By Kummer theory, we give an upper bound of the dimension \( \dim t_{R_{ord}/\Lambda} = \dim t_{R_x/W} \) by the dimension of the dual Selmer group, which turns out to be often optimal.
5.1. Local class field theory.
We summarize facts from local class field theory. Let $K/Q_p$ be a finite extension with algebraic closure $\overline{K}$ with integer ring $O$. Write $D := \text{Gal}(\overline{K}/K)$ fixing an algebraic closure $\overline{K}/K$. Let $D \triangleright I$ be the inertia subgroup and $D^{ab}$ be its maximal continuous abelian quotient.

- $x \mapsto [x, K] : K^\times \hookrightarrow D^{ab}$ (the local Artin symbol);
- $[\varpi, K]$ modulo the inertia subgroup $I^{ab} \subset D^{ab}$ is the Frobenius element $\text{Frob}$;
- For any integer $0 < m \in \mathbb{Z}$, $K^\times/(K^\times)^m = K^\times \otimes_{\mathbb{Z}} \mathbb{Z}/m\mathbb{Z} \cong D^{ab}/mD^{ab}$ by Artin symbol;
- $O^\times \cong I^{ab}$ by Artin symbol.
§5.2. Local cohomology.
We summarize facts from local cohomology.
• $inv : H^2(K, \mu_m(K)) \cong \mathbb{Z}/m\mathbb{Z}$ (the invariant map);
• $H^1(K, \mu_m) \cong K^\times/(K^\times)^m$ (Kummer theory valid for any field $K \supset \mathbb{Q}$).
This follows from the long exact sequence of $H^? (M) := H^? (K, M)$ associated to $\mu_m(K) \hookrightarrow K^\times \xrightarrow{x \mapsto x^m} K^\times$:

$$
\begin{array}{cccc}
H^0(K^\times) & \xrightarrow{x \mapsto x^m} & H^0(K^\times) & \rightarrow \\
\downarrow & & \downarrow & \\
K^\times & \xrightarrow{x \mapsto x^m} & K^\times & \rightarrow \end{array}
$$

$H^1(\mu_m) \rightarrow H^1(K^\times) \overset{(*)}{=} 0$

where the vanishing $(*)$ follows from Hilbert theorem 90.
§5.3. Local Tate duality.

For any finite (continuous) $D$-module $M$ killed by $0 < m \in \mathbb{Z}$, let $M^*(1) := \text{Hom}(M, \mu_m(\overline{K}))$ as Galois module acting by $\sigma \cdot \phi(x) = \sigma(\phi(\sigma^{-1}x))$ (called Tate dual). Then $M^*(1) \otimes_{\mathbb{Z}/m\mathbb{Z}} M \ni \phi \otimes x \mapsto \phi(x) \in \mu_m$ is a $\mathbb{Z}[D]$-morphism inducing a cup product pairing $H^r(M^*(1)) \times H^{2-r}(M) \to H^2(\mu_m)^{\text{inv}} \mathbb{Z}/m\mathbb{Z}$.

**Theorem 1** (J. Tate). *Cohomological dimension of $D$ is equal to 2 and the above pairing is perfect for $r = 0, 1, 2$.*

If $M = \mu_m(\overline{K})$, by definition $\mu_m = (\mathbb{Z}/m\mathbb{Z})^*(1)$. We know $H^1(\mu_m) = K^\times/(K^\times)^m$ and $H^1(\mathbb{Z}/m\mathbb{Z}) = \text{Hom}(D^{ab}/mD^{ab}, \mathbb{Z}/m\mathbb{Z})$. By local class field theory, $D^{ab}/mD^{ab} \cong K^\times/(K^\times)^m$; so, the duality in this case follows. One can deduce the proof of the duality in this special case basically by restricting to $\text{Gal}(\overline{K}/K(M))$ for the splitting field $K(M)$ of $M$ (see [MFG, Theorem 4.43]).
§5.4. Another example of local Tate duality. 
Consider $\text{Hom}(\text{Frob}^\widehat{\mathbb{Z}}, M) \subset H^1(K, M)$ for a finite $\mathbb{Z}/m\mathbb{Z}$-module $M$ on which $D$ acts trivially. Here Frob is the Frobenius element in $D/I$.

