \chi(a)\frac{q^a}{1-q^N} = \frac{\sum_a \chi(a)q^a}{1-q^N}.$$

Thus θ_χ is a meromorphic function on T. Since $\sum_a \chi(a) = 0$, $q = 1$ (that is, $z = 0$ is not a pole of θ_χ). Thus θ_χ can have a pole at nontrivial N-th roots of unity in $\mathbb{C}^\times$ (or at $\frac{a}{N} \in T$ for a prime to N).

Lemma 1.7. *If χ is a primitive character modulo N, θ_χ has a pole of order 1 only at primitive N-th roots of unity. The residue of θ_χ at $q = \zeta$ for a primitive N-th root ζ of unity is given by $\mathrm{Res}_{q=\zeta}\theta_\chi = -N^{-1}\zeta G(\chi,\zeta)$ for the Gauss sum $G(\chi,\zeta) = \sum_{a \bmod N} \chi(a)\zeta^a$.*

Proof. If ζ is a M-th root of unity for a proper divisor M of N with $MM' = N$. Then we have

$$\sum_{a=1}^{N} \chi(a)\zeta^a = \sum_{a \in \mathbb{Z}/M\mathbb{Z}} \zeta^a \sum_{b \equiv a \bmod M, b \in \mathbb{Z}/N\mathbb{Z}} \chi(b) = \sum_{a \in \mathbb{Z}/M\mathbb{Z}} \zeta^a \chi(a) \sum_{b \equiv 1 \bmod M, b \in \mathbb{Z}/N\mathbb{Z}} \chi(b) = 0,$$

because χ restricted to the subgroup $\{b \in (\mathbb{Z}/N\mathbb{Z})^\times | b \equiv 1 \mod M\}$ is a nontrivial character. This shows that the zero at $q = \zeta$ of $q^N - 1$ is canceled by the zero at the same location of the numerator of θ_χ.

Now take a primitive N-th root ζ of unity. The value of $(q^N - 1)/(q - \zeta)$ at ζ is given by $\frac{d(q^N-1)}{dq}|_{q=\zeta} = Nq^{N-1}|_{q=\zeta} = N\zeta^{-1}$. The value at $q = \zeta$ of the numerator at $q = \zeta$ is given by the Gauss sum $G(\chi,\zeta) = \sum_{a \bmod N} \chi(a)\zeta^a$ which is non-zero. Thus $\mathrm{Res}_{q=\zeta}\theta_\chi = -N^{-1}\zeta^{-1}G(\chi,\zeta)$. □

Exercise 1.8. *Prove the following facts:*

(1) *the expansion defining θ_χ is absolutely and locally uniformly convergent on the upper half plane $\mathfrak{H} = \{z \in \mathbb{C} | \mathrm{Im}(z) > 0\}$,*
(2) *the expansion of θ_χ convergent on the lower half plane $\overline{\mathfrak{H}}$ is given by*

$$-\sum_{n=-1}^{-\infty} \chi(n)q^n.$$

Write $z = x + iy$ for $i = \sqrt{-1}$. An important fact is

$$(1.3)\quad \int_0^\infty \theta(y\sqrt{-1})dy = \sum_{n=1}^\infty \chi(n)\int_0^\infty \exp(-2\pi y)y^{s-1}dy|_{s=1}$$
$$\overset{(*)}{=} (2\pi)^{-s}\Gamma(s)\sum_{n=1}^\infty \chi(n)n^{-s}|_{s=1} = (2\pi)^{-1}L(1,\chi).$$

Exercise 1.9. (1) *Justify the interchange of the summation and the integral in* (1.3) *when* $\mathrm{Re}(s) > 1$,
(2) *Prove the middle identity* $(*)$, *using Euler's formula* $\int_0^\infty e^{-y}y^{s-1}dy = \Gamma(s)$ *if* $\mathrm{Re}(s) > 1$,
(3) *Justify the formula* (1.3) *even if* $\mathrm{Re}(s) < 1$ *if* χ *is nontrivial. You may use the fact that* $\Gamma(s)$ *is a well defined meromorphic function on the whole complex plane* $\mathbb{C}$.

By a similar calculation using the expansion of θ_χ over the lower half plane, we have

$$(1.4)\quad \int_{-\infty}^0 \theta(y\sqrt{-1})dy = -\sum_{n=1}^\infty \chi(n)\int_{-\infty}^0 \exp(2\pi y)|y|^{s-1}dy|_{s=1}$$
$$= -\chi(-1)(2\pi)^{-s}\Gamma(s)\sum_{n=1}^\infty \chi(n)n^{-s}|_{s=1} = -\chi(-1)(2\pi)^{-1}L(1,\chi).$$

Exercise 1.10. *Justify the above formula* (1.4).

Thus we have
$$2(2\pi i)^{-1}L(1,\chi) = \int_{-\infty}^\infty \theta_\chi dz = \int_\gamma \theta_\chi dz,$$
because $dy = \sqrt{-1}dz$ on γ_0. Here $\gamma = \gamma_0$ is the vertical line on T passing through 0.

Our idea is to prove $L(0,\chi) = -2NG(\chi^{-1})^{-1}\frac{L(1,\chi^{-1})}{2\pi i} \in \mathbb{Q}(\chi)$ (if $\chi(-1) = -1$) by using cohomology theory. Here $G(\chi) = G(\chi, \mathbf{e}(\frac{1}{N}))$ is the standard Gauss sum $G(\chi) = \sum_{a=1}^N \chi(a)\mathbf{e}(\frac{a}{N})$. Here $\mathbb{Q}(\chi)$ is a finite extension of $\mathbb{Q}$ generated by the values $\chi(n)$ for $n = 1, 2, \ldots$. Since θ_χ for primitive χ has a pole at $\frac{a}{N}$, we need to consider $T_N = T - \{\frac{a}{N} | a\mathbb{Z} + N\mathbb{Z} = \mathbb{Z}\}$.

Exercise 1.11. *Prove* $G(\chi^{-1})G(\chi) = \chi(-1)N$.

Then $\omega_\chi = G(\chi)^{-1}\theta_\chi(z)dz$ gives rise to a cohomology class in $H^1_{DR}(T_N, \mathbb{C})$, since $d(\theta_\chi dz) = 0$. If we can prove that $\omega_\chi \in H^1(T_N, \mathbb{Q}(\chi))$, we will have $\int_\gamma \omega_\chi \in \mathbb{Q}(\chi)$.

A Dirichlet character χ is called even or odd according as $\chi(-1) = 1$ or $\chi(-1) = -1$. Here is a theorem of Euler (and Hurwitz):

Theorem 1.12. *If m is a positive integer whose parity is given by the parity of χ, we have* $L(1-m,\chi) \in \mathbb{Q}(\chi)$.

We are going to prove this in Subsection 1.4 when χ is odd and $m = 1$.

1.3. **Relative cohomology.** Since the vertical line γ_x passing through $x \in S^1 = \mathbb{R}/\mathbb{Z}$ is not a 1-chain, we need to generalize cohomology theory to manifolds with boundary.

Exercise 1.13. *Explain why γ_x is not a 1-chain on T_N for $x \notin \{\frac{a}{N} | a\mathbb{Z} + N\mathbb{Z} = \mathbb{Z}\}$.*

We add S^1 as the two boundaries of T_N at $y = \pm\infty$, and write the resulting space as $\overline{T}_N$. Thus $\partial\overline{T}_N$ is the disjoint union of S^1 located at $y = \pm\infty$. We define the subspace of n-chains having values in $\partial\overline{T}_N$ as $C_n(\partial\overline{T}_N; A) \subset C_n(\overline{T}_N; A)$. Then we define $C_n(\overline{T}_N, \partial\overline{T}_N; A) = C_n(\overline{T}_N; A)/C_n(\partial\overline{T}_N; A)$. We still have a chain complex

$$\cdots \xrightarrow{\partial} C_n(\overline{T}_N, \partial\overline{T}_N; A) \xrightarrow{\partial} C_{n-1}(\overline{T}_N, \partial\overline{T}_N; A) \xrightarrow{\partial} \cdots \xrightarrow{\partial} C_0(\overline{T}_N, \partial\overline{T}_N; A) \xrightarrow{\deg} A \to 0,$$

because the boundary map preserves $C_n(\partial\overline{T}_N; A)$. The A-linear dual complex of the above complex is written as $C^n(\overline{T}_N, \partial\overline{T}_N; A)$. Then we define

$$\begin{aligned} H_n(\overline{T}_N, \partial\overline{T}_N, A) &= \frac{\mathrm{Ker}(\partial : C_n(\overline{T}_N, \partial\overline{T}_N; A) \to C_{n-1}(\overline{T}_N, \partial\overline{T}_N; A))}{\mathrm{Im}(\partial : C_{n+1}(\overline{T}_N, \partial\overline{T}_N; A) \to C_n(\overline{T}_N, \partial\overline{T}_N; A))} \\ H^n(\overline{T}_N, \partial\overline{T}_N, A) &= \frac{\mathrm{Ker}(\partial^* : C^n(\overline{T}_N, \partial\overline{T}_N; A) \to C^{n+1}(\overline{T}_N, \partial\overline{T}_N; A))}{\mathrm{Im}(\partial^* : C^{n-1}(\overline{T}_N, \partial\overline{T}_N; A) \to C^n(\overline{T}_N, \partial\overline{T}_N; A))} \end{aligned} \tag{1.5}$$

We have the Poincaré duality pairing $\langle \cdot, \cdot \rangle : H_n(\overline{T}_N, \partial\overline{T}_N, A) \times H^n(\overline{T}_N, \partial\overline{T}_N, A) \to A$ similarly defined as in the case without boundary.

We consider the space of differential forms $\Omega^1(\overline{T}_N, \partial\overline{T}_N; \mathbb{C})$ given by C^∞ differential forms $\omega = f(z)dx + g(z)dy$ such that $\lim_{y\to\pm\infty} y^M g^{(m)}(z) = 0$ for m-th derivatives $g^{(m)}$ of g for all $m \geq 0$ and all $M \in \mathbb{Z}$. Similarly, we write $\Omega^2(\overline{T}_N, \partial\overline{T}_N; \mathbb{C})$ for the space spanned by $g(z)dx \wedge dy$ with $\lim_{y\to\pm\infty} g^{(m)}(z) = 0$ for m-th derivatives $g^{(m)}$ of g for all $m \geq 0$. The C^∞ function satisfying the above limit property is called rapidly decreasing towards $\pm\infty$. The space $\Omega^0(\overline{T}_N, \partial\overline{T}_N; \mathbb{C})$ is made up of functions rapidly decreasing towards $\pm\infty$. We then define

$$H^1_{DR}(\overline{T}_N, \partial\overline{T}_N; \mathbb{C}) = \frac{\mathrm{Ker}(d : \Omega^1(\overline{T}_N, \partial\overline{T}_N; \mathbb{C}) \to \Omega^2(\overline{T}_N, \partial\overline{T}_N; \mathbb{C}))}{\mathrm{Im}(\Omega^0(\overline{T}_N, \partial\overline{T}_N; \mathbb{C}) \to \Omega^1(\overline{T}_N, \partial\overline{T}_N; \mathbb{C}))}.$$

Exercise 1.14. *Prove that for $\omega \in \Omega^1(\overline{T}_N, \partial\overline{T}_N; \mathbb{C})$, the integral $\int_{\gamma_x} \omega$ converges absolutely.*

We thus have a pairing $H^1_{DR}(\overline{T}_N, \partial\overline{T}_N; \mathbb{C}) \times H_1(\overline{T}_N, \partial\overline{T}_N; \mathbb{C}) \to \mathbb{C}$ given by $([\omega], \gamma) = \int_\gamma \omega$. Thus we get (see [Du])

Lemma 1.15. *We have a canonical isomorphism:*

$$H^1_{DR}(\overline{T}_N, \partial\overline{T}_N; \mathbb{C}) \cong H^1(\overline{T}_N, \partial\overline{T}_N; \mathbb{C}).$$

Then the vertical line γ_x for $x \in S^1 - \{\frac{a}{N} | a\mathbb{Z} + \mathbb{N}\mathbb{Z} = \mathbb{Z}\}$ is a well defined 1 cycle. Let $(N^{-1}\mathbb{Z}/\mathbb{Z})^\times$ be the set of $\frac{a}{N}$ with $a\mathbb{Z} + N\mathbb{Z} = \mathbb{Z}$ in $S^1 = \mathbb{R}/\mathbb{Z}$, and write c_a with $a \in (N^{-1}\mathbb{Z}/\mathbb{Z})^\times$ for a small circle centered at a. Then c_a gives a homology class of $H^1(\overline{T}_N, \partial\overline{T}_N, A)$. We have the following theorem given in [LFE] Section 4.1, (1a):

