The dimension $d$ of the tangent space of a local ring $R$ over $\mathbb{F}$ gives the number of generators of the ring $R$. We describe this fact. Using this fact, we prove that $\Omega_{R/W}$ is generated by $d$ elements as $R$-modules. We fix a generator $\varpi$ of the maximal ideal $m_W$ of $W$.

Hereafter, we fix a finite set $S$ of rational primes (including infinite places), and we let $\mathfrak{S}_\mathbb{Q}$ denote the Galois group over $\mathbb{Q}$ of the maximal extension unramified outside $S$. 
§3.1. Tangent spaces of local rings

To study noetherian property of deformation ring, here is a useful lemma for an object $R$ in $CL_W$:

**Lemma 1.** If $t^*_{R/W} = m_R / (m_R^2 + m_W)$ is a finite dimensional vector space over $F$, then $R \in CL_W$ is noetherian.

The space $t^*_{R/W}$ is called the co-tangent space of $R$ at $m_R = (\varpi) \in \text{Spec}(R)$ over $\text{Spec}(W)$. Define $t^*_R$ by $m_R / m_R^2$, which is called the (absolute) co-tangent space of $R$ at $m_R$.

Since we have an exact sequence:

$$
\begin{array}{c}
F \xrightarrow{\sim} \frac{m_W}{m_W^2} \xrightarrow{a \mapsto a \varpi} t^*_R \xrightarrow{} t^*_{R/W} \xrightarrow{} 0,
\end{array}
$$

we conclude that $t^*_R$ is of finite dimension over $F$ if $t^*_{R/W}$ is of finite dimensional.
§3.2. Proof of the lemma, Step. 1.
First suppose that $m_N^R = 0$ for sufficiently large $N$. Let $\bar{x}_1, \ldots, \bar{x}_m$ be an $\mathbb{F}$–basis of $t^*_R$. Choose $x_j \in R$ so that $x_j \mod m^2_R = \bar{x}_j$. and consider the ideal $\mathfrak{a}$ generated by $x_j$. We have $\mathfrak{a} = \sum_j Rx_j \hookrightarrow m_R$ (the inclusion).

After tensoring $R/m_R$, we have the surjectivity of the induced linear map: $\mathfrak{a}/m_R \mathfrak{a} \cong \mathfrak{a} \otimes_R R/m_R \twoheadrightarrow m \otimes_R R/m_R \cong m/m^2_R$ because \{\bar{x}_1, \ldots, \bar{x}_m\} is an $\mathbb{F}$–basis of $t^*_R$. This shows that $m_R = \mathfrak{a} = \sum_j Rx_j$ (NAK: Nakayama’s lemma applied to the cokernel of $R^m \ni (a_1, \ldots, a_m) \mapsto \sum_j a_j x_j \in m_R$).

Therefore $m_R^k/m_{R}^{k+1}$ is generated by the monomials in $x_j$ of degree $k$ as an $\mathbb{F}$–vector space.

In particular, $m_R^{N-1}$ is generated by the monomials in $(x_0 := \varpi, x_1, \ldots x_m)$ of degree $N - 1$. 
§3.3. **Inductive step.** Define $\pi : B = W[[X_1, \ldots, X_m]] \to R$ by $\pi(f(X_1, \ldots, X_m)) = f(x_1, \ldots, x_m)$. Since any monomial of degree $> N$ vanishes after applying $\pi$, $\pi$ is a well defined $W$–algebra homomorphism. Let $m = m_B = (\varpi, X_1, \cdots, X_m)$ be the maximal ideal of $B$. By definition,

$$\pi(m^{N-1}) = m^{N-1}_R.$$ 

Suppose now that $\pi(m^{N-j}) = m^{N-j}_R$, and try to prove the surjectivity of $\pi(m^{N-j-1}) = m^{N-j-1}_R$.

