Assuming that $\rho_0 = \text{Ind}^Q_K \varphi$ for a quadratic field $K = \mathbb{Q}[\sqrt{D}]$ (with discriminant $D$) and a character $\varphi : \mathcal{O}_K \rightarrow W^\times$ of order prime to $p$, we explore the meaning of the cyclicity of $\text{Sel}(\rho_0)^\vee$ in terms of Iwasawa theory over $K$. Write $\overline{\varphi} := (\varphi \mod m_W)$ and $\overline{\rho} = \text{Ind}^Q_K \overline{\varphi}$. We denote by $O$ the integer ring of $K$. 
§6.1. Induced representation.
Let $A \in CL/W$ and $G$ be a profinite group with a subgroup $H$ of index 2. Put $\Delta := G/H$. Let $H$ be a character $\varphi : G \to A$. Let $A(\varphi) \cong A$ on which $H$ acts by $\varphi$.

Regard the group algebra $A[G]$ as a left and right $A[G]$-module by multiplication. Define $A(\text{Ind}_{H}^{G} \varphi) := A[G] \otimes_{A[H]} A(\varphi)$ (so, $\xi h \otimes a = \xi \otimes ha = \xi \otimes \varphi(h)a = \varphi(a)(\xi \otimes a)$) for $h \in H$. and let $G$ acts on $A(\text{Ind}_{H}^{G} \varphi)$ by $g(\xi \otimes a) := (g\xi) \otimes a$. The resulted $G$-module $A(\text{Ind}_{G}^{H} \varphi)$ is the induced module.

Similarly we can think of $A(\text{ind}_{G}^{H} \varphi) := \text{Hom}_{A[H]}(A[G], A(\varphi))$ (so, $\phi(h\xi) = h\phi(\xi) = \varphi(h)\phi(\xi)$) on which $g \in G$ acts by $g\phi(\xi) = \phi(\xi g)$. 
§6.2. **Matrix form of** \( \text{Ind}_H^G \varphi \).

Suppose that \( \varphi \) has order prime to \( p \). Then for \( \sigma \in G \) generating \( G \) over \( H \), \( \varphi_\sigma(h) = \varphi(\sigma^{-1}h\sigma) \) is again a character of \( H \). The module \( \text{Ind}_H^G \varphi \) has a basis \( 1_G \otimes 1 \) and \( \sigma \otimes 1 \) for the identity element \( 1_G \) of \( G \) and \( 1 \in A \cong A(\varphi) \).

We have

\[
g(1_G \otimes 1, \sigma \otimes 1) = (g \otimes 1, g\sigma \otimes 1)
= \begin{cases} (1_G \otimes g, \sigma \otimes \sigma^{-1}g\sigma) = (1_G \otimes 1, \sigma \otimes 1) \begin{pmatrix} \varphi(g) \\ 0 \\ \varphi_\sigma(g) \end{pmatrix} & \text{if } g \in H, \\
(\sigma \otimes \sigma^{-1}g, 1_G \otimes g\sigma) = (1_G \otimes 1, \sigma \otimes 1) \begin{pmatrix} 0 \\ \varphi(\sigma^{-1}g) \\ 0 \end{pmatrix} & \text{if } g\sigma \in H, \end{cases}
\]

Thus extending \( \varphi \) to \( G \) by 0 outside \( H \), we get

\[
\text{Ind}_H^G \varphi(g) = \begin{pmatrix} \varphi(g) & \varphi(\sigma^{-1}g) \\ \varphi(g\sigma) & \varphi(\sigma^{-1}g\sigma) \end{pmatrix}.
\]
§6.3. Two inductions are equal.
The induction $\text{ind}^G_H \varphi$ has basis $(\phi_1, \phi_\sigma)$ given by $\phi_1(\xi + \xi'\sigma) = \varphi(\xi) \in A = A(\varphi)$ and $\phi_\sigma(\xi + \xi'\sigma^{-1}) = \varphi(\xi') \in A = A(\varphi)$ for $\xi \in A[H]$; so, $(*) \phi_1(\xi' + \xi\sigma^{-1}) = \phi_\sigma(\xi + \xi'\sigma^{-1})$. Then we have

