ANSWER KEYS TO SELECTED EXERCISES IN THE LECTURE NOTES FOR MATH 205C IN 2007 WINTER

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Here is a sketch of or a hint for a solution to some more difficult exercises.

Exercise 1.1 Prove the following assertion:

(1) $\mathrm{Hom}_{\mathrm{alg}}(\mathbb{Z}[t,t^{-1}],A) \cong A^\times$ as sets for a commutative algebras A, where $\mathrm{Hom}_{\mathrm{alg}}$ denotes the set of algebra homomorphisms.

The isomorphism is given by $\mathrm{Hom}_{\mathrm{alg}}(\mathbb{Z}[t,t^{-1}],A) \ni \phi \mapsto \phi(t) \in A$. Since $\phi(t)\phi(t^{-1}) = \phi(tt^{-1}) = 1$, $\phi(t) \in A^\times$. □

Exercise 1.3 Prove that $H_0(X,A) \cong H^0(X,A) \cong A$ if X is connected.

Since X is connected, we can take a curve c connecting two points x and y. Regarding c as a 1-chain, we have $\partial c = x - y$. Thus the cohomology class of $[x]$ and $[y]$ is the same in $H_0(X,A)$. Thus fixing one point $x \in X$ and assigning $a \in A$ to the class $a[x]$, we get $H_0(X,A) \cong A$. As for cohomology group, the dual version of the above proof works. If X is non-connected and $\pi_0(X)$ is the set of connected component, $H_0(X,A) = \bigoplus_{Z\in\pi_0(X)} A$ and $H^0(X,A) = \prod_{Z\in\pi_0(X)} A$ by the same proof. □

Exercise 1.4 Prove that $H_1(S^1,\mathbb{Z}) \cong \mathbb{Z}$.

Fix a point $x \in S^1$, and the simplicial complex made of 1-chain c starting x and ending x circling S^1 once. Since each simplex on this complex is simply connected, we can compute $H_1(X,A)$ by using this complex. Since $\partial c = x - x = 0$, c is a cycle. Since S^1 is 1-dimensional, $C_2(S^1,A) = 0$; so, $[c] \neq 0$ and no relation among $m[c]$ and $n[c]$ for $m,n \in \mathbb{Z}$. Thus $H_1(X,\mathbb{Z}) = \mathbb{Z}[c]$. □

Exercise 1.5

(1) Check that $d\omega$ is a well defined differential form of degree $n+1$ if ω is of degree n. Here you need to verify that the chain-rule is satisfied by $d\omega$ if one changes coordinates.

(2) Prove $d \circ d = 0$.

This exercise is just a computation. □

Exercise 1.8 Prove the following facts:

(1) the expansion defining θ_χ is absolutely and locally uniformly convergent on the upper half plane $\mathfrak{H} = \{z \in \mathbb{C} |\ \mathrm{Im}(z) > 0\}$,

(2) the expansion of θ_χ convergent on the lower half plane $\overline{\mathfrak{H}}$ is given by

$$-\sum_{n=-1}^{-\infty} \chi(n) q^n.$$

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Since the answer is the same; so, we assume that $z = x + iy$ with $y > 0$. This follows from the fact $|\chi(n)| \leq 1$ and $|\exp(2\pi z)| \leq \exp(-2\pi y) < 1$, and the convergence of the geometric series $\sum_{n=0}^{\infty} q^n = \frac{1}{1-q}$ if $q = \exp(-2\pi y) < 1$. □

Exercise 1.9

Look at any Calculus book on the interchange of integration and sum. □

Exercise 1.11 Prove $G(\chi^{-1})G(\chi) = \chi(-1)N$.

See [IAT] Lemma 3.63 (and its proof) or [LFE] Exercise 2.3.5 for an indication of a more indirect proof. □

Exercise 1.14 Prove that for $\omega \in \Omega^1(\overline{T}_N, \partial\overline{T}_N; \mathbb{C})$, the integral $\int_{\gamma_x} \omega$ converges absolutely.

Write $\omega = f(z)dz$. Use $\int_1^{\infty} y^{-2}dy < \infty$ and $|f(z)| \leq Cy^{-2}$ if $y = \mathrm{Im}(z) \gg 0$. □

Theorem Let $\{c_a | a \in (N^{-1}\mathbb{Z}/\mathbb{Z})^{\times}\}$ and γ_0 gives a basis of $H_1(\overline{T}_N, \partial\overline{T}_N, A)$ over A for any commutative ring A.