Since $m^{N-j-1}_R/m^{N-j}_R$ is generated by monomials of degree $N-j-1$ in $x_j$, for each $x \in m^{N-j-1}_R$, we find a homogeneous polynomial $P \in m^{N-j-1}_R$ of $x_1, \ldots, x_m$ of degree $N-j-1$ such that $x - \pi(P) \in m^{N-j}_R = \pi(m^{N-j})$. This shows $\pi(m^{N-j-1}) = m^{N-j-1}_R$.

Thus by induction on $j$, we get the surjectivity of $\pi$. 
§3.4. General case
Write $R = \lim_{\leftarrow i} R_i$ for Artinian rings $R_i$. The projection maps are onto: $t_{R_{i+1}}^* \rightarrow t_{R_i}^*$. Since $t_R^*$ is of finite dimensional, for sufficiently large $i$,

$$t_{R_{i+1}}^* \cong t_{R_i}^*.$$ 

Thus choosing $x_j$ as above in $R$, we have its image $x_j^{(i)}$ in $R_i$.

Use $x_j^{(i)}$ to construct $\pi_i : W[[X_1, \ldots, X_m]] \rightarrow R_i$ in place of $x_j$. Then $\pi_i$ is surjective as already shown, and

$$\pi = \lim_{\leftarrow i} \pi_i : W[[X_1, \ldots, X_m]] \rightarrow R$$

remains surjective, because projective limit of continuous surjections, if all sets involved are compact sets, remain surjective; so, $R$ is noetherian as profinite sets are compact.
§3.5. Tangent space as cohomology group

Let $R = R_{\bar{\rho}}$ be the universal ring for a mod $p$-Galois absolutely irreducible representation $\bar{\rho} : \mathfrak{G}_Q \to GL_n(F)$.

We identify $t^*_R/W$ with a certain cohomology group $H^1(\mathfrak{G}_Q, ad(\bar{\rho}))$ and in this way, we prove $\dim_F t^*_R/W < \infty$ (and hence $R_{\bar{\rho}}$ is noetherian).
§3.6. **Adjoint module** \(ad(\overline{\rho})\).

Let \(M_n(\mathbb{F})\) be the space of \(n \times n\) matrices with coefficients in \(\mathbb{F}\). We let \(\mathcal{G}_Q\) acts on \(M_n(\mathbb{F})\) by \(gv = \overline{\rho}(g)v\overline{\rho}(g)^{-1}\). This action is called the **adjoint** action of \(\mathcal{G}_Q\), and this \(\mathcal{G}_Q\)–module will be written as \(ad(\overline{\rho})\).

Write \(Z\) for the center of \(M_n(\mathbb{F})\) and define \(sl_n(\mathbb{F}) = \{X \in M_n(\mathbb{F}) | \text{Tr}(X) = 0\}\). Since \(\text{Tr}(aXa^{-1}) = \text{Tr}(X)\), \(sl_n(\mathbb{F})\) is stable under the adjoint action. This Galois module will be written as \(Ad(\overline{\rho})\).

If \(p \nmid n\), \(X \mapsto \frac{1}{n}\text{Tr}(X) \oplus (X - \frac{1}{n}\text{Tr}(X))\) gives rise to \(M_n(\mathbb{F}) = Z \oplus sl_n(\mathbb{F})\) stable under the adjoint action.

So we have \(ad(\overline{\rho}) = 1 \oplus Ad(\overline{\rho})\) if \(p \nmid n\), where \(1\) is the trivial representation.
3.7. Tangent space as cohomology

Lemma 2. Let $R = R_{\overline{\rho}}$ for an absolutely irreducible representation $\overline{\rho} : \mathfrak{G}_Q \to GL_n(\mathbb{F})$. Then

$$t_{R/W} = \text{Hom}_F(t^*_{R/W}, \mathbb{F}) \cong H^1(\mathfrak{G}_Q, ad(\overline{\rho})),$$

where $H^1(\mathfrak{G}_Q, ad(\overline{\rho}))$ is the continuous first cohomology group of $\mathfrak{G}_Q$ with coefficients in the discrete $\mathfrak{G}_Q$–module $V(ad(\overline{\rho}))$.