$$g(\phi_1(\xi + \xi'\sigma^{-1}), \phi_\sigma(\xi + \xi'\sigma^{-1}))$$

$$= (\phi_1(\xi g + \xi'\sigma^{-1}g\sigma\sigma^{-1}), \phi_\sigma(\xi g + \xi'\sigma^{-1}g\sigma\sigma^{-1}))$$

$$= \begin{cases} 
(\phi_1(\xi), \varphi_\sigma(\xi')) \left( \begin{array}{cc} \varphi(g) & 0 \\
0 & \varphi_\sigma(g) \end{array} \right) \\
(\phi_1(\xi'\sigma^{-1}g), \phi_\sigma(\xi g\sigma)) \overset{(*)}{=} (\phi_1(\xi), \phi_\sigma(\xi')) \left( \begin{array}{cc} 0 & \varphi(g\sigma) \\
\varphi(\sigma^{-1}g) & 0 \end{array} \right) 
\end{cases} \quad (g \in H),$$

$$\begin{cases} 
\overset{(*)}{=} (\phi_1(\xi), \phi_\sigma(\xi')) \left( \begin{array}{cc} 0 & \varphi(g\sigma) \\
\varphi(\sigma^{-1}g) & 0 \end{array} \right) 
\end{cases} \quad (g\sigma \in H).$$

Thus we get

$$\text{Ind}^G_H \varphi \cong \text{ind}^G_H \varphi.$$
§6.4. Tensoring $\alpha: \Delta \cong \mu_2$.
Let $J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Extending $\varphi$ to $G$ by 0 outside $H$, we find

$$\text{Ind}_H^G \varphi \otimes \alpha(g) = \begin{cases} \begin{pmatrix} \varphi(g) & 0 \\ 0 & \varphi(\sigma^{-1}g\sigma) \end{pmatrix} = J \begin{pmatrix} \varphi(g) & 0 \\ 0 & \varphi(\sigma^{-1}g\sigma) \end{pmatrix} J^{-1} & (g \in H), \\ - \begin{pmatrix} 0 & \varphi(g\sigma) \\ \varphi(\sigma^{-1}g) & 0 \end{pmatrix} = J \begin{pmatrix} 0 & \varphi(g\sigma) \\ \varphi(\sigma^{-1}g) & 0 \end{pmatrix} J^{-1} & (g\sigma \in H). \end{cases}$$

Thus we get

$$\text{Ind}_H^G \varphi \otimes \alpha = J(\text{Ind}_H^G \varphi)J^{-1} \overset{i_\alpha}{\sim} \text{Ind}_H^G \varphi.$$ 

Thus $\text{Ad}(\text{Ind}_H^G) = \{ x \in \text{End}_A(\text{Ind}_H^G \varphi) | \text{Tr}(x) = 0 \}$ contains $i_\alpha$ as $\text{Tr}(J) = 0$. 

6.5. Characterization: \( \rho \otimes \alpha \cong \rho \iff \rho \cong \text{Ind}_{H}^{G} \varphi. \)

Let \( \varphi := (\varphi \mod m_{A}). \) Suppose \( \varphi_{\sigma} \neq \varphi. \) Since \( \text{Ind}_{H}^{G} \varphi(H) \) contains a diagonal matrices with distinct eigenvalues, its normalizer is \( \text{Ind}_{H}^{G} \varphi(G). \) Thus the centralizer \( Z(\text{Ind}_{H}^{G} \varphi) = \mathbb{F}^{\times} \) (scalar matrices). Since \( \text{Ind}_{H}^{G} \varphi(\sigma) \) interchanges \( \varphi \) and \( \varphi_{\sigma}, \) \( \text{Ind}_{H}^{G} \varphi \) is irreducible. Since \( \text{Aut}(\varphi) = \mathbb{F}^{\times}, i_{\alpha} \) for \( \varphi \) is unique up to scalars.