**Lemma 1.** The orthogonal complement of $\text{Hom}(\text{Frob}^\widehat{\mathbb{Z}}, M) \subset H^1(K, M)$ in the dual $H^1(K, M^*(1)) = K^\times \otimes_\mathbb{Z} M$ is given by $O^\times \otimes_\mathbb{Z} M$. In particular, the Tate duality between $H^1(K, \mu_m)$ and $H^1(K, \mathbb{Z}/m\mathbb{Z})$ gives rise to the tautological duality between $\text{Frob}^\widehat{\mathbb{Z}}/m\text{Frob}^\widehat{\mathbb{Z}}$ and $\text{Hom}(\text{Frob}^\widehat{\mathbb{Z}}, \mathbb{Z}/m\mathbb{Z})$.

The result for general $M$ follows from extending scalar to $M$; so, we may assume $M = \mathbb{Z}/m\mathbb{Z}$.
§5.5. Inflation-restriction.

To prove the lemma, we recall the inflation-restriction sequence. Let $G$ be a profinite group and $H$ is an open normal subgroup (so, $G/H$ is finite). If $M$ is a $G$-module, for a 1-cocycle $u : H \to M$, $g \cdot u := gu(g^{-1}hg)$ can be easily checked to be a one cocycle. If $u(h) = (h - 1)m$, we see $g \cdot u(h) = g(g^{-1}hg - 1)m = (hg - g)m = (h - 1)(gm)$; so, this preserves coboundaries, and hence $G/H$ acts on $H^1(H, M)$.

Since $H$ fixes $M^H = H^0(H, M)$, $M^H$ is a $G/H$-module. The inflation restriction exact sequence is

$$0 \to H^1(G/H, M^H) \overset{\text{Inf}}{\longrightarrow} H^1(G, M) \overset{\text{Res}}{\longrightarrow} H^1(H, M)^{G/H} \to H^2(G/H, M),$$

where $\text{Inf}(u)(g) = u(g \mod H)$ and $\text{Res}(u) = u|_H$ for cocycles. For a proof of this, see [MFG, Theorem 4.33].
§5.6. Proof of the lemma.
The last statement follows from the construction of pairing between $H^1(K, \mu_m)$ and $H^1(K, \mathbb{Z}/m\mathbb{Z})$ described in §5.2.

By the inflation-restriction sequence, we have an exact sequence $\text{Hom}(D/I, \mathbb{Z}/m\mathbb{Z}) \hookrightarrow \text{Hom}(D, \mathbb{Z}/m\mathbb{Z}) \twoheadrightarrow \text{Hom}(I, \mathbb{Z}/m\mathbb{Z})$ for the inertia group $I \triangleright D$. Since $D/I = \text{Frob}^\wedge \mathbb{Z}$, we have the following commutative diagram with exact rows:

\[
\begin{array}{ccc}
(O^\times/(O^\times)^m) & \xrightarrow{\iota} & (K^\times/(K^\times)^m) \\
\downarrow & & \downarrow \\
H^1(I, \mathbb{Z}/m\mathbb{Z}) & \rightarrow & H^1(K, \mathbb{Z}/m\mathbb{Z}) \\
\end{array}
\xrightarrow{\iota} \quad 
\begin{array}{ccc}
\text{Frob}^\wedge \mathbb{Z}/\text{Frob}^m \mathbb{Z} & \rightarrow & \\
\downarrow & & \downarrow \\
H^1(D/I, \mathbb{Z}/m\mathbb{Z}) & \rightarrow & H^1(D/I, \mathbb{Z}/m\mathbb{Z})
\end{array}
\]

