We study the universal deformation ring in the case of characters (i.e., representation into $\text{GL}_1$) and computes congruence modules $C_0$ and $C_1$. As before, we fix an odd prime $p$. 
§2.1. Deformation of a character.

Let \( F/\mathbb{Q} \) be a number field with integer ring \( O \). We fix a set \( \mathcal{P} \) of properties of Galois characters. The property \( \mathcal{P} \) is often unramified outside \( p \), or in addition, deformed characters has prime-to-\( p \) conductor a factor of a fixed ideal \( \mathfrak{c} \) prime to \( p \). Fix a continuous character \( \rho : \text{Gal}(\overline{\mathbb{Q}}/F) \to F^\times \) with the property \( \mathcal{P} \).

A character \( \rho : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to A^\times \) for \( A \in \text{CL}_W \) is called a \( \mathcal{P} \)-deformation of \( \overline{\rho} \) if \((\rho \mod m_A) = \overline{\rho} \) and \( \rho \) satisfies \( \mathcal{P} \).

A couple \((R, \rho)\) (universal couple) made of an object \( R \) of \( \text{CL}_W \) and a character \( \rho : \text{Gal}(\overline{\mathbb{Q}}/F) \to R^\times \) satisfying \( \mathcal{P} \) is called a universal couple for \( \overline{\rho} \) if for any \( \mathcal{P} \)-deformation \( \rho : \text{Gal}(\overline{\mathbb{Q}}/F) \to A^\times \) of \( \overline{\rho} \), we have a unique morphism \( \phi_\rho : R \to A \) in \( \text{CL}_W \) (so it is a local \( W \)-algebra homomorphism) such that \( \phi_\rho \circ \rho = \rho \). By the universality, if exists, the couple \((R, \rho)\) is determined uniquely up to isomorphisms.
§2.2. Ray class groups of finite level.

Fix an $O$-ideal $c$. Recall

$$Cl_F(c) = \frac{\{\text{fractional } O\text{-ideals prime to } c\}}{\{(\alpha)|\alpha \equiv 1 \text{ mod } ^{\times}c_\infty\}},$$

Here $\alpha \equiv 1 \text{ mod } ^{\times}c$ means that $\alpha = a/b$ for $a, b \in O$ is totally positive (i.e., $\sigma(\alpha) > 0$ for all real embedding $F \xrightarrow{\sigma} \mathbb{R}$) such that $(b) + c = O$ and $a \equiv b \text{ mod } c$ (or equivalently, for all primes $l|c$, $\alpha \in O_l^{\times}$ and $\alpha \equiv 1 \text{ mod } l^{v_l(c_\infty)}$ if the $l$-primary factor of $c$ has exponent $v_l(c)$ (if $l|\infty$, it just means $\alpha$ is positive at $l$).

Write $H_{cp^n}/F$ for the ray class field modulo $cp^n$. In other words, there exists a unique abelian extension $H_{cp^n}/F$ only ramified at $cp\infty$ exists such that we can identify $\text{Gal}(H_{cp^n}/F)$ with the strict ray class group $Cl_F(cp^n)$ by sending a class of prime $l$ in $Cl_F(cp^n)$ to the Frobenius element $\text{Frob}_l \in \text{Gal}(H_{cp^n}/F)$. This isomorphism is called the Artin symbol.
§2.3. Ray class group of infinite level.
The group $\text{Cl}_F(cp^n)$ is finite as we have an exact sequence:

$$(O/cp^n)^\times \xrightarrow{\alpha \rightarrow (\alpha)} \text{Cl}_F(cp^n) \rightarrow \text{Cl}_F^+ \rightarrow 1$$

for the strict class group $\text{Cl}_F^+$ (we write the usual class group without condition at $\infty$ as $\text{Cl}_F$). Note that $|\text{Cl}_F^+|/|\text{Cl}_F|$ is a factor of $2^e$ for the number $e$ of real embeddings of $F$.