Let \( \rho : G \to \text{GL}_{2}(A) \) be a deformation of \( \text{Ind}_{H}^{G} \varphi \) with \( \rho \otimes \alpha \cong \rho. \) Write \( j\rho j^{-1} = \rho \otimes \alpha. \) Since \( \alpha^{2} = 1, j^{2} \) is scalar. We may normalize \( j \equiv J \mod m_{A} \) as \( j \mod m_{A} = zJ \) for a scalar \( z \in A^{\times}. \) Thus \( j \) has two eigenvalues \( \epsilon_{\pm} \) with \( \epsilon_{\pm} \equiv \pm z \mod m_{A}. \) Let \( A_{\pm} \) be \( \epsilon_{\pm} \)-eigenspace of \( j. \) Since \( j\rho|_{H} = \rho|_{H}j, A_{\pm} \cong A \) is stable under \( H. \) Thus we find a character \( \varphi : H \to A^{\times} \) acting on \( A_{\pm}. \) Plainly \( H \) acts on \( A_{-} \) by \( \varphi_{\sigma}. \) This shows \( \rho \cong \text{Ind}_{H}^{G} \varphi \) as \( V(\rho) = A_{+} \oplus \rho(\sigma)A_{+}. \)
§6.6. Decomposition  \( Ad(\text{Ind}_H^G \varphi) \cong \alpha \oplus \text{Ind}_H^G \varphi^- \).

Here \( \varphi^-(g) = \varphi(g)\varphi_{\sigma}^{-1}(g) = \varphi(\sigma^{-1}g^{-1}\sigma g) \) and \( \text{Ind}_H^G \varphi^- \) is irreducible if \( \varphi^- \neq \varphi_{\sigma}^- = (\varphi^-)^{-1} \) (i.e., \( \varphi^- \) has order \( \geq 3 \)).

**Proof.** On \( H \), \( \rho := \text{Ind}_H^G \varphi = \begin{pmatrix} \varphi & 0 \\ 0 & \varphi_{\sigma} \end{pmatrix} \). Therefore

\[
Ad(\text{Ind}_H^G \varphi)(h) \begin{pmatrix} x & y \\ z & -x \end{pmatrix} = \rho(h) \begin{pmatrix} x & y \\ z & -x \end{pmatrix} \rho^{-1}(h) = \begin{pmatrix} x & \varphi^-(h)y \\ (\varphi^-)^{-1}(h)z & -x \end{pmatrix},
\]

and

\[
Ad(\text{Ind}_H^G \varphi)(\sigma) \begin{pmatrix} x & y \\ z & -x \end{pmatrix} = \begin{pmatrix} 0 & \varphi(\sigma^2) \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x & y \\ z & -x \end{pmatrix} \begin{pmatrix} 0 & 1 \\ \varphi(\sigma^{-2}) & 0 \end{pmatrix} = \begin{pmatrix} \alpha(\sigma)x & \varphi(\sigma)^2z \\ \varphi(\sigma)^{-2}y & -\alpha(\sigma)x \end{pmatrix}.
\]

Thus \( \alpha \) is realized on diagonal matrices, and \( \text{Ind}_H^G \varphi^- \) is realized on the anti-diagonal matrices. \( \square \)
6.7. Irreducibility of \( \text{Ind}^G_H \varphi^- \).

**Lemma 1.** \( \text{Ind}^G_H \varphi^- \) is irreducible if and only if \( \varphi^- \neq \varphi^-_\sigma = (\varphi^-)^{-1} \) (i.e., \( \varphi^- \) has order \( \geq 3 \)). If \( \varphi^- \) has order \( \leq 2 \), then \( \varphi^- \) extends to a character \( \overline{\phi} : G \to \mathbb{F}_\times \) and \( \text{Ind}^G_H \varphi^- \cong \overline{\phi} \oplus \overline{\phi}_\alpha \).

**Proof.** Note \( \varphi^-(\sigma^2) = \varphi(\sigma^2)\varphi(\sigma^{-1}\sigma^2\sigma)^{-1} = \varphi(1) = 1 \). The irreducibility of \( \text{Ind}^G_H \varphi^- \) under \( \varphi^- \neq \varphi^-_\sigma \) follows from the argument proving irreducibility of \( \text{Ind}^G_H \varphi \) under \( \varphi \neq \varphi_\sigma \) in §6.5. Suppose \( \varphi^- \) has order \( \leq 2 \) (so, \( \varphi^- = \varphi^-_\sigma \)). Choose a root \( \zeta = \pm 1 \) of \( X^2 - \varphi^-_\sigma(\sigma^2) = X^2 - 1 \) in \( \mathbb{F} \). Define \( \overline{\phi} = \varphi^- \) on \( H \) and \( \overline{\phi}(\sigma h) = \zeta \overline{\varphi^-}(h) \). For \( h, h' \in H \),

\[
\overline{\phi}(\sigma h \sigma h') = \overline{\phi}(\sigma^2 \sigma^{-1} h \sigma h') = \overline{\phi}^- (\sigma^2) \overline{\varphi^-}_\sigma(h) \overline{\varphi^-}(h') = \zeta^2 \overline{\varphi^-}(hh') = \overline{\varphi^-}(\sigma h) \overline{\varphi^-}(\sigma h').
\]