Theorem 1.16. *Let $\{c_a|a \in (N^{-1}\mathbb{Z}/\mathbb{Z})^\times\}$ and γ_0 gives a basis of $H_1(\overline{T}_N, \partial\overline{T}_N, A)$ over A for any commutative ring A.*

Here is a sketch of a proof. Taking a point $i\infty$ at the boundary S^1_∞ at ∞ on γ_0. Draw a straight line from ∞ to c_a and the boundary $S^1_{-\infty}$ at $-\infty$. We take γ_0 for the line from ∞ to $-\infty$ on $S^1_{-\infty}$. Cut $\overline{T}_N$ along this line. We get a simply connected polygon encircled by these lines and c_a, γ_0 and $S^1_{\pm\infty}$. The sum of all these cycles are zero. Modulo $S^1_{\pm\infty}$, we find that $H^1(\overline{T}_N, \partial\overline{T}_N, A)$ is generated by c_a and γ_0. We have the natural A-linear surjection

$$\pi : H^1(\overline{T}_N, \partial\overline{T}_N, A) \to H^1(\overline{T}, \partial\overline{T}, A),$$

regarding cycles in $\overline{T}_N$ as cycles in $\overline{T} = T \sqcup \partial\overline{T}_N$. Since $H^1(\overline{T}, \partial\overline{T}, A)$ is the dual of $H^1(T, A) = A \cdot S^1$ by the intersection product, we have $H^1(\overline{T}, \partial\overline{T}, A) = A\gamma_0$. The kernel of π is generated by c_as. The cycles c_as are independent by construction. □

Exercise 1.17. *Give details of the proof of the above theorem.*

Corollary 1.18. *For the cohomology class of $\omega_\chi = G(\chi)\theta_\chi dz$ in $H^1_{DR}(\overline{T}_N, \partial\overline{T}_N, \mathbb{C})$, the integral $\int_{c_a} \omega_\chi$ is nonzero and belongs to $\mathbb{Q}(\chi)$ for all $a \in (N^{-1}\mathbb{Z}/\mathbb{Z})^\times$.*

Proof. Identify $\mathrm{Gal}(\mathbb{Q}(\mu_N)/\mathbb{Q})$ with $(\mathbb{Z}/N\mathbb{Z})^\times$ $\sigma \leftrightarrow m$ if $\zeta^\sigma = \zeta^m$ for a primitive N-th root ζ of unity. Then $G(\chi,\zeta)^\sigma = \chi(\sigma)^{-1}G(\chi,\zeta)$ for $\sigma \in \mathrm{Gal}(\mathbb{Q}(\chi)[\mu_N]/\mathbb{Q}(\chi))$. Since $2\pi\sqrt{-1}dz = q^{-1}dq$, by Lemma 1.7, we find that $\int_{c_a} \omega_\chi = -N^{-1}G(\chi)^{-1}G(\chi,\zeta)$. Since $G(\chi,\zeta)^\sigma = \chi(\sigma)^{-1}G(\chi,\zeta)$, $\frac{G(\chi,\zeta)}{G(\chi)}$ is invariant under any $\sigma \in \mathrm{Gal}(\mathbb{Q}(\chi)[\mu_N]/\mathbb{Q}(\chi))$, we conclude the assertion. □

1.4. **Hecke operators.** We consider the multiplication $[n] : \mathbb{G}_m(\mathbb{C}) \to \mathbb{G}_m(\mathbb{C})$ given by $x \mapsto x^n$ for a positive integer n prime to N. This map is $z \mapsto nz$ on $\overline{T}_N$. Thus $[n]$ preserves $(N^{-1}\mathbb{Z}/\mathbb{Z})^\times$, $[n]$ acts on cycles and cocycles, by the direct image and the pull back. We write this action on $H_1(\overline{T}_N, \partial\overline{T}_N, A)$ and $H^1(\overline{T}_N, \partial\overline{T}_N, A)$ as $T(n)$. Thus for a differential form $\omega = f(z)dz$, $\omega|T(n) = [n]^*\omega$.

Exercise 1.19. *Prove that for $\omega = f(z)dz$, $[\omega]|T(n) = \frac{1}{n}[\sum_{a \bmod n} f(\frac{z+a}{n})dz]$.*

By the above formula, we have

$$\theta_\chi(z)|T(m) = \frac{1}{m}\sum_{n=1}^{\infty}\chi(n)\mathbf{e}(nz)|T(m) = \frac{1}{m}\sum_{n=1}^{\infty}\chi(n)\mathbf{e}(\frac{nz}{m})\sum_{a \bmod m}\mathbf{e}(\frac{na}{m})$$
$$= \frac{1}{m}\sum_{n=1}^{\infty}\chi(mn)m\mathbf{e}(\frac{nz}{m}) = \chi(m)\theta_\chi(z),$$

because $\sum_{a \bmod m}\mathbf{e}(\frac{na}{m}) = \begin{cases} 0 & \text{if } m \nmid n \\ m & \text{if } m|n. \end{cases}$ Thus we have

Lemma 1.20. *For each primitive character χ modulo N, we have $[\omega_\chi]|T(n) = \chi(n)[\omega_\chi]$.*

Exercise 1.21. *Give a detailed proof of Lemma 1.20.*

Theorem 1.22. *Let $H^1(\overline{T}_N, \partial\overline{T}_N, \mathbb{Q}(\chi))[\chi]$ be the subspace of $H^1(\overline{T}_N, \partial\overline{T}_N, \mathbb{Q}(\chi))$ on which $T(n)$ for all n prime to N acts by multiplication by $\chi(n)$. Then we have*

$$\dim_{\mathbb{Q}(\chi)} H^1(\overline{T}_N, \partial\overline{T}_N, \mathbb{Q}(\chi))[\chi] = 1$$

for a nontrivial primitive character χ modulo N.

Proof. By definition, $T(n)(c_a) = \sum_{u \bmod n} c_{\frac{a}{n}+\frac{u}{n}}$ for $a \in (N^{-1}\mathbb{Z}/\mathbb{Z})^\times$. If $\frac{a}{n}+\frac{u}{n}$ has reduced numerator bigger than N, the homology class of $c_{\frac{a}{n}+\frac{u}{n}}$ vanishes in $H_1(\overline{T}_N, \partial\overline{T}_N, \mathbb{Z})$. The fraction $\frac{a}{n}+\frac{u}{n}$ has exact numerator N if and only if $aN + nu \equiv 0 \mod n$. Such u is unique (because n is prime to N). In other words, $aN + nu = mn$; so, $\frac{aN+nu}{n} = m$ and $m \equiv [n]^{-1}aN \mod N$, writing $[n]$ for the multiplication by $n \mod N$ on $(N^{-1}\mathbb{Z}/\mathbb{Z})^\times$. We thus have $T(n)(c_a) = c_{[n]^{-1}a}$.

The morphism $\pi : H_1(\overline{T}_N, \partial\overline{T}_N, \mathbb{Q}(\chi)) \to H_1(\overline{T}, \partial\overline{T}, \mathbb{Q}(\chi)) = \mathbb{Q}(\chi)\gamma_0$ is equivariant under $T(n)$, though the section $H_1(\overline{T}, \partial\overline{T}, \mathbb{Q}(\chi))\gamma_0 \hookrightarrow H_1(\overline{T}_N, \partial\overline{T}_N, \mathbb{Q}(\chi))$ sending $\gamma_0 \in H_1(\overline{T}, \partial\overline{T}, \mathbb{Q}(\chi))$ to $\gamma_0 \in H_1(\overline{T}_N, \partial\overline{T}_N, \mathbb{Q}(\chi))$ is not equivariant. The action of $T(n)$ on $H_1(\overline{T}, \partial\overline{T}, \mathbb{Q}(\chi))$ is multiplication by n. Since c_a and γ_0 span $H_1(\overline{T}_N, \partial\overline{T}_N, \mathbb{Q}(\chi))$ and $T(n)$ permutes c_a, the eigenvalues of $T(n)$ on $\mathrm{Ker}(\pi)$ is given by $\{\chi(n)\}_\chi$ where χ runs over all characters of $\mathbb{Z}/N\mathbb{Z}$. Thus the commutative algebra $\mathcal{H}_N(\mathbb{Q}(\chi))$ generated over $\mathbb{Q}(\chi)$ by $T(n)$s in $\mathrm{End}(H^1(\overline{T}_N, \partial\overline{T}_N, \mathbb{Q}(\chi)))$ acts on $H^1(\overline{T}_N, \partial\overline{T}_N, \mathbb{Q}(\chi))$ semi-simply with multiplicity free, because $\dim_{\mathbb{Q}(\chi)} H^1(\overline{T}_N, \partial\overline{T}_N, \mathbb{Q}(\chi))$ is given by $1 + |(\mathbb{Z}/N\mathbb{Z})^\times|$, which is the number of distinct eigenvalues realized on $H^1(\overline{T}_N, \partial\overline{T}_N, \mathbb{Q}(\chi))$. Thus the result follows. ☐

Exercise 1.23. *Compute the action of $T(n)$ on $\gamma_0 \in H_1(\overline{T}_N, \partial\overline{T}_N, \mathbb{Q}(\chi))$ (for n prime to N).*

Corollary 1.24. *The cohomology class of ω_χ in $H^1_{DR}(\overline{T}_N, \partial\overline{T}_N, \mathbb{C})$ is rational over $\mathbb{Q}(\chi)$.*

Proof. The class $[\omega_\chi]$ generates $H^1_{DR}(\overline{T}_N, \partial\overline{T}_N, \mathbb{C})[\chi]$, which is one dimensional. If we find a 1-cycle c in $H_1(\overline{T}_N, \partial\overline{T}_N, \mathbb{Q}(\chi))$ such that $0 \neq \int_c \omega_\chi \in \mathbb{Q}(\chi)$, we may conclude that ω_χ is rational over $\mathbb{Q}(\chi)$. The cycle c_a as in Corollary 1.18 does the job. ☐

By this corollary, we have finished the proof of Theorem 1.12 for $m = 1$.

Exercise 1.25. *Show the following assertions:*

(1) *For an integer a prime to N, the operator $T(a) - a$ sends $H_1(\overline{T}_N, \partial\overline{T}_N, \mathbb{Z}[\chi])$ into* $\mathrm{Ker}(\pi : H_1(\overline{T}_N, \partial\overline{T}_N, \mathbb{Z}[\chi]) \to H_1(\overline{T}, \partial\overline{T}, \mathbb{Z}[\chi]))$*, where $\mathbb{Z}[\chi]$ is the subalgebra of $\mathbb{Q}(\chi)$ generated by all the values of χ.*
(2) $(a - \chi^{-1}(a))L(0, \chi) \in \mathbb{Z}[\chi]$ *for any integer a prime to N, where χ is a nontrivial primitive character.*

1.5. **p-Adic measure.** Let p be a prime and G be a topological group of the form $G = \mu \times \mathbb{Z}_p^r$ for a finite group μ. We put $G_i = (p^i\mathbb{Z}_p)^r$. We fix a finite extension $K/\mathbb{Q}_p$ and write A for its p-adic integer ring. We equip K a normalized p adic norm $|\cdot|_p$ such that $|p|_p = \frac{1}{p}$. For each topological space X, we write $LC(G; X)$ for the space of locally constant functions on G with values in X. Thus a function $\phi : G \to X$ is in $LC(G;X)$ if and only if for any point $g \in G$, there exists an open neighborhood V_g of g in G such that the restriction of ϕ to V_g is a constant function. By definition, it is plain that for any locally constant function ϕ and for any subset S of X, $\phi^{-1}(S) = \bigcup_{g\in\phi^{-1}(S)} V_g$ is open; in particular, ϕ is continuous. Since G is compact, $G = \bigcup_{g\in G} V_g$ implies that we can find finitely many points $g_1, \dots, g_s$ on G such that $G = \bigcup_{j=1}^{s} V_{g_j}$. By the definition of the topology of G, a basis of open sets of G is given by $\{g + G_i | g \in G, i = 0, 1, \dots\}$. Thus for large i, $V_{g_j} \supset g_j + G_i$ for all j, that is, ϕ induces a function

$$\phi_i : G/G_i \to X \quad \text{and} \quad \phi = \phi_i \circ \pi_i$$

for the projection $\pi_i : G \to G/G_i$. The space $C(G/G_i;X)$ of continuous functions of G/G_i into X is made of all functions on the finite group G/G_i with values in X and is isomorphic to the set $X[G/G_i]$ of formal linear combinations $\sum_{g\in G/G_i} x_g g$ with $x_g \in X$ via $\phi \mapsto \sum_g \phi(g)g$. Thus we see that

$$LC(G;X) = \varinjlim_i C(G/G_i;X) = \varinjlim_i X[G/G_i]. \tag{1.6}$$

For a topological ring R, we define the space of distributions $Dist(G;R)$ by

$$Dist(G;R) = \mathrm{Hom}_R(LC(G;R),R). \tag{1.7}$$

If $\varphi \in Dist(G;R)$ and if χ_S is the characteristic function of an open set S of G, we write $\varphi(S)$ for $\varphi(\chi_S)$. Since $\chi_{h+G_i} = \sum_{g\in G_i/G_j} \chi_{h+g+G_j}$ for $j \geq i$, we have the following distribution relation:

$$\varphi(h + G_i) = \sum_{g\in G_i/G_j} \varphi(h + g + G_j) \quad \text{for all } h \in G \text{ and } j \geq i. \tag{1.8}$$

On the other hand, if we are given a system φ assigning a value $\varphi(g + G_i) \in R$ for all $g \in G/G_i$ and for all i sufficiently large satisfying (1.8), we can extend φ to a distribution as follows. For a given $\phi \in LC(G;R)$, taking sufficiently large i so that $\varphi(g+G_i)$ is well defined and $\phi = \phi_i \circ \pi_i$ with $\phi_i : G/G_i \to R$, we define $\varphi(\phi) = \sum_{g\in G/G_i} \phi_i(g)\varphi(g + G_i)$.