Exercise 1.17 Give details of the proof of the above theorem.

A proof is given in Section 4.1 of [LFE]. □

Exercise 1.23 Compute the action of $T(n)$ on $\gamma_0 \in H_1(\overline{T}_N, \partial\overline{T}_N, \mathbb{Q}(\chi))$ (for n prime to N).

The formula is

$$T(n)(\gamma_0) = n\gamma_0 + \sum_{1 \leq a \leq N, (a,N)=1} (n - 1 - \left[\frac{na}{N}\right]) c_{a/N}.$$

A key point is that if we have two lines γ_0 and γ_b and m holes between γ_0 and γ_b encircled by $c_{x_1}, \ldots, c_{x_m}$, then $[\gamma_b] = [\gamma_0] + \sum_{j=1}^{m} [c_{x_j}]$, which is the boundary of the rectangle bounded by γ_0 and γ_b m interiors of c_{x_j} removed. Here $[c]$ indicates he cohomology class of a cycle c. Since $T(n)([\gamma_0]) = \sum_{u=0}^{n-1} [\gamma_{u/n}]$, replacing $[\gamma_{u/n}]$ by $[\gamma_b] = \gamma_0 + \sum_{j=1}^{m} [c_{x_j}]$ (for $b = u/n$), locating the holes between γ_0 and $\gamma_{u/n}$, one gets the desired formula. □

Exercise 1.25 Show the following assertions:

(1) For an integer a prime to N, the operator $T(a) - a$ sends $H_1(\overline{T}_N, \partial\overline{T}_N, \mathbb{Z}[\chi])$ into $\mathrm{Ker}(\pi : H_1(\overline{T}_N, \partial\overline{T}_N, \mathbb{Z}[\chi]) \to H_1(\overline{T}, \partial\overline{T}, \mathbb{Z}[\chi]))$, where $\mathbb{Z}[\chi]$ is the subalgebra of $\mathbb{Q}(\chi)$ generated by all the values of χ.

Using the previous exercise, we confirm that $T(a)([\gamma_0]) - a[\gamma_0]$ is a linear combination of $[c_{a/N}]$. Since $\{[c_{a/N}]\}$ generates the kernel, we get the result. □

(2) $(a - \chi^{-1}(a))L(0, \chi) \in \mathbb{Z}[\chi]$ for any integer a prime to N, where χ is a nontrivial primitive character.

Note here that $\int_{\gamma_0} \omega_{\chi^{-1}}$ is basically the L-value $L(0, \chi)$. By (1), $T(a)([\gamma_0]) - a[\gamma_0]$ is an integral linear combination of $c_{a/N}$. Since $\int_{c_{a/N}} \omega_{\chi} = \chi(a) \in \mathbb{Z}[\chi]$, the integration of $\omega_{\chi^{-1}}$ over $T(a)(\gamma_0) - a\gamma_0$ is therefore essentially $(a - \chi^{-1}(a))L(0, \chi)$, which is integral over $\mathbb{Z}[\chi]$. □

Exercise 1.28 Prove that $|\phi|_p$ gives a well defined norm on $C(G; R)$. Is $LC(G; R)$ (resp. $C(G; R)$) a Banach space under $|\cdot|_p$?

The norm by definition gives the topology of uniform convergence on $C(G;R)$. Since a uniform limit of continuous functions are continuous, $C(G;R)$ is complete under the norm; so, it is a Banach space.

If G is finite, we have $C(G;R) = LC(G;R)$; so, they are Banach spaces. If G is infinite, $LC(G;R)$ cannot be a Banach space because there are locally non-constant continuous functions (for example, non-constant polynomials on $\mathbb{Z}_p$) which are uniform limit of locally constant functions. Thus in this case, $LC(G;R)$ is not complete (and hence not a Banach space). $\square$

Exercise 1.30 If $\varphi \in Meas(G;K)$, prove

$$\mathrm{Sup}_{0\neq\phi\in C(G;K)} |\varphi(\phi)|_p/|\phi|_p = \mathrm{Sup}_{0\neq\phi\in LC(G;K)} |\varphi(\phi)|_p/|\phi|_p.$$

Use the density of $LC(G;K)$ in $C(G;K)$. $\square$

Exercise 1.31 Prove that $\mu = \{\zeta \in \mathbb{Z}_p^\times | \zeta^M = 1\}$, where $M = p-1$ or 2 according as p is odd or even.