Since the image of $I$ in $D^{ab}$ is given by $O^\times$, the result follows. □
§5.7. Dual Selmer group. By trace pairing \((x, y) = \text{Tr}(xy)\) the Galois modules \(ad(\bar{\rho})\) and \(Ad(\bar{\rho})\) are self dual; so, \(ad(\bar{\rho})^*(1) = ad(\bar{\rho})(1)\) and \(Ad(\bar{\rho})^*(1) = Ad(\bar{\rho})(1)\). The dual Selmer group of \(ad(\bar{\rho})\) and \(Ad(\bar{\rho})\) is defined as follows:

\[
\text{Sel}^\perp(Ad(\bar{\rho})(1)) := \text{Ker}(H^1(G_{\overline{Q}}, Ad(\bar{\rho})(1)) \rightarrow \prod_{l \in S \cup \{p\}} \frac{H^1(\mathbb{Q}_l, Ad(\bar{\rho})(1))}{D_{\chi, l}(\mathbb{F}[\epsilon])^\perp}),
\]

\[
\text{Sel}^\perp(ad(\bar{\rho})(1)) := \text{Ker}(H^1(G_{\overline{Q}}, ad(\bar{\rho})(1)) \rightarrow \prod_{l \in S \cup \{p\}} \frac{H^1(\mathbb{Q}_l, ad(\bar{\rho})(1))}{D_l(\mathbb{F}[\epsilon])^\perp}).
\]

Here “\(\perp\)” indicates the orthogonal complement under the Tate duality. We have the following bound due to R. Greenberg and A. Wiles:

**Lemma 2.** \(\dim_{\mathbb{F}} \text{Sel}(Ad(\bar{\rho})) \leq \dim_{\mathbb{F}} \text{Sel}^\perp(Ad(\bar{\rho})(1)).\)

This we admit. For a proof, see [MFG, Proposition 3.40].
§5.8. Details of $H^1(K, \mu_p) \cong K^\times \otimes_{\mathbb{Z}} \mathbb{F}_p$.

Here $K$ is any field. The connection map $\delta$ of the long exact sequence $H^0(K, M) \to H^0(N) \xrightarrow{\delta} H^1(L)$ of a short exact sequence $L \hookrightarrow M \twoheadrightarrow N$ is given as follows: Pick $n \in H^0(K, N)$ and lift it to $m \in M$. Then for $\sigma \in \text{Gal}(\overline{K}/K)$, $(\sigma - 1)m$ is sent to $(\sigma - 1)n = 0$ as $n$ is fixed by $\sigma$. Thus we may regard $u_m : \sigma \mapsto (\sigma - 1)m$ is a 1-cocycle with values in $L$. If we choose another lift $m'$, then $m' - m = l \in L$ and hence $u_{m'} - u_m = (\sigma - 1)l$ which is a coboundary. Thus we get the map $\delta$ sending $m$ to the class $[u_m]$.

Applying this, the cocycle $u_\alpha$ corresponding $\alpha \in K^\times/(K^\times)^p = K^\times \otimes \mathbb{F}_p$ is given by

$$u_\alpha(\sigma) = \sigma^{-1}(\sqrt[p]{\alpha}).$$
5.9. Unramifiedness of $u_\alpha$ at a prime $l \neq p$.

Let $K$ be an $l$-adic field which is a finite extension of $\mathbb{Q}_l$ for a prime $l \neq p$. If $\alpha \notin (K^\times)^p$, $\alpha' := l^p N \alpha \notin (K^\times)^p$ with $K[\sqrt[p]{\alpha}] = K[\sqrt[p]{\alpha'}]$ and $u_\alpha = u_{\alpha'}$. Replacing $\alpha$ by $\alpha'$ for a sufficiently large $N$, we may assume that $\alpha \in O \cap K^\times$.

The minimal equation of $\sqrt[p]{\alpha}$ is $f(X) = X^p - \alpha$. Since the derivative $f'(X) = pX^{p-1}$, the different of $K[\sqrt[p]{\alpha}]/K$ is a factor of $p\sqrt[p]{\alpha}^{p-1}$. Thus we find

\[
\boxed{u_\alpha \text{ is unramified } \iff \alpha \in O^\times}
\]

choosing $\alpha \in O \cap K^\times$. This can be also shown by noting that all conjugates of $\sqrt[p]{\alpha}$ is given by $\{\zeta \sqrt[p]{\alpha} | \zeta \in \mu_p\}$ which has $p$ distinct elements modulo $l$ if and only if $\alpha \in O^\times$. 
§5.10. Restriction to the splitting field of $Ad := Ad(\bar{\rho})$.