Exercise 1.26. *Prove that $\varphi(\phi)$ is well defined independent of the choice i of the index if i is sufficiently large.*

Thus we have

Proposition 1.27. *Let R be a topological ring. Then a function*

$$\varphi : \{g + G_i | i \geq M, \text{ and } g \in G\} \to R$$

is induced from a distribution if and only if φ satisfies (1.8) *for all $j \geq i \geq M$.*

Let R be a closed subring of K. Let $C(G;R)$ be the space of continuous functions of G into R. Define a norm on $C(G;R)$ by $|\phi|_p = \mathrm{Sup}_{x\in G} |\phi(x)|_p$. A measure φ is a R-linear functional $\varphi \in \mathrm{Hom}_R(C(G;R),R)$ such that $|\varphi(\phi)|_p \leq C|\phi|_p$ for a constant $C > 0$ independent of any continuous function $\phi \in C(G;R)$. We often write $\varphi(\phi)$ as $\int_G \phi d\varphi$ following the tradition of Libnitz.

Exercise 1.28. *Prove that* $|\phi|_p$ *gives a well defined norm on* $C(G;R)$*. Is* $LC(G;R)$ *(resp.* $C(G;R)$*) a Banach space under* $|\cdot|_p$*?*

Write $Meas(G;R)$ for the space of R-valued measures on G. For any measure $\varphi \in Meas(G;R)$, φ induces a distribution, again denoted by φ, by $\varphi(S) = \int \chi_S d\varphi$. Then $|\varphi(\phi)|_p \le C|\phi|_p$ for all $\phi \in C(G;R)$. Thus $|\varphi|_p = \mathrm{Sup}_{0\neq\phi\in LC(G;R)} |\varphi(\phi)|_p/|\phi|_p$ is finite. Now we want to show the converse. For any continuous function $\phi : G \to R$, we can find for each positive $\varepsilon > 0$ and $g \in G$ a small open neighborhood V_g of g such that $|\phi(h) - \phi(h')|_p < \varepsilon$ for all h and h' in V_g. Cover G by such V_g: $G = \bigcup_{g\in G} V_g$. Since G is compact, we can choose finitely many $g_1, \ldots, g_s \in G$ such that $\bigcup_{j=1}^{s} V_{g_j}$ and find an index i large such that $V_{g_j} \supset g + G_i$ for all $g \in V_{g_j}$. Choosing a complete representative set Ξ_i for G/G_i and defining $\phi_\varepsilon : G/G_i \to R$ by $\phi_\varepsilon(h) = \phi(g)$ if $h \in (g + G_i) \cap \Xi_i$, we see that $\phi_\varepsilon \in LC(G;R)$ and $|\phi_\varepsilon - \phi|_p < \varepsilon$. Thus $LC(G;R)$ is dense in $C(G;R)$ and

$$|\phi_{\varepsilon'} - \phi_\varepsilon|_p < |(\phi_{\varepsilon'} - \phi) + (\phi - \phi_\varepsilon)|_p \le \max(|\phi_{\varepsilon'} - \phi|_p, |\phi - \phi_\varepsilon|_p) \le \max(\varepsilon, \varepsilon'). \tag{1.9}$$

Let φ is a distribution with bounded norm $|\varphi|_p$. This is equivalent to saying that $|\varphi(g + G_i)|_p$ is bounded by $|\varphi|_p$ for all $i \ge M$ and all $g \in G$. Then (1.9) implies

$$\varphi(\phi_\varepsilon) - \varphi(\phi_{\varepsilon'})_p \le |\varphi|_p |\phi_{\varepsilon'} - \phi_\varepsilon|_p \le |\varphi|_p \max(\varepsilon, \varepsilon')$$

and $\{\varphi(\phi_{1/n})\}$ is a Cauchy sequence in R. We then define

$$\int_G \phi d\varphi = \lim_{n\to\infty} \varphi(\phi_{1/n}) \in R.$$

Then it is easy to verify that $\varphi \in Meas(G;R)$. Thus we have

Proposition 1.29. *For any closed subring* R *of* K*,* $LC(G;R)$ *is dense in* $C(G;R)$*. Any bounded distribution on* G *with values in* R *can be uniquely extended to a bounded measure with values in* R*. In particular,* $Meas(G;A) \cong Dist(G;A)$ *via the restriction to* $LC(G;A)$ *for the* p*-adic integer ring* A *of* K*.*

Exercise 1.30. *If* $\varphi \in Meas(G;K)$*, prove*

$$\mathrm{Sup}_{0\neq\phi\in C(G;K)} |\varphi(\phi)|_p/|\phi|_p = \mathrm{Sup}_{0\neq\phi\in LC(G;K)} |\varphi(\phi)|_p/|\phi|_p.$$

1.6. **p-Adic measure and Hecke operators.** We construct a p-adic measure which interpolates the values of Dirichlet L-functions via cohomology theory. This type of formalism (the formalism of modular symbols) was found by Mazur in [M] and [MS], where he applied it to L-functions of elliptic modular forms.

Let $N > 1$ be a positive integer prime to the fixed prime p. Let $\overline{T}_N = \mathbb{C}/\mathbb{Z} - (N^{-1}\mathbb{Z}/\mathbb{Z})^\times$ be as above. We have an A-linear map: $H^1(\overline{T}_N, \partial\overline{T}_N, A) \to A$ given by $\omega \mapsto \int_{\gamma_x} \omega$. Then we consider a map

$$c : p^{-\infty}\mathbb{Z} = \bigcup_{i=1}^{\infty} p^{-i}\mathbb{Z} \to \mathrm{Hom}_A(H^1(\overline{T}_N, \partial\overline{T}_N, A), K) \tag{1.10}$$

given by $c(x)(\omega) = \int_{\gamma_x} \omega$.

For $\omega \in H^1(\overline{T}_N, \partial\overline{T}_N, A)$, we write $c_\omega(r) = \int_{\gamma_r} \omega$. Then $c_\omega(r+1) = c_\omega(r)$ by definition, and c_ω factors through $\mathbb{Q}_p/\mathbb{Z}_p = p^{-\infty}\mathbb{Z}/\mathbb{Z}$. Supposing $\omega|T(p) = a\omega$ with $|a|_p = 1$, we define a distribution φ_ω on $\mathbb{Z}_p$ by

$$\varphi_\omega(z + p^m\mathbb{Z}_p) = a^{-m} c_\omega(\frac{z}{p^m}) \quad \text{for } z = 1, 2, \dots \text{ prime to } p. \tag{1.11}$$

This is well defined because $c_\omega(r+1) = c_\omega(r)$. We take the multiplicative group $G = \mathbb{Z}_p^\times$ and fix an isomorphism $G \cong \mu \times \mathbb{Z}_p$ for a finite group μ, where $\mathbb{Z}_p$ in the right-hand-side is an additive group.

Exercise 1.31. *Prove that* $\mu = \{\zeta \in \mathbb{Z}_p^\times | \zeta^M = 1\}$*, where* $M = p-1$ *or* 2 *according as* p *is odd or even.*

Then the multiplicative subgroup $G_i = 1 + p^i\mathbb{Z}_p$ corresponds to the additive group $p^i\mathbb{Z}_p$. To show that φ_ω actually gives a distribution, we need to check the distribution relation (1.8). We compute

$$\sum_{j=1}^{p} c_\omega(\frac{x+j}{p}) = \sum_j c(\frac{x+j}{p})(\omega) = c(x)(\omega|T(p)) = a \cdot c_\omega(x).$$

This shows

$$\sum_{j=1}^{p} \varphi_\omega(x + jp^m + p^{m+1}\mathbb{Z}_p) = \varphi_\omega(x + p^m\mathbb{Z}_p).$$

The general distribution relation (1.8) then follows from the iteration of this relation. By a similar argument, we see that

$$|\varphi_\omega(z + p^m\mathbb{Z}_p)|_p = |a^{-m} c_\omega(\frac{z}{p^m})|_p = |c(\frac{z}{p^m})(\omega)|_p \le |\omega|_p, \tag{1.12}$$

where $|\omega|_p = \mathrm{Sup}_x\, |c(x)(\omega)|_p$ with x running over $p^{-\infty}\mathbb{Z}$. Thus φ_ω is bounded and, by Proposition 1.29, we have a unique measure φ_ω extending the distribution φ_ω. Now we compute $\int_G \phi d\varphi_\omega(x)$. To do this, we may assume that $|\omega|_p \le 1$ by multiplying by a constant if necessary. For $\phi \in C(G/G_m; A)$, we have

$$\int_G \phi d\varphi_\omega = a^{-m} \sum_{z=1}^{p^m} \phi(z) c_\omega(\frac{z}{p^m}).$$

Let $N > 1$ be a positive integer prime to p. We take $\omega = \omega_{\chi^{-1}}$ for each primitive character χ modulo N (ω may not be p-integral but is bounded because $(a - \chi^{-1}(a))\omega$ is p-integral as seen in Exercise 1.25). Then we write φ_ω as $\varphi = \varphi_\chi$ and compute for any primitive character ϕ of $(\mathbb{Z}/p^r\mathbb{Z})^\times$ the integral $\int_G \phi d\varphi_\chi$. Note that $\omega|T(p) = \chi(a)^{-1}\omega$. We write $\alpha_x : \overline{T}_N \to \overline{T}_N$ be then translation $\alpha_x(z) = z + x$ for $x \in \mathbb{R}$. We see that, if

$\phi \neq 1$, then $\phi\chi$ is primitive modulo Np^r and

$$\begin{aligned}\int_G \phi d\varphi_\chi =&\chi(p)^r \sum_{x\in(\mathbb{Z}/p^r\mathbb{Z})^\times} \phi(x)c(\frac{x}{p^r})(\omega)\\ =&\chi(p)^r \int_{\gamma_0} \sum_{x\in(\mathbb{Z}/p^r\mathbb{Z})^\times} \phi(x)\alpha^*_{x/p^r}\omega\\ =&\chi(p)^r G(\chi)G(\phi)\int_{\gamma_0}\theta_{\phi^{-1}\chi^{-1}}(z)dz\\ =&-\chi(p)^r G(\chi)G(\phi)(2pi)^{-1}(1-\chi\phi(-1))L(1,(\chi\phi)^{-1})\\ =&-\chi(p)^r G(\chi)G(\phi)G(\chi\phi)^{-1}L(0,\chi\phi).\end{aligned} \tag{1.13}$$

Here we have used the following formulas

$$L(0,\chi^{-1}) = (1-\chi(-1))G(\chi^{-1})(2\pi i)^{-1}L(1,\chi)$$

by the functional equation of $L(s,\chi)$ and

$$G(\chi)G(\phi) = \chi^{-1}(p^r)\phi^{-1}(N)G(\chi\phi). \tag{1.14}$$

Exercise 1.32. *Prove* (1.14).