Sending a class $[a] \in \text{Cl}_F(cp^m)$ to the class $[a] \in \text{Cl}_F(cp^n)$ for $m > n$, we have a projective system $\{\text{Cl}_F(cp^n)\}_n$. Put $\text{Cl}_F(cp^\infty) = \lim_{\leftarrow n} \text{Cl}_F(cp^n)$. Then for $H_{cp^\infty} = \bigcup_n H_{cp^n}, \text{Cl}_F(cp^\infty) \cong \text{Gal}(H_{cp^\infty}/F)$ by $[l] \mapsto \text{Frob}_l$ for primes $l \nmid cp$.

If $F = \mathbb{Q}$ and $c = (N)$ for $0 < N \in \mathbb{Z}$, we have $H_{cp^n}$ is the cyclotomic field $\mathbb{Q}[\mu_{Np^n}]$ for the group $\mu_{Np^n}$ of $Np^n$-th roots of unity; so, $\text{Cl}_\mathbb{Q}(cp^n) \cong (\mathbb{Z}/Np^n\mathbb{Z})^\times$ and $\text{Cl}_\mathbb{Q}(cp^\infty) \cong (\mathbb{Z}/N\mathbb{Z})^\times \times \mathbb{Z}_p^\times$. 
§2.4. Groups algebra is universal. For a profinite abelian group $G$ with the maximal $p$-profinite ($p$-Sylow) quotient $\mathcal{G}_p$, consider the group algebra $W[[\mathcal{G}_p]] = \varprojlim_n W[\mathcal{G}_n]$ writing $\mathcal{G}_p = \varprojlim_n \mathcal{G}_n$ with finite $\mathcal{G}_n$. For example, $\Lambda = W[[\Gamma]] \ (\Gamma = 1 + p\mathbb{Z}_p = (1 + p)^\mathbb{Z}_p)$ (the Iwasawa algebra) is isomorphic to $W[[T]]$ by $1 + p \leftrightarrow t = 1 + T$. Suppose that $\mathcal{G}_p$ is finite. Fix a character $\chi: G \to \mathbb{F}^\times$. Since $\mathbb{F}^\times \hookrightarrow W^\times$, we may regard $\overline{\chi}$ as a character $\chi_0: G \to W^\times$ (Teichmüller lift of $\overline{\chi}$). Define $\kappa: \mathcal{G} \to W[[\mathcal{G}_p]]^\times$ by $\kappa(g) = \chi_0(g)g_p$ for the image $g_p$ of $g$ in $\mathcal{G}_p$. Note that $W[\mathcal{G}_p]$ is a local ring with residue field $\mathbb{F}$. For any continuous deformation $\chi: \mathcal{G} \to A^\times$ of $\overline{\chi}$, $\varphi: W[\mathcal{G}_p] \ni \sum_g a_g g \mapsto \sum_g a_g \chi\chi_0^{-1}(g) \in A$ gives a unique $W$-algebra homomorphism such that $\varphi \circ \kappa = \chi$. If $\mathcal{G}_p$ is infinite and $A = \varprojlim_n A_n$ for finite $A_n$ with $A_n = A/m_n$, $\chi_n := \chi\chi_0^{-1} \mod m_n: \mathcal{G} \to A_n^\times$ has to factor through $\mathcal{G}_{m(n)}$ by continuity, and we get $\varphi_n: W[\mathcal{G}_{m(n)}] \to A_n$ such that $\varphi_n \circ \kappa = \rho_n$. Passing to the limit, we have $\varphi \circ \kappa = \rho$ for $\varphi = \varprojlim_n \varphi_n: W[[\mathcal{G}_p]] \to A$. 
§2.5. Universal deformation ring for a Galois character $\bar{\rho}$.