Let $F$ be the splitting field of $Ad := Ad(\bar{\rho})$; so, $F = \mathbb{Q}^{\text{Ker}(Ad)}$, and $K := F[\mu_p]$ is the splitting field of $Ad(1)$. Write $G := \text{Gal}(F/\mathbb{Q})$. Let $\mathfrak{g}_F = \text{Ker}(Ad|_{\mathfrak{g}_F})$. We realize Sel$_1(Ad(1))$ inside $H^1(F, Ad(1)) = F^\times \otimes_{\mathbb{Z}} Ad$. Assume

(CV) $H^j(F/\mathbb{Q}, Ad(1)^{\mathfrak{g}_K}) = 0$ for $j = 1, 2$,

which follows if $K = F[\mu_p] \neq F$ or $p \nmid [F : \mathbb{Q}]$. If $F[\mu_p] \neq F$, we see $Ad(1)^{\mathfrak{g}_F} = 0$ as $Ad$ is trivial over $\mathfrak{g}_F$. If $p \nmid [F : \mathbb{Q}] = |G|$, we note $H^q(G, M) = 0$ for any $\mathbb{F}[G]$-module $M$ [MFG, Prop. 4.21]. Again by inflation-restriction,

$$H^1(G, Ad(1)^{\mathfrak{g}_F}) \hookrightarrow H^1(\mathbb{Q}, Ad(1)) \rightarrow H^1(F, Ad(1))^G \rightarrow H^2(G, Ad(1)^{\mathfrak{g}_F}).$$

is exact. So

$$H^1(\mathbb{Q}, Ad(1)) \cong (F^\times \otimes_{\mathbb{Z}} Ad)^G.$$
§5.11. Kummer theory.
We analyze how $G$ acts on $F^\times \otimes_{\mathbb{F}_p} \text{Ad}$. The action of $\tau \in G$ is given by $\tau u(g) = \tau u(\tau^{-1}g\tau) = \text{Ad}(\tau)u(\tau^{-1}g\tau) \ (\tau \in G)$ for cocycle $u$ giving rise to a class in $H^1(F, \text{Ad}(1))$. For a basis $(v_1, v_2, v_3)$ of $\text{Ad}$ giving an identification $\text{Ad} = \mathbb{F}^3$, and write $u = vu$ for $u := t(u_1, u_2, u_3)$ (column vector) for $v = (v_1, v_2, v_3)$ (row vector) as a $\mathbb{F}^3$ valued cocycle; so, $\tau v = (\tau v_1, \tau v_2, \tau v_3) = v^t \text{Ad}(\tau)$. Since $u_j(g) = u_{\alpha_j}(g) = g^{-1}\sqrt{\alpha_j}$ for $\alpha_j \in F^\times \otimes_{\mathbb{Z}} \mathbb{F}$, rewriting $u_{\alpha} := u$, we have $\tau(v^\tau u_{\alpha}(\tau^{-1}g\tau)) = v^t \text{Ad}(\tau)u_{\tau\alpha}(g)$. Thus $\tau$-invariance implies

$$u_{\tau\alpha} := t(u_{\tau\alpha_1}, u_{\tau\alpha_2}, u_{\tau\alpha_3}) = t \text{Ad}(\tau)^{-1}u_{\alpha} \iff v^t \text{Ad}(\tau)u_{\tau\alpha}(g) = vu_{\alpha}.$$ 

Therefore inside $F^\times \otimes_{\mathbb{Z}} \mathbb{F}$, $\alpha_j$s span an $\mathbb{F}$-vector space on which $G$ acts by a factor of $\text{Ad} \cong t \text{Ad}^{-1}$. Thus we get

$$H^1(Q, \text{Ad} \otimes \varpi) \cong \text{Hom}_{\mathbb{F}[G]}(\text{Ad}, F^\times \otimes_{\mathbb{Z}} \mathbb{F}) =: (F^\times \otimes_{\mathbb{Z}} \mathbb{F})[\text{Ad}]. \quad (1)$$
§5.12. Selmer group as a subgroup of $F^\times \otimes_{\mathbb{Z}} F$.