We have basically proved the following theorem of Kubota-Leopoldt-Iwasawa when $j = 1$:

Theorem 1.33. *Let p be a prime and N be a positive integer prime to p. For each primitive Dirichlet character $\chi \neq 1$ modulo N, we have a unique p-adic measure φ_χ on $\mathbb{Z}_p^\times$ such that for all finite order characters ϕ of $\mathbb{Z}_p^\times$ and $1 \leq j \in \mathbb{Z}$, we have*

$$\int \phi(z)z^j d\varphi_\chi = -\phi(N)^{-1}N^{-j}(1-\chi\phi(p)p^j)L(1-j,\chi\phi).$$

As for the identity character, fixing a prime q prime to p, we have a unique p-adic measure φ_q on $\mathbb{Z}_p^\times$ such that for all finite order characters ϕ of $\mathbb{Z}_p^\times$ and $1 \leq j \in \mathbb{Z}$, we have

$$\int \phi(z)z^j d\varphi_q = -(1-\phi^{-1}(q)q^{-1-j})(1-\phi(p)p^j)L(1-j,\phi).$$

We now define the p-adic Dirichlet L-function for each primitive character χ modulo Np^r (with values in K), writing χ_N (resp. χ_p) for the restriction of χ to $(\mathbb{Z}/N\mathbb{Z})^\times$ (resp. $(\mathbb{Z}/p^r\mathbb{Z})^\times$), by

$$L_p(s,\chi) = \begin{cases} -\chi_p\omega^{-1}(N)\langle N\rangle^{-s}\int_{\mathbb{Z}_p^\times}\chi_p\omega^{-1}(x)\langle x\rangle^{-s}d\varphi_{\chi_N}(x) & \text{if } \chi_N \neq 1,\\ -(1-\chi_p\omega^{-1}(\gamma)\langle\gamma\rangle^{s-1})^{-1}\int_{\mathbb{Z}_p^\times}\chi_p\omega^{-1}(x)\langle x\rangle^{-s}d\varphi_{\chi_N}(x) & \text{if } \chi_N = 1.\end{cases}$$

Here $\langle N\rangle \in 1+p\mathbb{Z}_p$ is given by $N\omega(N)^{-1}$ for the Teichmüller character $\omega : \mathbb{Z}_p^\times \to \mu$ which is given by $\omega(x) = \lim_{n\to\infty} x^{p^n}$ if p is odd, and if $p = 2$, $\omega(x) = \pm 1$ according as $x \equiv \pm 1 \mod 4$. Thus we get the following result:

Theorem 1.34. *For each primitive Dirichlet character* $\chi : (\mathbb{Z}/Np^r\mathbb{Z})^\times \to \overline{\mathbb{Q}}^\times$ *with* $\chi(-1) = 1$*, there exists a* p*-adic analytic function* $L_p(s, \chi)$ *on* $\mathbb{Z}_p$ *if* $\chi_N \neq 1$ *and on* $\mathbb{Z}_p - \{1\}$ *if* $\chi = 1$ *such that*

$$L_p(-m, \chi) = (1 - \chi\omega^{-m-1}(p)p^m)L(-m, \chi\omega^{-m-1})$$

for all non-negative integers m*.*

See [LFE] Chapter 4 for more details of these facts.

2. Modular p-adic L-functions

In this section, we will do the exactly the same construction of p-adic L-functions for elliptic Hecke eigenforms in place of rational functions on $\mathbb{G}_m$.

2.1. **Elliptic modular forms.** Let $\Gamma_0(N) = \left\{\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in SL_2(\mathbb{Z}) \middle| c \equiv 0 \mod N\right\}$. This a subgroup of finite index in $SL_2(\mathbb{Z})$.

Exercise 2.1. *Let $\mathbf{P}^1(A)$ be the projective space of dimension 1 over a ring A. Prove $[SL_2(\mathbb{Z}) : \Gamma_0(N)] = |\mathbf{P}^1(\mathbb{Z}/N\mathbb{Z})| = N\prod_{\ell|N}(1+\frac{1}{\ell})$ if N is square-free, where ℓ runs over all prime factors of N.* Hint: Let $\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in SL_2(\mathbb{Z})$ acts on $\mathbf{P}^1(A)$ by $z \mapsto \frac{az+b}{cz+d}$ and show that this is a transitive action if $A = \mathbb{Z}/N\mathbb{Z}$ and the stabilizer of ∞ is $\Gamma_0(N)$.

We let $\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in GL_2(\mathbb{C})$ acts on $\mathbf{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$ by $z \mapsto \frac{az+b}{cz+d}$ (by linear fractional transformation).

Exercise 2.2. *Prove the following facts:*

1. *there are two orbits of the action of $GL_2(\mathbb{R})$ on $\mathbf{P}^1(\mathbb{C})$: $\mathbf{P}^1(\mathbb{R})$ and $\mathfrak{H} \sqcup \overline{\mathfrak{H}}$, where $\mathfrak{H} = \{z \in \mathbb{C} | \operatorname{Im}(z) > 0\}$ and $\overline{\mathfrak{H}} = \{z \in \mathbb{C} | \operatorname{Im}(z) < 0\}$.*
2. *the stabilizer of $i = \sqrt{-1}$ is the center times $SO_2(\mathbb{R}) = \left\{\left(\begin{smallmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta)\end{smallmatrix}\right) \middle| \theta \in \mathbb{R}\right\}$,*
3. *$\gamma \in GL_2(\mathbb{R})$ with $\det(\gamma) < 0$ interchanges the upper half complex plane $\mathfrak{H}$ and lower half complex plane $\overline{\mathfrak{H}}$,*
4. *the upper half complex plane is isomorphic to $SL_2(\mathbb{R})/SO_2(\mathbb{R})$ by $SL_2(\mathbb{R}) \ni g \mapsto g(\sqrt{-1}) \in \mathfrak{H}$.*

Then $Y_0(N) = \Gamma_0(N)\backslash\mathfrak{H}$ is an open Riemann surface with hole at cusps. In other words, $X_0(N) = \Gamma_0(N)\backslash(\mathfrak{H} \sqcup \mathbf{P}^1(\mathbb{Q}))$ is a compact Riemann surface.

Exercise 2.3. *Show that $SL_2(K)$ acts transitively on $\mathbf{P}^1(K)$ for any field K by linear fractional transformation.* Hint: $\left(\begin{smallmatrix} 1 & a \\ 0 & 1\end{smallmatrix}\right)(0) = a$.

Let $f : \mathfrak{H} \to \mathbb{C}$ be a holomorphic functions with $f(z+1) = f(z)$. Since $\mathfrak{H}/\mathbb{Z} \cong D = \left\{z \in \mathbb{C}^\times \middle| |z| < 1\right\}$ by $z \mapsto q = \mathbf{e}(z) = \exp(2\pi i z)$, we may regard f as a function of q undefined at $q = 0 \Leftrightarrow z = i\infty$. Then the Laurent expansion of f gives

$$f(z) = \sum_n a(n, f)q^n = \sum_n a(n, f)\exp(2\pi i n z).$$

In particular, we may assume that q is the coordinate of $X_0(N)$ around the infinity cusp ∞. We call f is *finite* (resp. *vanishing*) at ∞ if $a(n, f) = 0$ if $n < 0$ (resp. if $n \leq 0$). By Exercise 2.3, we can bring any point $c \in \mathbf{P}^1(\mathbb{Q})$ to ∞; so, the coordinate around the cusp c is given by $q \circ \alpha$ for $\alpha \in SL_2(\mathbb{Q})$ with $\alpha(c) = \infty$.

Exercise 2.4. *Show that the above α can be taken in $SL_2(\mathbb{Z})$.* Hint: write $c = \frac{a}{b}$ as a reduced fraction; then, we can find $x, y \in \mathbb{Z}$ such that $ax - by = 1$.

We consider the space of holomorphic functions $f : \mathfrak{H} \to \mathbb{C}$ satisfying the following conditions:

(M1) $f(\frac{az+b}{cz+d}) = f(z)(cz+d)^2$ for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$.

If f satisfies the above conditions, we find that $f(z+1) = f(z)$ because $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)(z) = z+1$; so, we can say that f is finite or not.

Exercise 2.5. *Define* $f|\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)(z) = f(\frac{az+b}{cz+d})(cz+d)^{-2}$. *Prove the following facts:*

(1) $(f|\alpha)|\beta = f|(\alpha\beta)$ *for* $\alpha \in SL_2(\mathbb{R})$,

(2) *if* f *satisfies (M1),* $f|\alpha$ *satisfies (M1) replacing* $\Gamma_0(N)$ *by* $\Gamma = \alpha^{-1}\Gamma_0(N)\alpha$,

(3) *If* $\alpha \in SL_2(\mathbb{Z})$, *show that* Γ *contains* $\Gamma(N) = \{\gamma \in SL_2(\mathbb{Z}) | \gamma - 1 \in NM_2(\mathbb{Z})\}$.

By (3) of the above exercise, for $\alpha \in SL_2(\mathbb{Z})$, we find $f|\alpha(z+N) = f|\alpha(z)$; thus, $f|\alpha$ has expansion $f|\alpha = \sum_n a(n, f|\alpha)q^{Nn}$. We call f is finite (resp. vanishing) at the cusp $\alpha^{-1}(\infty)$ if $f|\alpha$ is finite (resp. vanishing) at ∞.

(M2) f is finite at all cusps of $X_0(N)$.

We write $M_2(\Gamma_0(N))$ for the space of functions satisfying (M1–2). Replace (M2) by

(S) f is vanishing at all cusps of $X_0(N)$,

we define subspace $S_2(\Gamma_0(N)) \subset M_2(\Gamma_0(N))$ by imposing (S). An element in $S_2(\Gamma_0(N))$ is called a holomorphic cusp form on $\Gamma_0(N)$ of weight 2.

2.2. **Modular cohomology group.** Take a holomorphic differential ω on $X_0(N)$. Then we pull back ω to $\mathfrak{H}$ and still write ω. We can write $\omega = f(z)dz$ on $\mathfrak{H}$ because $\mathfrak{H}$ is simply connected.

Exercise 2.6. *For* $\alpha = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{R})$, *prove* $\alpha^* dz = d(\frac{az+b}{cz+d}) = (cz+d)^{-2}dz$.

Since $\gamma^*\omega = \omega$ for all $\gamma \in \Gamma_0(N)$, we find

$$f(z)dz = \omega = \gamma^*\omega = f(\gamma(z))\gamma^* dz = f(\gamma(z))(cz+d)^{-2}dz$$

if $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Thus f has to satisfy (M1). At infinity, since $dz = 2\pi i \frac{dq}{q}$, ω with respect to the coordinate q is finite at ∞, and hence $f(z)dz = 2\pi i \frac{dq}{q} \sum_n a(n, f)q^n$ is finite at $q = 0$. This implies f has to be vanishing at ∞. Writing $H^0(X_0(N), \Omega_{X_0(N)/\mathbb{C}})$ for the space of holomorphic 1-forms on $X_0(N)$, we thus find

Proposition 2.7. *We have a canonical isomorphism* $S_2(\Gamma_0(N)) \cong H^0(X_0(N), \Omega_{X_0(N)/\mathbb{C}})$ *sending* f *to* $f(z)dz$.

Let C be the divisor on $X_0(N)$ which is the formal sum of all cusps. If we write $H^0(X_0(N), \Omega_{X_0(N)/\mathbb{C}}(-C))$ for the space of meromorphic 1-forms on $X_0(N)$ with at most simple poles at cusps, by the same argument, we have

Corollary 2.8. *We have an isomorphism* $M_2(\Gamma_0(N)) \cong H^0(X_0(N), \Omega_{X_0(N)/\mathbb{C}}(-C))$ *sending* f *to* $f(z)dz$.

For any compact Riemann surface X, general theory of Riemann surface tells us $H^1(X, \mathbb{C}) \cong H^0(X, \Omega_{X/\mathbb{C}}) \oplus H^0(X, \overline{\Omega}_{X/\mathbb{C}})$ (the Hodge decomposition, where $H^0(X, \Omega_{X/\mathbb{C}})$ is the space of holomorphic 1-forms on X and $H^0(X, \overline{\Omega}_{X/\mathbb{C}})$ is the space of antiholomorphic 1-forms on X. Since $H^0(X, \overline{\Omega}_{X/\mathbb{C}})$ is the complex conjugate of $H^0(X, \Omega_{X/\mathbb{C}})$, we get

Proposition 2.9. *We have a canonical isomorphism*

$$H^1(X_0(N),\mathbb{C}) \cong S_2(\Gamma_0(N)) \oplus \overline{S}_2(\Gamma_0(N)),$$

where $\overline{S}_2(\Gamma_0(N))$ is made up of complex conjugate $\overline{f}$ for $f \in S_2(\Gamma_0(N))$. In particular, $S_2(\Gamma_0(N))$ is finite dimensional, and its dimension is given by the genus of $X_0(N)$.