The space $t_{R/W}$ is called the tangent space of Spec$(R)/W$ at $m$.

In the following proof of the lemma, we write $G = \mathfrak{G}_Q$ and $R = R_{\overline{\rho}}$. 
§3.8. Proof, Step. 1, dual number

Let $A = \mathbb{F}[\varepsilon] = \mathbb{F}[X]/(X^2)$ with $X \leftrightarrow \varepsilon$.

Then $\varepsilon^2 = 0$.

We claim:

$$\text{Hom}_{W\text{-alg}}(R, A) \cong t_{R/W}.$$ 

Construction of the map.

Start with a $W$-algebra homomorphism $\phi : R \to A$. Write

$$\phi(r) = \phi_0(r) + \phi_\varepsilon(r)\varepsilon \quad \text{with } \phi_0(r), \phi_\varepsilon(r) \in \mathbb{F}.$$ 

Then the map is $\phi \mapsto \ell_\phi = \phi_\varepsilon|_{m_R}$.
§3.9. Step. 2, Well defined-ness of $\ell_\phi$

From $\phi(ab) = \phi(a)\phi(b)$, we get

$$\phi_0(ab) = \phi_0(a)\phi_0(b) \text{ and } \phi_\varepsilon(ab) = \phi_0(a)\phi_\varepsilon(b) + \phi_0(b)\phi_\varepsilon(a).$$

Thus $\phi_\varepsilon \in \text{Der}_W(R, F) \cong \text{Hom}_F(\Omega_{R/W} \otimes_R F, F)$. Since for any derivation $\delta \in \text{Der}_W(R, F)$, $\phi' = \phi_0 + \delta \varepsilon \in \text{Hom}_{W\text{-alg}}(R, A)$, we find

$$\text{Hom}_R(\Omega_{R/W} \otimes_R F, F) \cong \text{Der}_W(R, A) \cong \text{Hom}_{W\text{-alg}}(R, A).$$

and $\text{Ker}(\phi_0) = m_R$ because $R$ is local. Since $\phi$ is $W$–linear, $\phi_0(a) = \overline{a} = a \mod m_R$.

Thus $\phi$ kills $m_R^2$ and takes $m_R$ $W$–linearly into $m_A = F\varepsilon$; so, $\ell_\phi : t_R^* \to F$. For $r \in W$, $\overline{r} = r\phi(1) = \phi(r) = \overline{r} + \phi_\varepsilon(r)\varepsilon$, and hence $\phi_\varepsilon$ kills $W$; so, $\ell_\phi \in t_{R/W}$.
§3.10. Step. 3, $\phi \mapsto \ell_\phi$ is an injection.

Since $R$ shares its residue field $\mathbb{F}$ with $W$, any element $a \in R$ can be written as $a = r + x$ with $r \in W$ and $x \in m_R$.

Thus $\phi$ is completely determined by the restriction $\ell_\phi$ of $\phi_\varepsilon$ to $m_R$, which factors through $t_{R/W}^*$.

Thus $\phi \mapsto \ell_\phi$ induces an injective linear map $\ell : \text{Hom}_{W-\text{alg}}(R, A) \hookrightarrow \text{Hom}_{\mathbb{F}}(t_{R/W}^*, \mathbb{F})$.