Similarly \( \overline{\phi}(h \sigma h') = \overline{\phi}(\sigma \sigma^{-1} h c h') = \zeta \overline{\varphi^-}_\sigma(h) \overline{\varphi}(h') = \overline{\phi}(h) \overline{\phi}(\sigma h') \); so, \( \overline{\phi} \) is a character. Then \( \mathbb{F}[\zeta][G] \otimes_{\mathbb{F}[H]} \mathbb{F}[\zeta](\varphi^-) \cong \mathbb{F}[\zeta](\overline{\phi}) \) as \( G \)-modules by \( a \otimes b \mapsto \overline{\phi}(a)b \). \( \square \)
§6.8. Ordinality for $\bar{\rho} := \text{Ind}_{K}^{Q} \varphi$.

Let $\sigma \in \mathfrak{S}_{Q}$ induce a non-trivial field automorphism of $K/\mathbb{Q}$. Let $\bar{\rho} := \text{Ind}_{K}^{Q} \varphi = \text{Ind}_{\mathfrak{S}_{K}}^{\mathfrak{S}_{Q}} \varphi$ and assume that $p = pp^\sigma$ in $O$ (fixing the factor $p$ so that $\varphi$ is unramified at $p^\sigma$). Let $\epsilon$ be the conductor of $\varphi$; so, the ray class field $H_\epsilon/K$ of conductor $\epsilon$ is the smallest ray class field such that $\varphi$ factors through $\text{Gal}(H_\epsilon/K)$. Suppose $(\sigma p) \epsilon + \epsilon^\sigma = O$.

Pick a prime factor $l|\epsilon$. Then $l + l^\sigma = O$; so, $l$ splits in $K$. In particular, $I_l = I_l \subset \mathfrak{S}_K$ (for $(l) = l \cap \mathbb{Z}$), and $\varphi|I_l$ ramifies while $\varphi$ is unramified at $l^\sigma$. Thus $\bar{\rho}|I_l \cong \begin{pmatrix} \bar{\epsilon}_l & 0 \\ 0 & \delta_l \end{pmatrix}$ with $\bar{\epsilon}_l = \varphi|I_l$ and $\delta_l = \varphi^\sigma$ which is unramified.

Suppose $l|D$; so, $I_l$ is of index 2 in $I_l$. Then $\varphi|I_l = \varphi^\sigma|I_l = 1$. Similarly to §6.7, we find $\text{Ind}_{K}^{Q} \varphi|I_l = \text{Ind}_{I_l} I_l \varphi|I_l = \begin{pmatrix} \bar{\epsilon}_l & 0 \\ 0 & \delta_l \end{pmatrix}$ with $\bar{\epsilon}_l = \alpha|I_l$ and $\delta_l = 1$. In short, $\bar{\rho}$ satisfies $(\text{ord}_l)$ for $l \in S := \{l|DN(\epsilon)p\}$. 
§6.9. Identity of two deformation functors.
Let $\chi$ be the Teichmüller lift of $\det(\rho)$. For any Galois representation $\rho$, let $K(\rho)$ be the solitting field $\overline{\mathbb{Q}}^{\text{Ker}(\rho)}$ of $\rho$. Let $K(\bar{\rho})(p)/K(\bar{\rho})$ be the maximal $p$-profinite extension unramified outside $p$. Put $G = \text{Gal}(K(\bar{\rho})(p)/\mathbb{Q})$ and $H = \text{Gal}(K(\bar{\rho})(p)/K)$. Consider the deformation functor $\mathcal{D}_? : \text{CL}/B \to \text{SETS}$ for $\chi$ and $\kappa$. Since any deformation factors through $G$, we regard $\rho \in \mathcal{D}_?(A)$ is defined over $G$. Let

$$\mathcal{F}_H(A) = \{\varphi : H \to A^\times | \varphi \mod m_A = \overline{\varphi} \text{ unramified outside } c\}$$

and $\mathcal{D}_?^\Delta(A) = \{\rho \in \mathcal{D}_?(A) | \rho \otimes \alpha \cong \rho, \det \rho = ? \}/\text{GL}_2(A)$. Recall $\Delta = G/H$ and write $\widehat{\Delta} = \{\alpha, 1\}$ for its character group.