We add a small circle at each cusp of $Y_0(N)$ and getting a different compactification $\overline{Y}_0(N)$ of $Y_0(N)$ from $X_0(N)$. Taking the circle S around the cusp c. Then $\int_S \omega$ is essentially the residue of ω and if we write $\omega = f(z)dz$ it is given by $a(0, f|\alpha)$ for $\alpha \in SL_2(\mathbb{Z})$ taking the cusp to ∞. Thus we get

Corollary 2.10. *If we write $g_0(N)$ for the genus of $X_0(N)$ and $c_0(N)$ for the number of cusps of $X_0(N)$, the dimension of the space $M_2(\Gamma_0(N))$ is bounded by $g_0(N) + c_0(N)$. In fact, it is equal to $g_0(N) + c_0(N) - 1$*

The fact that the dimension is one less than $g_0(N) + c_0(N)$ follows from the fact that $M_2(SL_2(\mathbb{Z})) = 0$ (or $H^1(X_0(1), \Omega_{X_0(1)/\mathbb{C}}(-C)) = 0$, because a punctured sphere is still simply connected).

By the de Rham theorem, we have the following duality given by integration:

Proposition 2.11. *The space $H_1(X_0(N),\mathbb{C})$ and $H_1(\overline{Y}_0(N), \partial\overline{Y}_0(N); \mathbb{C})$ are dual to $H^1(X_0(N),\mathbb{C}) \cong S_2(\Gamma_0(N)) \oplus \overline{S}_2(\Gamma_0(N))$, and $H_1(\overline{Y}_0(N),\mathbb{C})$ is dual to $M_2(\Gamma_0(N)) \oplus \overline{S}_2(\Gamma_0(N))$.*

2.3. **Hecke operators.** Let $GL_2^+(\mathbb{R}) = \left\{\alpha \in GL_2(\mathbb{R}) \middle| \det(\alpha) > 0\right\}$ and put $GL_2^+(A) = GL_2^+(\mathbb{R}) \cap GL_2(A)$ for $A \subset \mathbb{R}$. For $\alpha = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL_2^+(\mathbb{R})$ and a function $f : \mathfrak{H} \to \mathbb{C}$, we define $f|\alpha(z) = \det(\alpha) f(\alpha(z))(cz+d)^{-2}$.

Exercise 2.12. *Prove $(f|\alpha)|\beta = f|(\alpha\beta)$ for $\alpha, \beta \in GL_2^+(\mathbb{R})$.*

Then $f \in S_2(\Gamma_0(N))$ (resp. $f \in M_2(\Gamma_0(N))$) if and only if f vanishes (resp. finite) at all cusps of $X_0(N)$ and $f|\gamma = f$ for all $\gamma \in \Gamma_0(N)$. Let $\Gamma = \Gamma_0(N)$. For $\alpha \in GL_2(\mathbb{R})$ with $\det(\alpha) > 0$, if $\Gamma\alpha\Gamma$ can be decomposed into a disjoint union of finite left cosets $\Gamma\alpha\Gamma = \bigsqcup_{j=1}^h \Gamma\alpha_j$, we can think of the finite sum $g = \sum_j f|\alpha_j$. If $\gamma \in \Gamma$, then $\alpha_j\gamma \in \Gamma\alpha_{\sigma(j)}$ for a unique index $1 \le \sigma(j) \le h$ and σ is a permutation of $1, 2, \ldots, h$. If further, $f|\gamma = f$ for all $\gamma \in \Gamma$, we have

$$g|\gamma = \sum_j f|\alpha_j\gamma = \sum_j f|\gamma_j\alpha_{\sigma(j)} = \sum_j (f|\gamma_j)|\alpha_{\sigma(j)} = \sum_j f|\alpha_{\sigma(j)} = g.$$

Thus under the condition that $f|\gamma = f$ for all $\gamma \in \Gamma$, $f \mapsto g$ is a linear operator only dependent on the double coset $\Gamma\alpha\Gamma$; so, we write $g = f|[\Gamma\alpha\Gamma]$. More generally, if we have a set $T \subset GL_2^+(\mathbb{R})$ such that $\Gamma T\Gamma = T$ with finite $|\Gamma\backslash T|$, we can define the operator $[T]$ by $f \mapsto \sum_j f|T_j$ if $T = \bigsqcup_j \Gamma t_j$. We define

$$\Delta_0(N) = \left\{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in M_2(\mathbb{Z}) \cap GL_2^+(\mathbb{R}) \middle| c \equiv 0 \mod N,\ a\mathbb{Z} + N\mathbb{Z} = \mathbb{Z}\right\}.$$

Exercise 2.13. *Prove that $\Gamma\Delta_0(N)\Gamma = \Delta_0(N)$ for $\Gamma = \Gamma_0(N)$.*

Lemma 2.14. *Let* $\Gamma = \Gamma_0(N)$.

(1) *If* $\alpha \in M_2(\mathbb{Z})$ *with positive determinant,* $|\Gamma\backslash(\Gamma\alpha\Gamma)| < \infty$*;*

(2) *If* p *is a prime,*

$$\Gamma \left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right) \Gamma = \left\{\alpha \in \Delta_0(N) \middle| \det(\alpha) = p\right\} = \begin{cases} \Gamma \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) \sqcup \bigsqcup_{j=0}^{p-1} \Gamma \left(\begin{smallmatrix} 1 & j \\ 0 & p \end{smallmatrix}\right) & \text{if } p \nmid N, \\ \bigsqcup_{j=0}^{p-1} \Gamma \left(\begin{smallmatrix} 1 & j \\ 0 & p \end{smallmatrix}\right) & \text{if } p|N. \end{cases}$$

(3) *for an integer* $n > 0$,

$$\begin{aligned} T_n &:= \left\{\alpha \in \Delta_0(N) \middle| \det(\alpha) = n\right\} \\ &= \bigsqcup_a \bigsqcup_{b=0}^{d-1} \Gamma_0(N) \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \quad (a > 0,\ ad = n,\ (a, N) = 1,\ a, b, d \in \mathbb{Z}), \end{aligned}$$

(4) *Write* $T(n)$ *for the operator corresponding to* T_n*. Then we get the following identity of Hecke operators for* $f \in M_2(\Gamma_0(N))$*:*

$$a(m, f|T(n)) = \sum_{0<d|(m,n),(d,N)=1} d \cdot a(\frac{mn}{d^2}, f).$$

(5) $T(m)T(n) = T(n)T(m)$ *for all integers* m *and* n.

Proof. Note that (1) and (2) are particular cases of (3). We only prove (2), (4) when $n = p$ for a prime p and (5), leaving the other cases as an exercise (see [IAT] Proposition 3.36 and and (3.5.10) for a detailed proof of (3) and (4)).

We first deal with (2). Since the argument in each case is essentially the same, we only deal with the case where $p \nmid N$ and $\Gamma = \Gamma_0(N)$. Take any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in M_2(\mathbb{Z})$ and $ad - bc = p$. If c is divisible by p, then ad is divisible by p; so, one of a and d has a factor p. We then have

$$\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \begin{pmatrix} a/p & b \\ c/p & d \end{pmatrix} \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma_0(N) \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)$$

if a is divisible by p. If d is divisible by p and a is prime to p, choosing an integer j with $0 \le j \le p-1$ with $ja \equiv b \mod p$, we have $\gamma \left(\begin{smallmatrix} 1 & j \\ 0 & p \end{smallmatrix}\right)^{-1} \in GL_2(\mathbb{Z})$. If c is not divisible by p but a s divisible by p, we can interchange a and c via multiplication by $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ from the left-side. If a and c are not divisible by p, choosing an integer j so that $ja \equiv -c \mod p$, we find that the lower left corner of $\left(\begin{smallmatrix} 1 & 0 \\ j & 1 \end{smallmatrix}\right) \gamma$ is equal to $ja + c$ and is divisible by p. This finishes the proof of (2).

We now deal with (4) assuming $n = p$. By (2), we have

$$f|T(p)(z) = \begin{cases} p \cdot f(pz) + \sum_{j=0}^{p-1} f\left(\frac{z+j}{p}\right) & \text{if } p \nmid N, \\ \sum_{j=0}^{p-1} f\left(\frac{z+j}{p}\right) & \text{if } p|N. \end{cases} \tag{2.1}$$

Writing $f = \sum_{n=1}^{\infty} a(n, f)q^n$ for $q = \mathbf{e}(z)$, we find

$$a(m, f|T(p)) = a(mp, f) + p \cdot a(\frac{m}{p}, f).$$

Here we put $a(r, f) = 0$ unless r is a non-negative integer.

The formula of Lemma 2.14 (4) is symmetric with respect to m and n; so, we conclude $T(m)T(n) = T(n)T(m)$. This proves (5). □

Exercise 2.15. *Give a detailed proof of the above lemma.*

The following exercise is more difficult:

Exercise 2.16. *Let* $\Gamma = SL_2(\mathbb{Z})$. *Prove that* $|\Gamma\backslash(\Gamma\alpha\Gamma)| < \infty$ *for* $\alpha \in GL_2(\mathbb{R})$ *if and only if* $\alpha \in M_2(\mathbb{Q})$ *modulo real scalar matrices.*

Write $\pi : \mathfrak{H} \to Y_0(N) = \Gamma\backslash\mathfrak{H}$ for the quotient map.

Lemma 2.17. *If* $\Gamma\alpha\Gamma = \bigsqcup_{j=1}^{h} \Gamma\alpha_j$ *for* $\Gamma = \Gamma_0(N)$ *and* $\alpha \in GL_2^+(\mathbb{R})$, *then for a chain* $c \in C_1(\mathfrak{H}; A)$ *with* $\partial(\pi(c)) = \pi(\partial c) = 0$, $\partial(\pi(\sum_{j=1}^{h} \alpha_j(c))) = 0$.

Proof. If $\pi(\partial c)$ in $Y_0(N)$ vanishes, writing $\partial c = \sum_z a_z[z]$ for points z in $\mathfrak{H}$, we may assume that $a_z + a_{\gamma(z)} = 0$ for some $\gamma \in \Gamma$. If $\Gamma\alpha\Gamma = \bigsqcup_{j=1}^{h} \Gamma\alpha_j$, then $\alpha_j\gamma = \gamma_j\alpha_{\sigma(j)}$ and $\sum_j (a_{\alpha_j(z)} + a_{\alpha_j\gamma(z)}) = \sum_j (a_{\alpha_j(z)} + a_{\gamma_j\alpha_{\sigma(j)}(z)}) = 0$. This shows that $\pi(\partial(\sum_j \alpha_j(c))) = 0$, which finishes the proof. □

Obviously, for any 2-chain c, $\pi(\sum_j \partial\alpha_j(c)) = \pi(\sum_j \alpha_j(\partial(c)))$, and therefore, the operator $c \mapsto \sum_j \alpha_j(c)$ preserves boundaries and cycles. In this way, the Hecke operator $[\Gamma\alpha\Gamma]$ acts on $H_1(Y_0(N), A)$, and hence on $H^1(Y_0(N), A)$ by the definition of cohomology group. On $H^1_{DR}(Y_0(N), \mathbb{C})$, the action of $[\Gamma\alpha\Gamma]$ is given by $[\omega] \mapsto [\sum_{j=1}^{h} \alpha_j^*\omega]$. Since $z \mapsto \alpha_j(z)$ takes cusps to cusps, we can show similarly that Hecke operators act on $H_1(X_0(N), A)$ and $H_1(\overline{Y}_0(N), \partial\overline{Y}_0(N), A)$. Since we can verify $\alpha^* f(z)dz = (f|\alpha)dz$ by the chain rule, the Eichler-Shimura isomorphism $S_2(\Gamma_0(N)) \oplus \overline{S}_2(\Gamma_0(N)) \cong H^1(X_0(N), \mathbb{C})$ is equivariant under Hecke operators.

2.4. **Duality.** Let $A \subset \mathbb{C}$ be a subring, and define

$$S_2(\Gamma_0(N), A) = \left\{ f \in S_2(\Gamma_0(N)) \middle| a(n, f) \in A \right\}.$$

By definition, $S_2(\Gamma_0(N), \mathbb{C}) = S_2(\Gamma_0(N))$. By Lemma 2.14, $T(n)$ preserves the A-submodule $S_2(\Gamma_0(N), A)$ of $S_2(\Gamma_0(N))$. Define

$$\begin{aligned} h(N, A) &= A[T(n)|n = 1, 2, \ldots] \subset \mathrm{End}_A(S_2(\Gamma_0(N), A)), \\ H(N, A) &= A[T(n)|n = 1, 2, \ldots] \subset \mathrm{End}_A(M_2(\Gamma_0(N), A)) \end{aligned} \tag{2.2}$$

and call $h(N, A)$ the Hecke algebra on $\Gamma_0(N)$. By Lemma 2.14 (5), $h(N, A)$ is a commutative A-algebra.