**Theorem 2.** Let $O$ be the integer ring of $F$. If $p \nmid h_F = |\text{Cl}_F|$, we have the following inclusion

$$\text{Sel}^\perp(\text{Ad}(\bar{\rho})(1)) \hookrightarrow O^\times \otimes_{\mathbb{Z}} F[\text{Ad}(\bar{\rho})].$$

We start the proof of the theorem which ends in §5.15. Let $[u] \in \text{Sel}^\perp(\text{Ad}(\bar{\rho})(1))$ for a cocycle $u : \mathcal{O}_Q \to \text{Ad}(\bar{\rho})(1)$. Thus $u|_{\mathcal{O}_F}$ gives rise to $u_\alpha$ for $\alpha \in F^\times \otimes_{\mathbb{Z}} F[\text{Ad}(\bar{\rho})]$ by Kummer theory. Consider the fractional ideal $(\alpha) = \alpha O[\frac{1}{p}]$. Make a prime decomposition $(\alpha) = \prod_l l^{e(l)}$ in $O[\frac{1}{p}]$. Since $u_\alpha$ is unramified at all $l \neq p$, we find $p | e(l)$ as otherwise, $l$ ramifies in $F[\sqrt{\alpha}]$. So $(\alpha) = a^p$ for $a = \prod_l l^{e(l)/p}$.
§5.13. $l$-integrality ($l \neq p$).

If a local Kummer cocycle $u_\alpha$ associated to $\alpha \in F_v^\times \otimes_\mathbb{Z} F_p$ for $v \nmid p$ is unramified, then $\alpha$ vanishes in $(F_v^\times / O_v^\times) \otimes_\mathbb{Z} F_p$. The local cocycle is trivial if and only if $\alpha$ vanishes in $F_v^\times \otimes_\mathbb{Z} F_p$. If a global Kummer cocycle $u_\alpha$ for $\alpha \in F^\times \otimes_\mathbb{Z} F_p$ is trivial at $v | N$ and unramified outside $p$, then the principal ideal $\alpha O[\frac{1}{p}]$ is a $p$-power.

If $p \nmid h := h_F = |Cl_F|$, replacing $\alpha$ by $\alpha^h$ does not change the Kummer cocycle up to non-zero scalar. We do this replacement and write $\alpha$ instead of $\alpha^h$. Then $a$ is replaced by the principal ideal $a^h = (\alpha')$, and we find that $\alpha = \varepsilon a^p$ for $\varepsilon \in O[\frac{1}{p}]^\times$. Thus $u_\alpha = u_\varepsilon$. Therefore

\[
\text{Sel}^\perp(Ad(1)) \subset (O[\frac{1}{p}]^\times \otimes_\mathbb{Z} F)[Ad].
\]
§5.14. Case where $\overline{\rho}|_D$ is indecomposable for $D = \text{Gal}(F_p/\mathbb{Q}_p)$. By indecomposability, the matrix form of $Ad(\sigma)$ if $\overline{\rho}(\sigma) = \begin{pmatrix} \overline{\epsilon} & a \\ 0 & \overline{\delta} \end{pmatrix}$ ($a \neq 0$) with respect to the basis $\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \}$ is

$$\begin{pmatrix} \overline{\epsilon}\overline{\delta}^{-1} & -2\overline{\delta}^{-1}a - (\overline{\epsilon}\overline{\delta})^{-1}a^2 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

in short, $Ad$ is also an indecomposable $D$-module without trivial quotient. We have an exact sequence of $D$-modules:

$$O^\times \otimes_{\mathbb{Z}} F \hookrightarrow O[\frac{1}{p}]^\times \otimes_{\mathbb{Z}} F \xrightarrow{\xi \mapsto (\xi)} \bigoplus_{\sigma \in G/D} F e^e \sigma \cong \text{Ind}^G_D 1,$$

where $e$ is the order of the class of $p$ in $Cl_F$. By Shapiro’s lemma [MFG, Lemma 4.20, (4.27)], $\text{Ind}^G_D 1[Ad] = \text{Hom}_{F[G]}(Ad, \text{Ind}^G_D 1) = \text{Hom}_D(Ad|_D, 1) = 0$ by indecomposability; so, $\text{Sel}^\perp(Ad(1)) \subset (O^\times \otimes F)[Ad]$. 
§5.15. Case where $\overline{\rho}|_D$ is completely reducible.