**Lemma 2.** Let $\widehat{\Delta}$ act on $\mathcal{F}$ by $\rho \mapsto \rho \otimes \alpha$. Then $\mathcal{F}_H(A) \ni \varphi \mapsto \text{Ind}_H^G \varphi \in \mathcal{D}(A)^\widehat{\Delta}$ induces an isomorphism: $\mathcal{F}_H \cong \mathcal{D}_?^\Delta$ of the functors if $\varphi \neq \overline{\varphi}_c$. 

Note $\hat{D}\hat{\Delta}(A) = \{\rho \in D\hat{\Delta}(A) | J(\rho \otimes \alpha)J^{-1} \sim \rho\} / (1 + M_2(m_A))$ (realizing $D\hat{\Delta}$ under strict equivalence and choosing $\text{Ind}_{G}^{H} \varphi$ specified §6.2) as $J(\bar{\rho} \otimes \alpha)J^{-1} = \bar{\rho}$ (see §6.4). By the characterization in §6.5, we find a character $\varphi : H \rightarrow A^\times$ such that $\text{Ind}_{G}^{H} \varphi \cong \rho$.

We choose $j \in \text{GL}_2(A)$ with $j \equiv J \mod m_A$ as in §6.5. Then $A_+ = A(\varphi)$ for a character $\varphi : H \rightarrow A^\times$. Note that $\varphi \mod m_A = \bar{\varphi}$ by the construction in §6.5. By $(\text{ord}_l)$ for $l \in S$, $\bar{\varphi}_\sigma$ acting on $A_-$ is unramified at $l|cp$. Thus we conclude $\mathcal{F}_H \cong \hat{D}\hat{\Delta}$. \qed

By $\rho \mapsto \rho \otimes \alpha$, $\hat{\Delta}$ acts on $D\hat{\Delta}$. For the universal representation $\rho_? \in D\hat{\Delta}(R?)$, therefore, we have an involution $[\alpha] \in \text{Aut}_{B\text{-alg}}(R?)$ such that $[\alpha] \circ \rho_? \cong \rho_? \otimes \alpha$. Define $R^\pm_? := \{x \in R_?[\alpha](x) = \pm x\}$. 


§6.11. Induced Selmer groups.
For a character $\phi : H \to \mathbb{F}^\times$, let $K^{(p)}$ be the maximal $p$-abelian extension of $K$ unramified outside $p$. Let $\Gamma_p = \text{Gal}(K^{(p)}/K)$ which is a $p$-profinite abelian group.

**Corollary 1.** We have a canonical isomorphism $R_\kappa/R_\kappa([\alpha]-1)R_\kappa \cong W[[\Gamma_p]]$, where $R_\kappa([\alpha]-1)R_\kappa$ is the $R_\kappa$-ideal generated by $[\alpha](x) - x$ for all $x \in R_\kappa$.

If a finite group $\langle \gamma \rangle$ acts on $R \in CL/B$ fixing $B$, then

$$\text{Hom}_{B\text{-alg}}(R, A)^{\langle \gamma \rangle} = \text{Hom}_{B\text{-alg}}(R/R(\gamma - 1)R, A).$$

Indeed, $f \in \text{Hom}_{B\text{-alg}}(R, A)^{\gamma}$, then $f \circ \gamma = f$; so, $f(R(\gamma-1)R) = 0$. Thus $\text{Hom}_{B\text{-alg}}(R, A)^{\gamma} \hookrightarrow \text{Hom}_{B\text{-alg}}(R/R(\gamma-1)R, A)$. Surjectivity is plain.
Since $\mathcal{F}_H = D_κ^\Delta$, we find

$$\mathcal{F}_H(A) = \text{Hom}_{\Lambda\text{-alg}}(R_κ, A)^\Delta = \text{Hom}_{\Lambda\text{-alg}}(R_κ/(R_κ(\alpha - 1)R_κ), A).$$

Thus $\mathcal{F}_H$ is represented by $R_κ/(R_κ(\alpha - 1)R_κ)$.

