Overview of Math 207b Spring 2018: Elementary modular Iwasawa theory

This course is independent from the material dealt with in Winer 2018, though they are related. **No** knowledge from Winter course is assumed, and everything can be done in a quite elementary way (through basics of complex analysis and algebraic number theory).

We discuss the first three topics of the following list:

(1) Explicit construction of modular forms and modular functions;
(2) Determination of units in the elliptic modular function fields (modular units);
(3) The cuspidal class group of modular curves including a proof of the cuspidal class number formula;
(4) Construction of units (called elliptic units) of the Hilbert ring class field of imaginary quadratic fields as specialization of modular units;
(5) Iwasawa theory for imaginary quadratic fields via elliptic units.

If time allows, we further go into the topics (4) and (5). A modular curve is an affine curve (i.e., an open Riemann surface) classifying elliptic curves with certain additional structure (called level structure) naturally defined over $\mathbb{Q}$. As a Riemann surface, it is a quotient of the upper half complex plane by $\mathrm{SL}_2(\mathbb{Z})$ (and its subgroups). Adding finite number of points (called cusps), we can complete the curve into a projective curve (i.e., a compact Riemann surface). Most of recent progress in number theory and arithmetic geometry is based on the study of modular curves and modular forms defined on them; e.g., proof of Iwasawa's conjectures (Mazur–Wiles) and Fermat's last theorem (Wiles).

Modular units are the units in the ring of holomorphic functions of the affine modular curve. Divisors supported on cusps modulo principal divisors (divisors of modular units) give the cuspidal class group (which is the torsion subgroup of rational points of the Jacobian of the modular curve). We can determine explicitly the group of modular units via classical results of Weierstrass and Siegel (like the determination of cyclotomic units in the field generated by roots of unity by Dirichlet–Kummer).

Striking points are that the class group is finite and that we have an explicit class number formula of Kubert–Lang in terms of the Dirichlet L-values at $s = 2$ (while the classical class number formula of Dirichlet/Kummer of the cyclotomic field is in terms of the values at $s = 1$). This is done by generalizing Stickelberger's theory of cyclotomic class groups to the setting of modular curves. This theory gives a base of the proof of the Iwasawa main conjecture by Mazur–Wiles.

For each prime $p > 2$, $\{\sin(\pi a/p)/\sin(\pi/p)\}_{0<a<p/2}$ gives independent units in the integer ring of the field of p-th root of unity for each prime p (the cyclotomic units). Analogously, the value of modular units at a point on the upper half complex plane belonging to an imaginary quadratic field K gives independent units in the (certain) Hilbert ring class field over K, and this is a base of the generalization of the cyclotomic Iwasawa theory to the elliptic Iwasawa theory of imaginary quadratic fields. If time allows, we describe these finer results (possibly including the class number formulas of the ring class field as an index of elliptic units inside the entire units).