Let $\Gamma = 1 + p\mathbb{Z}_p$ if $p > 2$ and $\Gamma = 1 + 4\mathbb{Z}_2$ if $p = 2$. Then we know that $\mathbb{Z}_p^\times = \Gamma \times \mu_{p-1}$ if p is odd and $\mathbb{Z}_2^\times = \Gamma \times \{\pm 1\}$. The logarithm Taylor expansion $\log_p(z) = \sum_{n=1}^\infty (-1)^{n+1}\frac{(z-1)^n}{n}$ converges p-adically to an element in $p\mathbb{Z}_p$, on which $\exp_p(z) = \sum_{n=0}^\infty \frac{z^n}{n!}$ also converges p-adically. Since $\exp \circ \log_p$ and $\log_p \circ \exp_p$ are the identity maps, we find $\log_p$ is an injection of Γ into $\mathbb{Q}_p$ which satisfies $\log_p(xy) = \log_p(x) + \log_p(y)$. Since $\mathbb{Q}_p$ is torsion-free, Γ cannot have an element of finite order except for 1. $\square$

Exercise 2.1 Let $\mathbf{P}^1(A)$ be the projective space of dimension 1 over a ring A. Prove $[SL_2(\mathbb{Z}) : \Gamma_0(N)] = |\mathbf{P}^1(\mathbb{Z}/N\mathbb{Z})| = N\prod_{\ell|N}(1+\frac{1}{\ell})$ if N is square-free, where ℓ runs over all prime factors of N.

See for example [IAT] Proposition 1.43 (and its proof) or [MFM] Theorem 4.2.5. You may also use the fact $\mathbf{P}^1(A) = \{(x,y) \in A^2 | xA + yA = A\}/A^\times$. $\square$

Exercise 2.2 Prove the following facts:

1. there are two orbits of the action of $GL_2(\mathbb{R})$ on $\mathbf{P}^1(\mathbb{C})$: $\mathbf{P}^1(\mathbb{R})$ and $\mathfrak{H} \sqcup \overline{\mathfrak{H}}$, where $\mathfrak{H} = \{z \in \mathbb{C} | \operatorname{Im}(z) > 0\}$ and $\overline{\mathfrak{H}} = \{z \in \mathbb{C} | \operatorname{Im}(z) < 0\}$.
2. the stabilizer of $i = \sqrt{-1}$ is the center times $SO_2(\mathbb{R}) = \left\{\left(\begin{smallmatrix}\cos(\theta) & \sin(\theta)\\ -\sin(\theta) & \cos(\theta)\end{smallmatrix}\right) \middle| \theta \in \mathbb{R}\right\}$,
3. $\gamma \in GL_2(\mathbb{R})$ with $\det(\gamma) < 0$ interchanges the upper half complex plane $\mathfrak{H}$ and lower half complex plane $\overline{\mathfrak{H}}$,
4. the upper half complex plane is isomorphic to $SL_2(\mathbb{R})/SO_2(\mathbb{R})$ by $SL_2(\mathbb{R}) \ni g \mapsto g(\sqrt{-1}) \in \mathfrak{H}$.

This is standard. See, for example, [MFM] Sections 1.2–3. $\square$

Exercise 2.3 Show that $SL_2(K)$ acts transitively on $\mathbf{P}^1(K)$ for any field K by linear fractional transformation.

This is because $\left(\begin{smallmatrix}1 & a\\ 0 & 1\end{smallmatrix}\right)(0) = a$ and $\left(\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right)(0) = \infty$. $\square$

Exercise 2.4 Show that the above α can be taken in $SL_2(\mathbb{Z})$.