Let $φ_0 : H \to W^\times$ be the Teichmüller lift of $\bar{φ}$. Define $φ : H \to W[[\Gamma_p]]^\times$ by $φ(h) = φ_0(h)h|_{K(φ)} \in W[[\Gamma_p]]$. We show that $(W[[\Gamma_p]], φ)$ is a universal couple for $\mathcal{F}_H$, which implies the identity of the corollary. Pick a deformation $φ \in \mathcal{F}_H(A)$. Then $(μ_A \circ φ_0)^{-1}φ$ has values in $1 + m_A$ unramified outside $p$ as the ramification at $l \in S$ different from $p$ is absorbed by that of $φ$ by the fact that the inertia group at $l$ in $H$ is isomorphic to the inertia group at $l$ of $\text{Gal}(K(φ)/K)$. Thus $(μ_A \circ φ_0)^{-1}φ$ factors through $\Gamma_p$, and induces a unique $W$-algebra homomorphism $W[[\Gamma_p]] \xrightarrow{φ} A$ with $φ = φ \circ φ$. ∎
§6.13. What is $\Gamma_p$?

**Proposition 1.** If $p > 2$, we have an exact sequence

$$1 \to (1 + p\mathbb{Z}_p)/\varepsilon^{(p-1)p} \to \Gamma_p \to Cl_K \otimes_{\mathbb{Z}} \mathbb{Z}_p \to 1,$$

where $\varepsilon = 1$ if $K$ is imaginary, and $\varepsilon$ is a fundamental unit of $K$ if $K$ is real. Thus $\Gamma_p$ is finite if $K$ is real.

**Proof.** Since $\Gamma_p = Cl_K(p^{\infty}) \otimes_{\mathbb{Z}} \mathbb{Z}_p$, the exact sequence is the $p$-primary part of the exact sequence of the class field theory:

$$1 \to O_p^\times / \overline{O}^\times \to Cl_K(p^{\infty}) \to Cl_K \to 1.$$

Thus tensoring $\mathbb{Z}_p$ over $\mathbb{Z}$, we get the desired exact sequence, since $O_p \cong \mathbb{Z}_p$ canonically. Note here $\varepsilon^{p-1} \in 1 + p\mathbb{Z}_p = 1 + pO_p$. \(\square\)
§6.14. Iwasawa theoretic interpretation of \( \text{Sel}(\text{Ad}(\text{Ind}_{K}^{\mathbb{Q}} \varphi)) \).

Pick a deformation \( \varphi \in \mathcal{F}_{H}(A) \). By \( \text{Ad}(\text{Ind}_{K}^{\mathbb{Q}} \varphi) = \alpha \oplus \text{Ind}_{K}^{\mathbb{Q}} \varphi^{−} \), the cohomology is decomposed accordingly:

\[
H^{1}(G, \text{Ad}(\text{Ind}_{K}^{\mathbb{Q}} \varphi)) = H^{1}(G, \alpha) \oplus H^{1}(G, \text{Ind}_{K}^{\mathbb{Q}} \varphi^{−}).
\]

Since Selmer cocycles are upper triangular over \( D_{p} \) and upper nilpotent over \( I_{p} \), noting the fact that \( \alpha \subset \text{Ad}(\text{Ind}_{H}^{G} \varphi) \) is realized on diagonal matrices, and \( \text{Ind}_{H}^{G} \varphi^{−} \) is realized on anti-diagonal matrices, the Selmer condition is compatible with the above factorization; so, we have

**Theorem 1.** We have \( \text{Sel}(\text{Ad}(\text{Ind}_{H}^{G} \varphi)) = \text{Sel}(\alpha) \oplus \text{Sel}(\text{Ind}_{H}^{G} \varphi^{−}) \), where \( \text{Sel}(\alpha) \) is made of classes in \( H^{1}(G, \alpha) \) unramified everywhere and \( \text{Sel}(\text{Ind}_{H}^{G} \varphi^{−}) \) is isomorphic to the subgroup \( \text{Sel}(\varphi^{−}) \) of \( H^{1}(H, \varphi^{−}) \) made of classes unramified outside \( p \) and vanishes over \( D_{p}^{\sigma} \). In particular,

\[
\text{Sel}(\alpha) = \text{Hom}(\text{Cl}_{K}, A^{\vee}) = \text{Hom}(\text{Cl}_{K} \otimes_{\mathbb{Z}} A, \mathbb{Q}_{p}/\mathbb{Z}_{p}).
\]
§6.15. Proof.