We define an A–bilinear pairing

$$\langle\ ,\ \rangle : h(N, A) \times S_2(\Gamma_0(N), A) \to A$$

by $\langle h, f\rangle = a(1, f|h)$.

Proposition 2.18. (1) *We have the following canonical isomorphism:*

$$\mathrm{Hom}_A(S_2(\Gamma_0(N),A),A) \cong h(N,A) \quad \textit{and} \quad \mathrm{Hom}_A(h(N,A),A) \cong S_2(\Gamma_0(N),A),$$

and the latter is given by sending an A–linear form $\phi : h(N,A) \to A$ to the q–expansion $\sum_{n=1}^{\infty} \phi(T(n))q^n$.

(2) (Shimura) *We have*

$$S_2(\Gamma_0(N),A) = S_2(\Gamma_0(N),\mathbb{Z}) \otimes A \quad \textit{and} \quad h(N,A) = h(N,\mathbb{Z}) \otimes A.$$

Proof. We start with proving the result for a subfield A of $\mathbb{C}$. Since $h(N,A)$ and $S_2(\Gamma_0(N),A)$ are both finite dimensional, we only need to show the non-degeneracy of the pairing. By Lemma 2.14 (4), we find $\langle T(n),f\rangle = a(n,f)$; so, if $\langle h,f\rangle = 0$ for all n, we find $f = 0$. If $\langle h,f\rangle = 0$ for all f, we find

$$0 = \langle h, f|T(n)\rangle = a(1,f|T(n)h) = a(1,f|hT(n)) = \langle T(n), f|h\rangle = a(n,f|h).$$

Thus $f|h = 0$ for all f, implying $h = 0$ as an operator.

We have the Poincaré duality pairing $(\cdot,\cdot) : H^1(X_0(N),A) \times H^1(X_0(N),A) \to A$ which is a perfect pairing. Define $\Theta : H^1(X_0(N),A) \otimes_A H^1(X_0(N),A) \to S_2(\Gamma_0(N),A)$ by $\Theta(\xi \otimes \eta) = \sum_{n=1}^{\infty}(\xi,\eta|T(n))q^n$. Indeed, $h \mapsto (\xi,\eta|h)$ is an A-linear form on $h(N,A)$, and by the result already proven, we have $\sum_{n=1}^{\infty}(\xi,\eta|T(n))q^n \in S_2(\Gamma_0(N),A)$ if A is a field. By Proposition 2.9, Θ is surjective if $A = \mathbb{C}$. Indeed, by the self-duality of $H^1(X_0(N),\mathbb{C})$, the projection $H^1(X_0(N),\mathbb{C}) \twoheadrightarrow S_2(\Gamma_0(N))$ induces a $T(n)$-equivariant inclusion $h(N,\mathbb{C}) = \mathrm{Hom}_{\mathbb{C}}(S_2(\Gamma_0(N),\mathbb{C}) \hookrightarrow H^1(X_0(N),\mathbb{C})$, and thus any linear form on $h(N,\mathbb{C})$ is a linear combination of $h \mapsto \langle \xi|h,\eta\rangle$. This is equivalent to the surjectivity of Θ over $\mathbb{C}$.

Since $H^1(X_0(N),\mathbb{Z}) \otimes A = H^1(X_0(N),A)$, the image under Θ of $H^1(X_0(N),\mathbb{Z}) \otimes H^1(X_0(N),\mathbb{Z})$ spans $S_2(\Gamma_0(N),\mathbb{C})$. Thus $S_2(\Gamma_0(N),\mathbb{Z})$ span $S_2(\Gamma_0(N),\mathbb{C})$. This shows

$$S_2(\Gamma_0(N),\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} = S_2(\Gamma_0(N),\mathbb{C}),$$

and therefore

$$S_2(\Gamma_0(N),\mathbb{Z}) \otimes_{\mathbb{Z}} A = S_2(\Gamma_0(N),A)$$

for any ring A. In particular, $h(N,A)$ is a subalgebra of $\mathrm{End}_{\mathbb{C}}(S_2(\Gamma_0(N)))$ generated over A by $T(n)$ for all n. Then by definition $h(N,A) = h(N,\mathbb{Z}) \otimes_{\mathbb{Z}} A$ for any subring $A \subset \mathbb{C}$.

As for $A = \mathbb{Z}$, we only need to show that $\phi \mapsto \sum_{n=1}^{\infty} \phi(T(n))q^n$ is well defined and is surjective onto $S_2(\Gamma_0(N),\mathbb{Z})$ from $h(N,\mathbb{Z})$, because this is the case if we extend scalar to $A = \mathbb{Q}$. The cusp form $f \in S_2(\Gamma_0(N),A)$ corresponding to ϕ satisfies $\langle h,f\rangle = \phi(h)$; so, $a(n,f) = \langle T(n),f\rangle = \phi(T(n))$. Thus $f = \sum_{n=1}^{\infty} \phi(T(n))q^n \in S_2(\Gamma_0(N),A)$. However

$$f \in S_2(\Gamma_0(N),\mathbb{Z}) \iff \phi \in \mathrm{Hom}(h(N,\mathbb{Z}),\mathbb{Z}),$$

because $h(N,\mathbb{Z})$ is generated by $T(n)$ over $\mathbb{Z}$. This is enough to conclude surjectivity.

Since $h(N,A) = h(N,\mathbb{Z}) \otimes A$ and $S_2(\Gamma_0(N),\mathbb{Z}) \otimes_{\mathbb{Z}} A = S_2(\Gamma_0(N),A)$, the duality over $\mathbb{Z}$ implies that over A. □

Corollary 2.19. *We have the following assertions.*

(1) *For any* $\mathbb{C}$*-algebra homomorphism* $\lambda : h(N,\mathbb{C}) \to \mathbb{C}$*,* $\lambda(h(N,\mathbb{Z}))$ *is in the integer ring of an algebraic number field. In other words,* $\lambda(T(n))$ *for all* n *generates an algebraic number field* $\mathbb{Q}(\lambda)$ *over* $\mathbb{Q}$ *and* $\lambda(T(n))$ *is an algebraic integer.*

(2) *For any* $\mathbb{Z}$*-algebra homomorphism* $\lambda : h(N,\mathbb{Z}) \to \mathbb{Q}(\lambda)$*,*

$$S_2(\Gamma_0(N),\mathbb{Q}(\lambda))[\lambda] = \big\{f \in S_2(\Gamma_0(N),\mathbb{Q}(\lambda)\big| f|T(n) = \lambda(T(n))f \text{ for all } n\big\}$$

is one dimensional and is generated by $\sum_{n=1}^{\infty} \lambda(T(n))q^n$*.*

(3) *For any* $\mathbb{Z}$*-algebra homomorphism* $\lambda : h(N,\mathbb{Z}) \to \mathbb{Q}(\lambda)$*,*

$$H^1(X_0(N),\mathbb{Q}(\lambda))[\lambda] = \big\{c \in H^1(X_0(N),\mathbb{Q}(\lambda)\big| c|T(n) = \lambda(T(n))c \text{ for all } n\big\}$$

is two dimensional, and is isomorphic to

$$S_2(\Gamma_0(N),\mathbb{Q}(\lambda))[\lambda] \oplus \overline{S_2(\Gamma_0(N),\mathbb{Q}(\overline{\lambda}))[\overline{\lambda}]}.$$

Proof. Since $h(N,\mathbb{Z})$ is of finite rank over $\mathbb{Z}$, $R = \lambda(h(N,\mathbb{Z}))$ has finite rank d over $\mathbb{Z}$. Then the characteristic polynomial $P(X)$ of multiplication by $r \in R$ (regarding $R \cong \mathbb{Z}^d$) is satisfied by r, that is, $P(r) = 0$. Since $P(X) \in \mathbb{Z}[X]$, r is an algebraic integer. Then $R \otimes_{\mathbb{Z}} \mathbb{Q}$ is a finite extension $\mathbb{Q}(\lambda)$ of degree d over $\mathbb{Q}$.

Let K be a field. For any finite dimensional commutative K-algebra A, a K-algebra homomorphism $\lambda : A \to K$ gives rise to a generator of λ-eigenspace of the linear dual $\mathrm{Hom}_K(A,K)$. Applying this fact to $\mathrm{Hom}_K(h(N,K),K) = S_2(\Gamma_0(N),K)$ for $K = \mathbb{Q}(\lambda)$, we get the second assertion.

The third assertion then follows from Proposition 2.9. □

Let $\varepsilon = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Define $\varepsilon(z) = -\overline{z}$. Then ε takes $\mathfrak{H}$ to $\mathfrak{H}$. Define the action of $GL_2^+(\mathbb{R})\varepsilon$ on $\mathfrak{H}$ by $\gamma\varepsilon(z) = \gamma(-\overline{z})$. Since $GL_2(\mathbb{R}) = GL_2^+(\mathbb{R}) \sqcup \varepsilon GL_2^+(\mathbb{R})$, we have well defined map $\gamma : \mathfrak{H} \to \mathfrak{H}$ for all $\gamma \in GL_2(\mathbb{R})$.

Exercise 2.20. *Prove the following fact.*

(1) *The above action is an action of the group* $GL_2(\mathbb{R})$ *on* $\mathfrak{H}$*. In other words,* $\alpha(\beta(z)) = (\alpha\beta)(z)$ *for* $\alpha, \beta \in GL_2(\mathbb{R})$*.*

(2) *We have* $\mathfrak{H} \cong GL_2(\mathbb{R})/\mathbb{R}^\times O_2(\mathbb{R})$ *by* $g \mapsto g(\sqrt{-1})$*.*

Since $\varepsilon^2 = 1$ and $\varepsilon T_n \varepsilon^{-1} = T_n$, the action of ε commutes with Hecke operators. Thus if A is a ring in which 2 is invertible, we have a decomposition $H_1(X_0(N),A)$ and $H^1(X_0(N),A)$ into the direct sum of ± 1 eigenspaces of ε. We write $H_1^{\pm}$ or $H^1_{\pm}$ for the eigenspace.

Proposition 2.21. *Let* $A \subset \mathbb{C}$ *be a PID. If* $\lambda : h(N,\mathbb{Z}) \to A$ *be an algebra homomorphism. Then the cohomology and homology groups* $H_1^{\pm}(X_0(N),A)[\lambda]$*,* $H^1_{\pm}(X_0(N),A)[\lambda]$ *and* $H_1^{\pm}(\overline{Y}_0(N),\partial\overline{Y}_0(N),A)[\lambda]$ *are free of rink 1 over* A*.*

Proof. Since $\varepsilon^* f = f(\varepsilon(z)) \in \overline{S}_2(\Gamma_0(N),\mathbb{C})$ if $f \in S_2(\Gamma_0(N),\mathbb{C})$, $\pi_{\pm}(f) = \frac{f \pm \varepsilon^* f}{2} \neq 0$ if $f \neq 0$. Moreover $\pi_{\pm}$ induces an isomorphism of $S_2(\Gamma_0(N),\mathbb{C})[\lambda]$ onto $(S_2(\Gamma_0(N),\mathbb{C}) \oplus \overline{S}_2(\Gamma_0(N),\mathbb{C}))^{\pm}[\lambda]$, where the superscript "$\pm$" indicates the $\pm$ eigenspace for ε. Since $S_2(\Gamma_0(N),\mathbb{C})[\lambda]$ is one dimensional, we conclude that $(S_2(\Gamma_0(N),\mathbb{C}) \oplus \overline{S}_2(\Gamma_0(N),\mathbb{C}))^{\pm}[\lambda]$

is one dimensional. By Proposition 2.9, we have $(S_2(\Gamma_0(N),\mathbb{C})\oplus\overline{S}_2(\Gamma_0(N),\mathbb{C}))^{\pm}[\lambda]\cong H^1_{\pm}(X_0(N),\mathbb{C})[\lambda]$, which is therefore one dimensional. Then by the Poincaré duality, $H_1^{\pm}(X_0(N),\mathbb{C})[\lambda]$ is one dimensional.

By Proposition 2.11, the same argument tell us that $H_1^{\pm}(\overline{Y}_0(N),\partial\overline{Y}_0(N),\mathbb{C})[\lambda]$ is one dimensional.