Let $C_F(\mathfrak{c}p^\infty)$ for the maximal $p$-profinite quotient of $Cl_F(\mathfrak{c}p^\infty)$. If $\bar{\rho}$ has prime-to-$p$ conductor equal to $\mathfrak{c}$, we define a deformation $\rho$ to satisfy $\mathcal{P}$ if $\rho$ is unramified outside $\mathfrak{c}p$ and has prime-to-$p$ conductor a factor of $\mathfrak{c}$ (i.e., $\rho$ factors through $\text{Gal}(H_{\mathfrak{c}p^\infty}/F)$).

For the Teichmüller lift $\rho_0$ of $\bar{\rho}$ and the inclusion $\kappa : C_F(\mathfrak{c}p^\infty) \hookrightarrow W[[C_F(\mathfrak{c}p^\infty)]]$, we define $\rho(\sigma) := \rho_0(\sigma)\kappa(\sigma)$. Then the universality of the group algebra tells us

**Theorem 1.** The couple $(W[[C_F(\mathfrak{c}p^\infty)]], \rho)$ is universal among all $\mathcal{P}$-deformations. If $\bar{\rho}$ is unramified everywhere, $(W[[C_F]], \rho)$ for $C_F := Cl_{F,p}$ is universal among everywhere unramified deformations.
§2.6. Congruence modules for group algebras.
Let $H$ be a finite $p$-abelian group. If $m$ is a maximal ideal of $W[H]$, then for the inclusion $\kappa : H \hookrightarrow W[H]^{\times}$ with $\kappa(\sigma) = \sigma$, $\kappa \mod m$ is trivial as the finite field $W[H]/m$ has no non-trivial $p$-power roots of unity; so, $m$ is generated by $\{\sigma - 1\}_{h \in H}$ and $m_W$. Thus $m$ is unique and $W[H]$ is a local ring.

We have a canonical algebra homomorphism: $W[H] \to W$ sending $\sigma \in H$ to 1. This homomorphism is called the augmentation homomorphism of the group algebra. Write this map $\pi : W[H] \to W$. Then $b = \text{Ker}(\pi)$ is generated by $\sigma - 1$ for $\sigma \in H$. Thus

$$b = \sum_{\sigma \in H} W[H](\sigma - 1)W[H].$$

We compute the congruence module and the differential module $C_j(\pi, W) \ (j = 0, 1)$. 
§2.7. Theorem. We have

\[ C_0(\pi; W) \cong W/|H|W \quad \text{and} \quad C_1(\pi; W) = H \otimes_{\mathbb{Z}} W. \]

Proof for the congruence module.

Let \( K := \text{Frac}(W) \). Then \( \pi \) gives rise to the algebra direct factor \( K\varepsilon \subset K[H] \) for the idempotent \( \varepsilon = \frac{1}{|H|} \sum_{\sigma \in H} \sigma \). Thus \( a = K\varepsilon \cap W[H] = (\sum_{\sigma \in H} \sigma) \) and \( \pi(W(H))/a = (\varepsilon)/a \cong W/|H|W. \)
§2.8. Proof of $C_1(\pi; W) = H \otimes_{\mathbb{Z}} W$, 1st step.
Consider the functor $\mathcal{F} : CL_W \to SETS$ given by

$$\mathcal{F}(A) = \text{Hom}_{\text{group}}(H, A^\times) = \text{Hom}_{W\text{-alg}}(W[H], A).$$

Thus $R := W[H]$ and the character $\rho : H \to W[H]$ (the inclusion: $H \hookrightarrow W[H]$) are universal among characters of $H$ with values in $A \in CL_W$.

Then for any $R$-module $X$, consider $R[X] = R \oplus X$ with algebra structure given by $rx = 0$ and $xy = 0$ for all $r \in R$ and $x, y \in X$.

Define $\Phi(X) = \{\rho \in \mathcal{F}(R[X]) | \rho \mod X = \rho\}$. Write $\rho(\sigma) = \rho(\sigma) \oplus u'_\rho(\sigma)$ for $u'_\rho : H \to X$. 