Pick a Selmer cocycle $u : G \to Ad(\rho_0)^\ast$. Projecting down to $\alpha$, it has diagonal form; so, the projection $u_\alpha$ restricted to $D_p$ is unramified. Starting with an unramified cocycle $u_\alpha$ and regard it as having values in diagonal matrices in $Ad(\rho_0)^\ast$, its class falls in $\text{Sel}(Ad(\rho_0))$.

Similarly, the projection $u^{\text{Ind}}$ of $u$ to the factor $\text{Ind}_H^G \varphi^-$ is anti-diagonal of the form $\begin{pmatrix} 0 & u^+ \\ u^- & 0 \end{pmatrix}$. Noting $H^j(\Delta, ((\text{Ind}_H^G \varphi^-)^\ast)^H) = 0$ $(j = 1, 2)$, by inflation-restriction sequence,

$$H^1(G, (\text{Ind}_H^G \varphi^-)^\ast) \cong (H^1(H, (\varphi^-)^\ast) \oplus H^1(H, (\varphi_\sigma^-)^\ast))^{\Delta}.$$ 

So $u^-(\sigma^{-1}g\sigma) = u^+(g)$ as $\sigma \in \Delta$ interchanges $H^1(H, (\varphi^-)^\ast)$ and $H^1(H, (\varphi_\sigma^-)^\ast)$. Moreover $u^+$ is unramified outside $p$ as an element of $H^1(H, (\varphi^-)^\ast)$. Since $u^-|_{D_p} = 0$, $u^+$ vanishes on $D_p\sigma$ by $u^-(\sigma^{-1}g\sigma) = u^+(g)$. \qed
§6.16. Anti-cyclotomic \( \mathbb{Z}_p \)-extension.

Regard \( \varphi : G \to W[[\Gamma_p]]^\times \). Then define \( K^- \) by the fixed subfield of \( K(\overline{\rho})^{(p)} \) by \( \text{Ker}(\varphi^-) \). Let \( \Gamma^- \) be the maximal \( p \)-abelian quotient of \( \text{Gal}(K^-/K) \cong \text{Im}(\varphi^-) \); so, \( \text{Gal}(K^-/K) \cong \Gamma^- \times \text{Gal}(K(\overline{\varphi_0})/K) \).

Note that \( \varphi^-(h) = \varphi(h)\varphi(\sigma^{-1}h\sigma)^{-1} \in \Gamma_p \) if \( h \in \Gamma^- \). Thus we have an exotic homomorphism \( \Gamma^- \to \Gamma_p \). We have an exact sequence:

\[
1 \to (1 + pO_p/\varepsilon(p^{-1}\mathbb{Z}_p))^{\sigma=-1} \to \Gamma^- \to Cl_K \otimes_{\mathbb{Z}} \mathbb{Z}_p \to 1,
\]

which is the "-"-eigenspaces of the action of \( \sigma \) on the exact sequence.

\[
1 \to (1 + pO_p)/\varepsilon(p^{-1}\mathbb{Z}_p) \to Cl_K(p^\infty) \otimes_{\mathbb{Z}} \mathbb{Z}_p \to Cl_K \otimes_{\mathbb{Z}} \mathbb{Z}_p \to 1.
\]

Therefore the above homomorphism induces an isomorphism \( \Gamma^- \cong \Gamma_p \), and in this way, we identify \( W[[\Gamma^-]] \) with \( W[[\Gamma_p]] \).
§6.17. Iwasawa modules.

Let $L/K^−$ be the maximal $p$-abelian extension unramified outside $p$ totally split at $p^\sigma$. Put $\mathcal{Y} := \text{Gal}(L/K^−)$. Note that $K^− \subset K(\bar{\rho})^{(p)}$. By conjugation, $\text{Gal}(K(\varphi^−)/K) \times \Gamma_− = \text{Gal}(K^−/K)$ acts on $\mathcal{Y}$; so, we put $\mathcal{Y}(\varphi^−_0) = \mathcal{Y} \otimes_{\mathbb{Z}_p[\text{Gal}(K(\varphi^−)/K)]} \varphi^−_0$ (the maximal quotient of $\mathcal{Y}$ on which the subgroup $\text{Gal}(K(\varphi^−_0)/K) \subset \text{Gal}(K^−/K)$ acts by $\varphi^−_0$). By the action of $\Gamma_−$, $\mathcal{Y}(\varphi_0)$ is a module over $W[[\Gamma_−]]$ (an Iwasawa module), whose module structure we study.