Write $c = \frac{a}{b}$ as a reduced fraction; then, we can find $x, y \in \mathbb{Z}$ such that $ax - by = 1$. Thus for $\alpha = \left(\begin{smallmatrix}a & y\\ b & x\end{smallmatrix}\right) \in SL_2(\mathbb{Z})$, we have $\alpha(\infty) = c$. $\square$

Exercise 2.5 Define $f|\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)(z) = f(\frac{az+b}{cz+d})(cz+d)^{-2}$. Prove the following facts:

1. $(f|\alpha)|\beta = f|(\alpha\beta)$ for $\alpha \in SL_2(\mathbb{R})$,
2. if f satisfies (M1), $f|\alpha$ satisfies (M1) replacing $\Gamma_0(N)$ by $\Gamma = \alpha^{-1}\Gamma_0(N)\alpha$,
3. If $\alpha \in SL_2(\mathbb{Z})$, show that Γ contains $\Gamma(N) = \{\gamma \in SL_2(\mathbb{Z}) | \gamma - 1 \in NM_2(\mathbb{Z})\}$.

This is standard. See [IAT] Section 3.2. $\square$

Exercise 2.16 Let $\Gamma = SL_2(\mathbb{Z})$. Prove that $|\Gamma\backslash(\Gamma\alpha\Gamma)| < \infty$ for $\alpha \in GL_2(\mathbb{R})$ if and only if $\alpha \in M_2(\mathbb{Q})$ modulo real scalar matrices.

The direction: $\Leftarrow$ is easy. Here is a proof of the converse (there are many different arguments): If $|\Gamma\backslash(\Gamma\alpha\Gamma)| < \infty$, we can check $\Gamma \cap \alpha\Gamma\alpha^{-1}$ and $\Gamma \cap \alpha^{-1}\Gamma\alpha$ are of finite index in Γ both containing $\Gamma(N)$ for a suitable integer N. Then prove that $\Gamma(N)$ contains a basis of $M_2(\mathbb{Q})$ over $\mathbb{Q}$. This shows that $\alpha M_2(\mathbb{Q})\alpha^{-1} = M_2(\mathbb{Q})$. Thus $x \mapsto \alpha x \alpha^{-1}$ gives an automorphism of the central simple algebra $M_2(\mathbb{Q})$ over $\mathbb{Q}$. By the theorem of Skolem-Noether, this automorphism is inner; in other words, we can find $\beta \in GL_2(\mathbb{Q})$ such that $\alpha x \alpha^{-1} = \beta x \beta^{-1}$ for all $x \in M_2(\mathbb{Q})$. Thus $\beta^{-1}\alpha$ commutes with all $x \in M_2(\mathbb{Q})$ and has to be a scalar matrix. This shows the result. $\square$

Exercise 2.20 Prove the following fact.

1. The above action is an action of the group $GL_2(\mathbb{R})$ on $\mathfrak{H}$. In other words, $\alpha(\beta(z)) = (\alpha\beta)(z)$ for $\alpha, \beta \in GL_2(\mathbb{R})$.
2. We have $\mathfrak{H} \cong GL_2(\mathbb{R})/\mathbb{R}^\times O_2(\mathbb{R})$ by $g \mapsto g(\sqrt{-1})$.

This is just a computation. $\square$

Exercise 2.24 Prove $L(s, \lambda)$ converges absolutely if $\mathrm{Re}(s) > 2$.

Show $|L(s, \lambda)| \leq B\zeta(\mathrm{Re}(s) - 1)$ and the convergence of Riemann zeta function if $\mathrm{Re}(s) > 1$. $\square$

Exercise 2.29 Let N' be the LCM of N and m^2. Prove $(f|R_\chi)|\gamma = \chi^2(a)f|R_\chi$ for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N')$.

This is standard. See for example [IAT] Proposition 3.64 and its proof or [MFM] Lemma 4.3.10. $\square$

Exercise 2.31 If $f \in S_2(\Gamma_0(N))$, prove that $f(pz) \in S_2(\Gamma_0(Np))$.

This is because $\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)^{-1} \Gamma_0(N) \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) \supset \Gamma_0(pN)$. $\square$

Exercise 2.32

1. Prove $T(m)T(n) = \sum_{0<d|(m,n),(d,N)=1} d \cdot T(mn/d^2)$.
2. Prove the above Euler factorization of $L(s, \lambda \otimes \chi)$.

This is again standard. See [IAT] Section 3.2. $\square$

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