Since

\[ \rho(\sigma \tau) \oplus u'_\rho(\sigma \tau) = \rho(\sigma \tau) \]
\[ = (\rho(\sigma) \oplus u'_\rho(\sigma))(\rho(\tau) \oplus u'_\rho(\tau)) \]
\[ = \rho(\sigma \tau) \oplus (u'_\rho(\sigma)\rho(\tau) + \rho(\sigma)u'_\rho(\tau)), \]

we have \( u'_\rho(\sigma \tau) = u'_\rho(\sigma)\rho(\tau) + \rho(\sigma)u'_\rho(\tau), \) and thus \( u_\rho := \rho^{-1}u'_\rho : H \to X \) is a homomorphism from \( H \) into \( X \).

This shows

\[ \text{Hom}(H, X) = \Phi(X). \]
§2.10. Proof, Third step.

Any $W$-algebra homomorphism $\xi : R \to R[X]$ with $\xi \mod X = \text{id}_R$ can be written as $\xi = \text{id}_R \oplus d\xi$ with $d\xi : R \to X$.

Since $(r \oplus x)(r' \oplus x') = rr' \oplus rx' + r'x$ for $r, r' \in R$ and $x.x' \in X$, we have $d\xi(rr') = rd\xi(r') + r'd\xi(r)$; so, $d\xi \in \text{Der}_W(R, X)$. By universality of $(R, \rho)$, we have

$$\Phi(X) \cong \{\xi \in \text{Hom}_{W\text{-alg}}(R, R[X]) | \xi \mod X = \text{id}\} = \text{Der}_W(R, X) = \text{Hom}_R(\Omega_{R/W}, X).$$
§2.11. Proof, Fourth step, good choice of $X$.

Thus taking $X = K/W$, we have

\[
\text{Hom}_W(H \otimes \mathbb{Z} W, K/W) = \text{Hom}(H, K/W) = \text{Hom}_R(\Omega_{R/W}, K/W) = \text{Hom}_W(\Omega_{R/W} \otimes_{R, \pi} W, K/W).
\]

By taking Pontryagin dual back, we have

\[
H \cong \Omega_{R/W} \otimes_{R, \pi} W = C_1(\pi; W).
\]
§2.12. Class group and Selmer group.

Let \( \text{Ind}^{\mathbb{Q}}_{\mathbb{F}} \text{id} = \text{id} \oplus \chi \) and \( H = C_F \). Then for \( \Omega_F \) given basically by the regulator and some power of \((2\pi i)\),

\[
|L(1, \chi)/\Omega_F|_p = \left| |C_F| \right|_p.
\]

We can identify \( C^\vee_F = \text{Hom}(C_F, \mathbb{Q}_p/\mathbb{Z}_p) \) with the Selmer group of \( \chi \) given by

\[
\text{Sel}_\mathbb{Q}(\chi) := \text{Ker}(H^1(\mathbb{Q}, V(\chi)^*)) \to \prod_l H^1(I_l, V(\chi)^*))
\]

for the inertia group \( I_l \subset \text{Gal}(\overline{\mathbb{Q}}_l/\mathbb{Q}_l) \).
§2.13. Class number formula.

Theorem 2 (Class number formula). Assume that $F/Q$ is a Galois extension and $p \nmid [F: \mathbb{Q}]$. For the augmentation homomorphism $\pi: W[C_F] \to W$, we have, for $r(W) = \text{rank}_{\mathbb{Z}_p} W$,

$$\left| \frac{L(1, \chi)}{\Omega_F} \right|_p^{r(W)} = |C_1(\pi; W)|^{-1} = |C_0(\pi; W)|^{-1} = |\text{Sel}_Q(\chi)|_p^{r(W)}$$

and $C_1(\pi; W) = C_F \otimes W$ and $C_0(\pi; W) = W/|C_F|W$. 