Recall $\varphi \in \mathcal{F}_H(A)$. By inflation-restriction sequence:

$$H^1(K(\varphi^−)/K, (\varphi^−)^∗) \hookrightarrow H^1(H, (\varphi^−)^∗) \to \text{Hom}_{\text{Gal}(K(\varphi^−)/K)}(\text{Gal}(K(\bar{\rho})^{(p)}/K(\varphi^−)), (\varphi^−)^∗) \to H^2(K(\varphi^−)/K, (\varphi^−)^∗)$$

is exact.
§6.18. Vanishing of $H^j(K(\varphi^-)/K, (\varphi^-)^*)$ if $\Gamma^-$ is cyclic.

Assume that $\varphi^- \neq 1$. For a finite cyclic group $C$ generated by $\gamma$

$$H^1(C, M) = \text{Ker}(\text{Tr})/\text{Im}(\gamma - 1), \quad H^2(C, M) = \text{Ker}(\gamma - 1)/\text{Im}(\text{Tr}),$$

where $\text{Tr}(x) = \sum_{c \in C} cx$ and $(\gamma - 1)(x) = \gamma x - x$ for $x \in M$. If $C$ is infinite with $M$ discrete, $H^q(C, M) = \varprojlim_{C' \subseteq C} H^q(C/C', M^{C'})$.

Thus if $\varphi^-(\gamma) \neq 1$ for a generator of $\Gamma^-$, we find

$$H^j(K(\varphi^-)/K, (\varphi^-)^*) = 0$$

as $\gamma - 1 : (\varphi^-)^* \rightarrow (\varphi^-)^*$ is a bijection. Thus

$$H^1(H, (\varphi^-)^*) \cong \text{Hom}_{\text{Gal}}(K(\varphi^-)/K)(\text{Gal}(K(\overline{\rho})^{(p)}/K(\varphi^-)), (\varphi^-)^*).$$

Then Selmer cocycles factor through $\mathcal{Y}$; so, for $\mathcal{G} := \text{Gal}(K(\varphi^-)/K)$,

$$\text{Sel}(\varphi^-) = \text{Hom}_{\mathcal{G}}(\mathcal{Y}, (\varphi^-)^*) \cong \text{Hom}_{W[[\Gamma^-]]}(\mathcal{Y}(\varphi^-_0), (\varphi^-)^*)$$

$$\cong \text{Hom}_W(\mathcal{Y}(\varphi^-_0) \otimes_{W[[\Gamma^-]]} A(\varphi^-), \mathbb{Q}_p/\mathbb{Z}_p).$$
§6.19. Cyclicity of \( \mathcal{Y}(\varphi_0^-) \).

Since \( R_\kappa / R_\kappa ([\alpha] - 1) \cong W[[\Gamma_p]] = W[[\Gamma^-]] \), we write this morphism as \( \theta : R_\kappa \to W[[\Gamma^-]] \).

**Theorem 2.** If \( \Gamma^- \) is cyclic (i.e., without torsion), we have

\[
\text{Sel}(\varphi^-) \cong \text{Hom}_{W[[\Gamma^-]]}(\mathcal{Y}(\varphi_0^-), (\varphi^-)^*), \quad \Omega_{R_\kappa/\Lambda \otimes R_\kappa, \lambda} W[[\Gamma^-]] \cong \mathcal{Y}(\varphi_0^-)
\]
as \( W[[\Gamma^-]] \)-modules.

This follows from the discussion in the previous section §6.18.

Since \( p \nmid [K(\varphi_0^-) : K] \), the \( p \)-Hilbert class field \( H/K \) and \( K(\varphi_0^-) \) is linearly disjoint over \( K \); so, we have \([H : K] = [HF, F] \); so, \( p \nmid h_F \) implies \( p \nmid h_K \). Thus combining the above theorem with the cyclicity result in §5.17, we get

**Corollary 2.** If \( p \nmid h_F \), \( \mathcal{Y}(\varphi_0^-) \) is a cyclic module over \( W[[\Gamma^-]] \) if \( \varphi_0 \neq 1 \).