Note $R/(m_R^2 + m_W) = \mathbb{F} \oplus t_{R/W}^* = \mathbb{F}[t_{R/W}^*]$ with the projection $\pi : R \rightarrow t_{R/W}^*$ to the direct summand $t_{R/W}^*$. Indeed, writing $\bar{r} = (r \mod m_R)$, for the inclusion $\iota : \mathbb{F} = W/m_W \hookrightarrow R/(m_R^2 + m_W)$, $\pi(r) = r - \iota(\bar{r})$. 
§3.11. Step. 4, $\phi \mapsto \ell_\phi$ is a surjection.
For any $\ell \in \text{Hom}_F(t_{R/W}^*, F)$, we extends $\ell$ to $R$ by putting $\ell(r) = \ell(\pi(r))$. Then we define $\phi : R \to A$ by $\phi(r) = \overline{r} + \ell(\pi(r))\varepsilon$. Since $\varepsilon^2 = 0$ and $\pi(r)\pi(s) = 0$ in $F[t_{R/W}^*]$, we have

$$rs = (\overline{r} + \pi(r))(\overline{s} + \pi(s)) = \overline{rs} + \overline{s}\pi(r) + \overline{r}\pi(s)$$

is an $W$–algebra homomorphism. In particular, $\ell(\phi) = \ell$, and hence $\ell$ is surjective.

By $\text{Hom}_R(\Omega_{R/W} \otimes_R F, F) \cong \text{Hom}_{W\text{-alg}}(R, A)$, we have

$$\text{Hom}_R(\Omega_{R/W} \otimes_R F, F) \cong \text{Hom}_F(t_{R/W}^*, F);$$

so, if $t_{R/W}^*$ is finite dimensional, we get

$$\Omega_{R/W} \otimes_R F \cong t_{R/W}^*.$$
§3.12. Step. 5, use of universality.

By the universality, we have

\[ \text{Hom}_{W-alg}(R, A) \cong \{ \rho : G \to GL_n(A) | \rho \mod m_A = \bar{\rho} \} / \sim. \]

Write \( \rho(g) = \bar{\rho}(g) + u'_\phi(g)\varepsilon \) for \( \rho \) corresponding to \( \phi : R \to A \). From the mutiplicativity, we have

\[ \begin{align*}
\bar{\rho}(gh) + u'_\phi(gh)\varepsilon &= \rho(gh) = \rho(g)\rho(h) \\
&= \bar{\rho}(g)\bar{\rho}(h) + (\bar{\rho}(g)u'_\phi(h) + u'_\phi(g)\bar{\rho}(h))\varepsilon,
\end{align*} \]

Thus as a function \( u' : G \to M_n(\mathbb{F}) \), we have

\[ u'_\phi(gh) = \bar{\rho}(g)u'_\phi(h) + u'_\phi(g)\bar{\rho}(h). \] (1)
§3.13. Step. 6, Getting 1-cocycle.

Define a map \( u_\rho = u_\phi : G \to ad(\bar{\rho}) \) by
\[
    u_\phi(g) = u'_\phi(g)\bar{\rho}(g)^{-1}.
\]
Then by a simple computation, we have
\[
    gu_\phi(h) = \bar{\rho}(g)u_\phi(h)\bar{\rho}(g)^{-1}
\]
from the definition of \( ad(\bar{\rho}) \). Then from the above formula (1), we conclude that
\[
    u_\phi(gh) = gu_\phi(h) + u_\phi(g).
\]
Thus \( u_\phi : G \to ad(\bar{\rho}) \) is a 1–cocycle. Thus we get an \( \mathbb{F} \)-linear map
\[
    t_{R/W} \cong \text{Hom}_{W-\text{alg}}(R, A) \to H^1(G_{\mathbb{Q}}, ad(\bar{\rho}))
\]
by \( \ell_\phi \mapsto [u_\phi] \).

By computation, for $x \in ad(\rho)$

$$\rho \sim \rho' \iff \rho(g) + u'_\rho(g)\varepsilon = (1 + x\varepsilon)(\rho(g) + u'_\rho(g)\varepsilon)(1 - x\varepsilon)$$

$$\iff u'_\rho(g) = x\rho(g) - \rho(g)x + u'_\rho(g)$$

$$\iff u_\rho(g) = (1 - g)x + u'_\rho(g).$$

Thus the cohomology classes of $u_\rho$ and $u'_\rho$ are equal if and only if $\rho \sim \rho'$. This shows:

$$\text{Hom}_F(t^*_R/W, \mathbb{F}) \cong \text{Hom}_{W-\text{alg}}(R, A) \cong \{\rho : G \to \text{GL}_n(A) | \rho \mod m_A = \rho\} / \sim \cong H^1(G, ad(\rho)).$$

In this way, we get a bijection between $\text{Hom}_F(t^*_R/W, \mathbb{F})$ and $H^1(G, ad(\rho))$. 
§3.15. $p$-Frattini condition.