As long as A contains the eigenvalues $\lambda(T(n))$, we have $H_1^{\pm}(X_0(N),A)[\lambda]\otimes_A\mathbb{C}\cong H_1^{\pm}(X_0(N),\mathbb{C})[\lambda]$. Since A is PID, $H_1^{\pm}(X_0(N),A)[\lambda]$ is A-free. Then by the above identity, we conclude that the rank of $H_1^{\pm}(X_0(N),A)[\lambda]$ is equal to 1. The same argument prove the assertion for other homology and cohomology groups. □

2.5. **Modular Hecke L-functions.** Let $\lambda: h(N,\mathbb{Z})\to\mathbb{C}$ be an algebra homomorphism. Then we define $L(s,\lambda)=\sum_{n=1}^{\infty}\lambda(T(n))n^{-s}$, which is the modular Hecke L-function of λ. We now prove that this Dirichlet series converges absolutely if $\mathrm{Re}(s)>2$. We start with

Lemma 2.22. *If $f\in S_2(\Gamma_0(N))$, then $|f(x+iy)|\le Cy^{-1}$ for a constant independent of x and y.*

Proof. Let $\alpha=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in GL_2^+(\mathbb{R})$. By definition,

$$\alpha\left(\begin{smallmatrix}z&\overline{z}\\1&1\end{smallmatrix}\right)=\left(\begin{smallmatrix}\alpha(z)&\alpha(\overline{z})\\1&1\end{smallmatrix}\right)\left(\begin{smallmatrix}cz+d&0\\0&c\overline{z}+d\end{smallmatrix}\right).$$

Taking the determinant of this, we get $\det(\alpha)\,\mathrm{Im}(z)=\mathrm{Im}(\alpha(z))|cz+d|^2$. Thus $g(z)=|f(z)\,\mathrm{Im}(z)|$ factors through $X_0(N)$. Since $f(z)$ vanishes at cusps, $g(z)$ also; so, $g(z)$ is a continuous function on the compact space $X_0(N)$. Thus the positive function $g(z)$ is bounded by a positive constant C: $|g(z)|\le C$, which proves the lemma. □

Lemma 2.23. *There exists a constant $B>0$ such that $|\lambda(T(n))|\le B\cdot n$ for all integers $n>0$.*

Proof. Since $f=f_\lambda=\sum_{n=1}^{\infty}\lambda(T(n))q^n$ is a cusp form in $S_2(\Gamma_0(N))$ with $f_\lambda|T(n)=\lambda(T(n))f_\lambda$, picking any $f\in S_2(\Gamma_0(N))$, we need to prove $|a(n,f)|\le Bn$ for all n. Since $a(n,f)=(2\pi i)^{-1}\int_{|q|=r}f(q)q^{-n-1}dq=\int_0^1 f(z)\exp(-2\pi inz)dz$ for any $r>0$ by the residue formula (and it is independent of $y=\mathrm{Im}(z)$), taking $r=\exp(-1/n)$ ($\Leftrightarrow \mathrm{Im}(z)=\frac{1}{2\pi n}$, by Lemma 2.22, we get $|a(n,f)|\le 2Ce\pi n$. Thus $B=2Ce\pi$. □

Exercise 2.24. *Prove, by a standard argument, $L(s,\lambda)$ converges absolutely if* $\mathrm{Re}(s)>2$.

Exercise 2.25. *Let $\tau=\left(\begin{smallmatrix}0&-1\\N&0\end{smallmatrix}\right)$. Prove the following facts:*

(1) $\tau\Gamma_0(N)\tau^{-1}=\Gamma_0(N)$.
(2) *If $f\in S_2(\Gamma_0(N))$, then $g=f|\tau\in S_2(\Gamma_0(N))$.*

Lemma 2.26. *If $f\in S_2(\Gamma_0(N)$, then $L(s,f)=\sum_{n=1}^{\infty}a(n,f)n^{-s}$ converges absolutely if* $\mathrm{Re}(s)>2$ *and can be continuated analytically to a holomorphic function on the whole complex plane.*

Proof. By the same computation as in (1.3), we have

$$\int_0^\infty f(iy)y^{s-1}dy = (2\pi)^{-s}\Gamma(s)L(s,f)$$

if $\mathrm{Re}(s) > 2$. However the integral $\int_0^\infty f(iy)y^{s-1}dy$ is convergent for all $s \in \mathbb{C}$, because for any power y^s, $\lim_{y\to\infty} f(iy)y^{s-1} = \lim_{y\to 0} f(iy)y^{s-1} = 0$ (by the q-expansion). This gives us the analytic continuation of $L(s,f)$ to the whole complex plane. □

By the above expression, we have

$$L(1,\lambda) = -\int_0^\infty (2\pi i)f_\lambda(z)dz \tag{2.3}$$

Since $\varepsilon^*(f(z)dz) = -f(-\overline{z})d\overline{z}$, we have

$$L(1,\lambda) = -\int_0^\infty \varepsilon^*((2\pi i)f_\lambda(z)dz) \tag{2.4}$$

2.6. **Rationality of Hecke L-values.** Let $A \subset \mathbb{C}$ be a PID and $\lambda : h(N,\mathbb{Z}) \to A$ be an algebra homomorphism. Define $\omega_\pm(\lambda) = \frac{1}{2}((2\pi i)f_\lambda(z)dz \pm \varepsilon^*((2\pi i)f_\lambda(z)dz))$ and $\omega_\pm(f) = \frac{1}{2}((2\pi i)f(z)dz \pm \varepsilon^*((2\pi i)f(z)dz))$ for $f \in S_2(\Gamma_0(N))$. Let $\delta_\pm(\lambda)$ be a generator of $H^1_\pm(X_0(N),A)[\lambda]$ over A; so,

$$H^1_\pm(X_0(N),A)[\lambda] = A\delta_\pm(\lambda).$$

Then by Proposition 2.21, we have $[\omega_\pm(\lambda)] = \Omega_\pm(\lambda;A)\delta_\pm(\lambda)$. We call $\Omega_\pm(\lambda;A)$ the $\pm$ period of λ. Let $\gamma_a \in H_1(\overline{Y}_0(N),\partial\overline{Y}_0(N),\mathbb{Z})$ for $a \in \mathbb{Q}$ be the relative 1-cycle represented by vertical line in $\mathfrak{H}$ passing through $a \in \mathbb{Q}$.

Lemma 2.27. *We have $\frac{L(1,\lambda)}{\Omega_+(\lambda;A)} \in M^{-1}A$ for a positive integer M only dependent on N. Moreover $\int_{\gamma_a}\delta_\pm(\lambda) \in M^{-1}A$ for all $a \in \mathbb{Q}$.*

Proof. We have a natural map $\iota : H_1(X_0(N),A) \to H_1(\overline{Y}_0(N),\partial\overline{Y}_0(N),A)$, and after tensoring $\mathbb{C}$, ι becomes an isomorphism by Proposition 2.11, $\mathrm{Coker}(\iota)$ is finite of order M. Since the vertical line γ_a passing through $a \in \mathbb{Q}$ is a 1-cycle in $H_1(\overline{Y}_0(N),\partial\overline{Y}_0(N),A)$, we have $M\gamma_a \in H_1(X_0(N),A)$. Thus by (2.3) and (2.4), we get

$$\frac{M\int_{\gamma_0}\omega_+(\lambda)}{\Omega_+(\lambda;A)} = \int_{M\gamma_0}\delta_+(\lambda) \in A.$$

The same argument also applies to γ_a. This finishes the proof. □

Since $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)(z) = z+1$, $\gamma_a = \gamma_{a+1}$. Thus the cycle γ_a only depends $a \in \mathbb{Q}/\mathbb{Z}$. Let $\chi : (\mathbb{Z}/m\mathbb{Z}) \to A$ be a primitive Dirichlet character. Consider $\gamma(\chi) = \sum_{u \bmod m}\chi^{-1}(u)\gamma_{\frac{u}{m}} \in H_1(\overline{Y}_0(N),\partial\overline{Y}_0(N),A)$.

Lemma 2.28. *We have $\varepsilon(\gamma(\chi)) = \chi(-1)\gamma(\chi)$.*

Proof. Note that $\varepsilon(\gamma_a) = \gamma_{-a}$, and from this, we have

$$\varepsilon(\gamma(\chi)) = \sum_{u \mod m} \chi^{-1}(u)\varepsilon(\gamma_{u/m}) = \sum_{u \mod m} \chi^{-1}(u)(\gamma_{-u/m}) \overset{u \mapsto -u}{=} \chi(-1)\gamma(\chi).$$

□

We consider $f|R_\chi(z) = \sum_{u \mod m} \chi^{-1}(u) f(z + \frac{u}{m})$. The following exercise is a bit difficult.

Exercise 2.29. *Let N' be the LCM of N and m^2. Prove $(f|R_\chi)|\gamma = \chi^2(a) f|R_\chi$ for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N')$.*

Note that

$$\int_{\gamma(\chi)} \omega_\pm(\lambda) = \int_{\gamma_0} \omega_\pm(f|R_\chi).$$

Then we have

$$\begin{aligned} f|R_\chi(z) &= \sum_{u \mod m} \chi^{-1}(u) f(z + \frac{u}{m}) = \sum_{n=1}^{\infty} a(n,f) q^n \sum_u \chi^{-1}(u) \exp(\frac{2\pi i n u}{m}) \\ &= G(\chi^{-1}) \sum_{n=1}^{\infty} a(n,f)\chi(n) q^n. \end{aligned} \tag{2.5}$$

Then by the same argument proving Lemma 2.27 applied to $\gamma(\chi)$, we get

Proposition 2.30. *Let χ be a primitive Dirichlet character modulo m with values in a PID $A \subset \mathbb{C}$. Then there exists a constant M only depending on N such that $\frac{G(\chi^{-1})L(1,\lambda\otimes\chi)}{\Omega_{\chi(-1)}(\lambda;A)} \in M^{-1}A$, where $L(s,\lambda\otimes\chi) = \sum_{n=1}^{\infty} \chi(n)\lambda(T(n))n^{-s}$.*

2.7. **p-Old and p-new forms.** Let N be a positive integer prime to p.

Exercise 2.31. *If $f \in S_2(\Gamma_0(N))$, prove that $f(pz) \in S_2(\Gamma_0(Np))$.*

We consider an algebra homomorphism $\lambda : h(N,\mathbb{Z}) \to \overline{\mathbb{Q}}$. Then we have a Hecke eigenform $f = \sum_{n=1}^{\infty} \lambda(T(n))q^n \in S_2(\Gamma_0(N))$ with $f|T(n) = \lambda(T(n))f$. The L-function for a Dirichlet character χ modulo M $L(s,\lambda\otimes\chi) = \sum_{n=1}^{\infty} \lambda(T(n))\chi(n)n^{-s}$ has the following Euler product:

$$\prod_\ell \frac{1}{(1-\lambda(T(\ell))\chi(\ell)\ell^{-s} + \chi(\ell)^2\ell^{1-2s})} = \prod_\ell \left[(1 - \frac{\alpha_\ell\chi(\ell)}{\ell^s})(1 - \frac{\beta_\ell\chi(\ell)}{\ell^s})\right]^{-1},$$

where α_ℓ and β_ℓ are two roots of $X^2 - \lambda(T(\ell))X + \ell = 0$.

Exercise 2.32.

1. *Prove $T(m)T(n) = \sum_{0<d|(m,n),(d,N)=1} d \cdot T(mn/d^2)$ by* Lemma 2.14 (4).
2. *Prove the above Euler factorization of $L(s,\lambda\otimes\chi)$.*

Lemma 2.33. *Let $\alpha = \alpha_p$ and $\beta = \beta_p$, and put $f_\alpha(z) = f(z) - \beta f(pz)$. Then we have $f_\alpha \in S_2(\Gamma_0(Np))$ and $f_\alpha|U(p) = \alpha f_\alpha$ and $f_\alpha|T(n) = \lambda(T(n))f_\alpha$ for all $n > 0$ prime to p, where $U(p)$ is the Hecke operator $T(p)$ acting on $S_2(\Gamma_0(Np))$.*

Proof. By Lemma 2.14 (4), we have

$$a(m,f|T(n)) = \sum_{0<d|(m,n),\ (d,N)=1} d\cdot a(\frac{mn}{d^2},f).$$

From this, it is easy to see that $T(m)T(n) = T(mn)$ if $m\mathbb{Z}+n\mathbb{Z} = \mathbb{Z}$. Note that $a(n,f) = \lambda(T(n))$ and hence $a(mn,f) = a(m,f)a(n,f)$ if $m\mathbb{Z}+n\mathbb{Z} = \mathbb{Z}$. Since $f_\alpha = \sum_{m=1}^{\infty} a(m,f)q^m - \beta\sum_{m=1}^{\infty} a(m,f)q^{mp}$, we see $a(m,f_\alpha) = a(m,f)$ if $p\nmid m$. In particular, if $p\nmid m$ and $p\nmid n$, we have

$$\begin{aligned}a(m,f_\alpha|T(n)) &= \sum_{0<d|(m,n),\ (d,Np)=1} d\cdot a(\frac{mn}{d^2},f_\alpha)\\ &= \sum_{0<d|(m,n),\ (d,N)=1} d\cdot a(\frac{mn}{d^2},f) = \lambda(T(n))a(m,f) = \lambda(T(n))a(m,f_\alpha).\end{aligned}$$

If $m = m_0p$, we have $a(m,f_\alpha) = a(m,f) - \beta\cdot a(m_0,f)$. Thus if $p|m$ and $p\nmid n$, we have

$$\begin{aligned}a(m,f_\alpha|T(n)) &= \sum_{0<d|(m,n),\ (d,Np)=1} d\cdot a(\frac{mn}{d^2},f_\alpha)\\ &= \sum_{0<d|(m,n),\ (d,N)=1} d\cdot (a(\frac{mn}{d^2},f) - \beta\cdot a(\frac{m_0n}{d^2},f))\\ &= \lambda(T(n))(a(m,f) - \beta\cdot a(m_0,f)) = \lambda(T(n))a(m,f_\alpha).\end{aligned}$$

This shows that $f_\alpha|T(n) = \lambda(T(n))f_\alpha$ for n prime to p.