In this case, we have

\[
\text{Ind}^G_D 1[Ad] = \text{Hom}_{F[G]}(Ad, \text{Ind}^G_D 1) = \text{Hom}_D(Ad|_D, 1) = F.
\]

If a cocycle $u : D_p \to Ad(1)$ restricted to the decomposition group $D_p = \text{Gal}(\overline{Q}/F_p)$ at $p$ project down non-trivially to $F_p^\times \otimes F[1]$ (i.e., $u \in H^1(Q_p, \mu_p \otimes F)$), by the lemma in §5.4, if $u$ is a dual Selmer cocycle it corresponds to an element in $O_p^\times \otimes F$. Since $p|p$ is arbitrary, we conclude again

\[
\text{Sel}^\perp(Ad(1)) \subset (O^\times \otimes_{\mathbb{Z}} F)[Ad].
\]

This finishes the proof of the theorem.
§5.16. *Dirichlet’s unit theorem.*

Fix a complex conjugation \( c \in G \) and \( C \) be the subgroup generated by \( c \). Let \( \infty \) be the set of complex places of \( F \). Dirichlet’s unit theorem is proven by considering \( \mathcal{O}^{\times} \xrightarrow{\text{Log}} \mathbb{R}^\infty := \prod_{\infty} \mathbb{R} \)

given by \( \text{Log}(\varepsilon) = (\log |\varepsilon|_v)_{v \in \infty} \) and showing \( \text{Im}(\text{Log}) \otimes_\mathbb{Z} \mathbb{R} = \text{Ker}(\mathbb{R}^\infty \xrightarrow{\text{Tr}} \mathbb{R}) \) for \( \text{Tr}(x_v)_v = \sum_v x_v \). The Galois group \( G \) acts by permutation on \( \infty \cong G/C \). Therefore \( \mathbb{R}^\infty \cong \text{Ind}_C^G 1 \). Thus \( (\mathcal{O}^{\times} \otimes \mathbb{Q}) \oplus 1 \cong \text{Ind}_C^G \mathbb{Q} 1 \).

If \( p \nmid |G| \), any \( \mathbb{F}[G] \)-module over \( \mathbb{F} \) is semi-simple; so, characterized by its trace. Therefore this descends to \( \mathcal{O}^{\times}/\mu_p(F) \otimes_\mathbb{Z} \mathbb{F} \) and

\[
\text{Ind}_C^G \mathbb{F} 1 \cong (\mathcal{O}^{\times}/\mu_p(F) \otimes_\mathbb{Z} \mathbb{F}) \oplus \mathbb{F} 1.
\]
§5.17. **Theorem** $\dim_F \text{Sel}^\perp(\text{Ad}(1)) \leq 1$ if $p \nmid |G|h_F$.

By Shapiro’s lemma, we have

$$(O^\times / \mu_p(F) \otimes \mathbb{Z} F)[\text{Ad}] = \text{Hom}_G(\text{Ad}, (O^\times / \mu_p(F)) \otimes \mathbb{Z} F)$$

$$\cong \text{Hom}_G(\text{Ad}, \text{Ind}_C^G F_1) \cong \text{Hom}_C(\text{Ad}|_C, F_1) \cong F,$$

since $\text{Ad}(c) \sim \text{diag}[-1, 1, -1]$. By irreducibility, $\mu_p(F)[\text{Ad}] = 0$; so, $(O^\times \otimes \mathbb{Z} F)[\text{Ad}] \cong F$. By §5.12, we have

$\text{Sel}^\perp(\text{Ad}(1)) \hookrightarrow (O^\times \otimes \mathbb{Z} F)[\text{Ad}] \cong F,$

we conclude $\boxed{\dim_F \text{Sel}^\perp(\text{Ad}(1)) \leq 1}$. Thus

**Theorem 3.** If $p \nmid |G|h_F$, then for any deformation $\rho \in D_X(A)$, $\text{Sel}(\text{Ad}(\rho))$ is generated by at most one element over $A$. 