For each open subgroup $H$ of a profinite group $G$, we write $H_p$ for the maximal $p$–profinite quotient. Define $p$–Frattini quotient $\Phi(H_p)$ of $H$ by $\Phi(H_p) = H_p / (H_p)^p(H_p, H_p)$ for the the commutator subgroup $(H_p, H_p)$ of $H_p$. We consider the following condition:

($\Phi$) For any open subgroup $H$ of $G$, $\Phi(H_p)$ is a finite group.

**Proposition 1** (Mazur). By class field theory, $\mathcal{G}_\mathbb{Q}$ satisfies ($\Phi$), and $R_{\mathcal{P}}$ is a noetherian ring. In particular, $t^*_{R/W}$ is finite dimensional over $\mathbb{F}$ and is isomorphic to $\Omega_{R/W} \otimes_R \mathbb{F}$ (see §3.11).

By this fact, hereafter we always assume that the deformation functor is defined over $CNL/W$. 
\[ \text{3.16. Proof.} \] Let \( H = \text{Ker}(\bar{\rho}) \). Then the action of \( H \) on \( \text{ad}(\bar{\rho}) \) is trivial. By the inflation-restriction sequence for \( G = \mathbb{G}_\mathbb{Q} \), we have the following exact sequence:

\[
0 \to H^1(G/H, H^0(H, \text{ad}(\bar{\rho}))) \to H^1(G, \text{ad}(\bar{\rho})) \\
\quad \to \text{Hom}(\Phi(H_p), M_n(F)).
\]

From this, it is clear that

\[
\dim_F H^1(G, \text{ad}(\bar{\rho})) < \infty.
\]

The fact that \( \mathbb{G}_\mathbb{Q} \) satisfies (\( \Phi \)) follows from class field theory. Indeed, if \( F \) is the fixed field of \( H \), then \( \Phi(H_p) \) fixes the maximal abelian extension \( M/F \) unramified outside \( p \). By class field theory, \([M : F]\) is finite. \( \square \)
§3.17. **Corollary:** \( \Omega_{R/W} \) is an \( R \)-module of finite type, and its minimal number of generators over \( R \) is equal to

\[
\dim_{\mathbb{F}} \Omega_{R/W} \otimes_R \mathbb{F} = \dim_{\mathbb{F}} t_{R/W}.
\]

**Proof.** For any \( R \)-module \( M \), Nakayama’s lemma tells us \( M \otimes_R \mathbb{F} = 0 \Rightarrow M = 0 \). Choose a basis \( B = \{ \bar{b} \} \) of \( M / \mathfrak{m}_R M = M \otimes_R \mathbb{F} \) and suppose \( B \) is finite. Lift \( \bar{b} \) to \( b \in M \), and consider the \( R \)-linear map \( \pi : \bigoplus_{g \in B} R \ni (a_{\bar{b}})_{\bar{b} \in B} \mapsto \sum_{\bar{b}} a_{\bar{b}} b \in M \). Tensoring \( \mathbb{F} \) over \( R \), we find \( \text{Coker}(\pi) \otimes_R \mathbb{F} = 0 \); so, \( \text{Coker}(\pi) = 0 \). This implies that \( \{ b | \bar{b} \in B \} \) is the minimal generators of \( M \) over \( R \). Apply this to \( M = \Omega_{R/W} \), we get the result by Mazur’s proposition (in §3.16). \( \square \)