Now we have $a(m,f_\alpha|U(p)) = a(mp,f_\alpha) = a(mp,f) - \beta\cdot a(m,f)$. On the other hand,

$$(\alpha+\beta)a(m,f) = a(m,f|T(p)) = p\cdot a(\frac{m}{p},f) + a(mp,f) = (\alpha\beta)\cdot a(\frac{m}{p},f) + a(mp,f).$$

This shows

$$a(m,f_\alpha|U(p)) = (\alpha+\beta)a(m,f) - (\alpha\beta)\cdot a(\frac{m}{p},f) - \beta\cdot a(m,f) = \alpha\cdot a(m,f_\alpha).$$

This finishes the proof. □

Corollary 2.34. *Let the notation be as in* Lemma 2.33. *If $p\nmid N$ and $\lambda : h(N,\mathbb{Z}) \to \overline{\mathbb{Q}}$ is an algebra homomorphism, we have an algebra homomorphism $\lambda_\alpha : h(pN,\mathbb{Z}) \to \overline{\mathbb{Q}}$ such that $\lambda_\alpha(U(p)) = \alpha$ and $\lambda_\alpha(T(n)) = \lambda(T(n))$ if $p\nmid n$. Moreover, we have $L(s,\lambda_\alpha\otimes\chi) = (1-\beta\chi(p)p^{-s})L(s,\lambda\otimes\chi)$.*

Exercise 2.35. *Give a detailed proof of the above corollary.*

2.8. **Elliptic modular p-adic measure.** Take an algebra homomorphism of the Hecke algebra $\lambda : h(pN,\mathbb{Z}) \to \overline{\mathbb{Q}}$. Then we have $f_\lambda = \sum_{n=1}^{\infty} \in S_2(\Gamma_0(Np))$ with $f_\lambda|T(n) = \lambda(T(n))f_\lambda$. We write the Hecke operator $T(p)$ on $S_2(\Gamma_0(Np))$ as $U(p)$; so, we have

$$f|U(p)(z) = \frac{1}{p}\sum_{j=0}^{p-1} f(\frac{z+j}{p}).$$

Thus the action of $U(p)$ is exactly the same as in the case of $\mathbb{G}_m$. We suppose $a = \alpha_p = \lambda(U(p))$ is a p-adic unit in $\mathbb{Q}_p(\lambda)$. Such λ and f_λ are called *p-ordinary.*

We have an A-linear map: $H^1(\overline{Y}_0(Np), \partial\overline{Y}_0(Np), A) \to A$ given by $\omega \mapsto \int_{\gamma_x} \omega$. Then we consider a map

$$c : p^{-\infty}\mathbb{Z} = \bigcup_{i=1}^{\infty} p^{-i}\mathbb{Z} \to \mathrm{Hom}_A(H^1(\overline{Y}_0(Np), \partial\overline{Y}_0(Np), A), K) \tag{2.6}$$

given by $c(x)(\omega) = \int_{\gamma_x} \omega$.

For $\omega \in H^1(\overline{Y}_0(Np), \partial\overline{Y}_0(Np), A)$, we write $c_\omega(r) = \int_{\gamma_r} \omega$. Then $c_\omega(r + 1) = c_\omega(r)$ by definition, and c_ω factors through $\mathbb{Q}_p/\mathbb{Z}_p = p^{-\infty}\mathbb{Z}/\mathbb{Z}$. Supposing $\omega|U(p) = a\omega$ with $|a|_p = 1$, we define a distribution φ_ω on $\mathbb{Z}_p$ by

$$\varphi_\omega(z + p^m\mathbb{Z}_p) = a^{-m} c_\omega(\frac{z}{p^m}) \text{ for } z = 1, 2, \ldots \text{ prime to } p. \tag{2.7}$$

This is well defined because $c_\omega(r+1) = c_\omega(r)$. We take the multiplicative group $G = \mathbb{Z}_p^\times$ and fix an isomorphism $G \cong \mu \times \mathbb{Z}_p$ for a finite group μ, where $\mathbb{Z}_p$ in the right-hand-side is an additive group. Then the multiplicative subgroup $G_i = 1 + p^i\mathbb{Z}_p$ corresponds to the additive group $p^i\mathbb{Z}_p$. To show that φ_ω actually gives a distribution, we need to check the distribution relation (1.8). We compute

$$\sum_{j=1}^{p} c_\omega(\frac{x+j}{p}) = \sum_j c(\frac{x+j}{p})(\omega) = c(x)(\omega|U(p)) = a \cdot c_\omega(x).$$

This shows

$$\sum_{j=1}^{p} \varphi_\omega(x + jp^m + p^{m+1}\mathbb{Z}_p) = \varphi_\omega(x + p^m\mathbb{Z}_p).$$

The general distribution relation (1.8) then follows from the iteration of this relation. By a similar argument, we see that

$$|\varphi_\omega(z + p^m\mathbb{Z}_p)|_p = |a^{-m} c_\omega(\frac{z}{p^m})|_p = |c(\frac{z}{p^m})(\omega)|_p \le |\omega|_p, \tag{2.8}$$

where $|\omega|_p = \mathrm{Sup}_x |c(x)(\omega)|_p$ with x running over $p^{-\infty}\mathbb{Z}$. Thus φ_ω is bounded (by the proof of Lemma 2.27) and, by Proposition 1.29, we have a unique measure φ_ω extending the distribution φ_ω. Now we compute $\int_G \phi d\varphi_\omega(x)$. To do this, we may assume that $|\omega|_p \le 1$ by multiplying by a constant if necessary. For $\phi \in C(G/G_m; A)$, we have

$$\int_G \phi d\varphi_\omega = a^{-m} \sum_{z=1}^{p^m} \phi(z) c_\omega(\frac{z}{p^m}).$$

Let $N > 1$ be a positive integer prime to p. We take $\omega = \delta_+(\lambda)$ for each algebra homomorphism $\lambda : h(Np, \mathbb{Z}) \to \overline{\mathbb{Q}}$. Then we write φ_ω as $\varphi = \varphi_\lambda$ and compute for any primitive character ϕ of $(\mathbb{Z}/p^r\mathbb{Z})^\times$ the integral $\int_G \phi d\varphi_\lambda$. Note that $\omega|U(p) = a\omega$. We write $\alpha_x : \overline{Y}_0(Np) \to \overline{Y}_0(Np)$ be then translation $\alpha_x(z) = z + x$ for $x \in \mathbb{R}$. We see that, if $\phi \neq 1$,

$$\begin{aligned}\int_G \phi d\varphi_\lambda =& a^{-r} \sum_{x \in (\mathbb{Z}/p^r\mathbb{Z})^\times} \phi(x) c(\frac{x}{p^r})(\omega) \\ =& a^{-r} \int_{\gamma_0} \sum_{x \in (\mathbb{Z}/p^r\mathbb{Z})^\times} \phi(x) \alpha^*_{x/p^r}\omega \\ =& a^{-r} \int_{\gamma_0} \omega | R_{\phi^{-1}} \\ =& a^{-r} \frac{G(\phi)L(1, \lambda \otimes \phi^{-1})}{\Omega_{\phi(-1)}(\lambda; A)}\end{aligned} \tag{2.9}$$

We have basically proved the following theorem of Mazur:

Theorem 2.36. *Let p be a prime and N be a positive integer prime to p. Let $A = \{x \in \mathbb{Q}(\lambda) \big| |x|_p \leq 1\}$ be the discrete valuation ring in $\mathbb{Q}(\lambda)$ for a p-adic valuation $|\cdot|_p$ of $\mathbb{Q}(\lambda)$. For each algebra homomorphism $\lambda : h(Np, \mathbb{Z}) \to A$ with $|\lambda(U(p))|_p = 1$, we have a unique p-adic measure φ_λ on $\mathbb{Z}_p^\times$ such that for all finite order characters ϕ of $\mathbb{Z}_p^\times$ and $1 \leq j \in \mathbb{Z}$, we have*

$$\int_{\mathbb{Z}_p^\times} \phi(z) d\varphi_\lambda = \lambda(U(p))^{-r} \frac{G(\phi)L(1, \lambda \otimes \phi^{-1})}{\Omega_{\phi(-1)}(\lambda; A)}.$$

When ϕ is the trivial character $\mathbf{1}$, we need to explain what $G(\mathbf{1})$ means. We have

$$\begin{aligned} f|R_{\mathbf{1}}(z) =& \sum_{x \in (\mathbb{Z}/\|\mathbb{Z})^\times} f(z + \frac{x}{p}) \\ =& \sum_{n=1}^\infty a(n,f)q^n (\sum_{x \in (\mathbb{Z}/\|\mathbb{Z})^\times} \exp(\frac{2\pi n x}{p})) = (p-1) \sum_{n=1}^\infty a(np, f) q^{np} - \sum_{n=1, p\nmid n}^\infty a(n,f) q^n \\ & = -f(z) + p \sum_{n=1}^\infty a(np, f) q^{np}. \end{aligned}$$

Thus we get $\int_0^\infty f_\lambda | R_{\mathbf{1}} dy = (-2\pi)^{-1}(1 - \alpha_p) L(1, \lambda)$ for $a = \alpha_p = \lambda(U(p))$. Since we have

$$\omega_+(f) = 2^{-1}(2\pi i)(f(z)dz + \varepsilon^*(f(z)dz)) = 2^{-1}(2\pi i)(f(z)dz - (f(-\overline{z})d\overline{z}))$$

whose restriction to γ_0 is $(-2\pi)f(iy)dy$, and replacing f by $f|R_{\mathbf{1}}$, we get

$$\int_{\mathbb{Z}_p^\times} d\varphi_\lambda = \lambda(U(p))^{-1} \frac{(1 - \lambda(U(p)))L(1, \lambda)}{\Omega_+(\lambda; A)}.$$

Thus essentially $G(\mathbf{1}) = 1$. We leave you to formulat the corresponding p-adic L-functions. See [LFE] Chapter 6 for more details of these facts.

References

Books

[BCM] N. Bourbaki, *Algèbre Commutative*, Hermann, Paris, 1961–83

[CGP] K. S. Brown, *Cohomology of Groups*, Graduate texts in Math. **87**, Springer, 1982

[CPI] K. Iwasawa, *Collected Papers*, Vol. 1-2, Springer, 2001

[CRT] H. Matsumura, *Commutative Ring Theory*, Cambridge studies in advanced mathematics **8**, Cambridge Univ. Press, 1986

[IAT] G. Shimura, *Introduction to the Arithmetic Theory of Automorphic Functions*, Princeton University Press and Iwanami Shoten, 1971, Princeton-Tokyo

[ICF] L. C. Washington, *Introduction to Cyclotomic Fields*, Graduate Text in Mathematics, **83**, Springer, 1980

[LFE] H. Hida, *Elementary Theory of L–functions and Eisenstein Series*, LMSST **26**, Cambridge University Press, Cambridge, 1993

[MFG] H. Hida, *Modular Forms and Galois Cohomology*, Cambridge Studies in Advanced Mathematics **69**, 2000, Cambridge University Press

Articles

[Du] G. F. D. Duff, Differential forms in manifolds with boundary. Ann. of Math. **56**, (1952) 115–127

[M] B. Mazur, Courbes elliptiques et symbols modulaires, Sém. Bourbaki Exposé 414 (1972, juin)

[MS] B. Mazur and H. P. F. Swinnerton-Dyer, Arithmetic of Weil curves, Inventiones Math. **25** (1974), 1–61