Overview of Math 207a Winter 2018

We give an overview of what we will do in this topic course. Any finite extension of $\mathbb{Q}$ inside $\mathbb{C}$ is called a number field. Write μ_N for the group of N-th roots of unity inside $\mathbb{C}^\times$ and $\mathbb{Q}(\mu_N)$ for the field generated by roots of unity in μ_N, which is called a cyclotomic field.

For any given number field K, the class group Cl_K defined by the quotient of the group of fractional ideals of K modulo principal ideals is a basic invariant of K. It is a desire of many algebraic number theorists to know the module structure of Cl_K. Or if $K/\mathbb{Q}$ is a Galois extension, $G_{K/\mathbb{Q}} := \mathrm{Gal}(K/\mathbb{Q})$ acts on Cl_K. Thus it might be easier to see the module structure of Cl_K over the group ring $\mathbb{Z}[G_{K/\mathbb{Q}}]$ larger than $\mathbb{Z}$.

The first step towards this goal of determining Cl_K for $K = \mathbb{Q}[\mu_N]$ was given in 1839 by Dirichlet as a formula of the order of the class group. The cyclotomic field K has its maximal real subfield K^+ and K/K^+ is a quadratic extension if N is odd with G_{K/K^+} generated by complex conjugation c. The norm map gives rise to a homomorphism $Cl_K \to Cl_K^+ := Cl_{K^+}$ whose kernel is written by Cl_K^- (the minus part of Cl_K). By the formula, if N is an odd prime p, the order of the Cl_K^- is given by

$$2p \prod_{\chi:(\mathbb{Z}/p\mathbb{Z})^\times \to \overline{\mathbb{Q}}^\times;\chi(-1)=-1} \frac{1}{p}\left(\sum_{j=1}^{p-1} \chi^{-1}(a)a\right) \quad \text{(Dirichlet/Kummer).}$$

Since $G_{K/\mathbb{Q}} \cong (\mathbb{Z}/p\mathbb{Z})^\times$ sending $\sigma_a \in G_{K/\mathbb{Q}}$ with $\sigma_a(\zeta) = \zeta^a$ $(\zeta \in \mu_p)$ to $a \in (\mathbb{Z}/p\mathbb{Z})^\times$, we have $\mathbb{Z}[G_{K/\mathbb{Q}}] \cong \mathbb{Z}[(\mathbb{Z}/p\mathbb{Z})^\times]$. Since each character χ of $G_{K/\mathbb{Q}}$ extends to an algebra homomorphism $\chi : \mathbb{Z}[G_{K/\mathbb{Q}}] \to \overline{\mathbb{Q}}$ sending σ_a to $\chi(a)$, Stickelberger guessed that

$$\theta_1 := \sum_{a=1}^{p-1} \frac{a}{p}\sigma_a^{-1} \text{ annihilates } Cl_K^- \text{ as } \chi(\theta_1) = \frac{1}{p}\left(\sum_{j=1}^{p-1} \chi^{-1}(a)a\right).$$

This "symbolic" statement means that $\mathfrak{A}^{\beta\theta_1}$ (for any fractional ideal $\mathfrak{A}$ of K) is principal as long as $\beta\theta_1 \in \mathbb{Z}[G_{K/\mathbb{Q}}]$ for $\beta \in \mathbb{Z}[G_{K/\mathbb{Q}}]$. Writing $\mathfrak{a}$ for the $\mathbb{Z}[G_{K/\mathbb{Q}}]$-ideal generated by elements of the form $\beta\theta_1 \in \mathbb{Z}[G_{K/\mathbb{Q}}]$, we might expect:

$$Cl_K^- \cong \mathbb{Z}[G_{K/\mathbb{Q}}]/\mathfrak{a}? \quad \text{(Cyclicity over } \mathbb{Z}[G_{K/\mathbb{Q}}])$$

which is not generally true. After supplying basics of cyclotomic fields, we will prove in the course Stickelberger's theorem:

$$Cl_K^- \otimes_{\mathbb{Z}} \mathbb{Z}_p \cong \mathbb{Z}_p[G_{K/\mathbb{Q}}]^-/(\mathfrak{a} \otimes \mathbb{Z}_p)^- \quad \text{(}p\text{-Cyclicity of the minus part)}$$

assuming Kummer–Vandiever conjecture: $p \nmid |Cl_K^+|$. Here $\mathfrak{A}^- = \{x \in \mathfrak{A} | cx = -x\}$ for complex conjugation c for an ideal $\mathfrak{A}$ of $\mathbb{Z}_p[G_{K/\mathbb{Q}}]$. Set $\Lambda = \mathbb{Z}_p[[T]]$ (one variable power series ring). Then we can easily prove that

$$\varprojlim_n \mathbb{Z}_p[G_{\mathbb{Q}[\mu_{p^n}]/\mathbb{Q}}] \cong \Lambda[\mu_{p-1}] \quad (\varprojlim_n \sigma_{1+p} \mapsto t = 1 + T),$$

where the limit is taken via restriction maps $G_{\mathbb{Q}[\mu_{p^{n+1}}]/\mathbb{Q}} \ni \sigma \mapsto \sigma|_{Q[\mu_{p^n}]} \in G_{\mathbb{Q}[\mu_{p^n}]/\mathbb{Q}}$. Then, assuming again Kummer–Vandiever conjecture, we further go on to show Iwasawa's way of proving his main conjecture and cyclicity of his Iwasawa module $X := \varprojlim_n (Cl^-_{\mathbb{Q}[\mu_{p^n}]} \otimes \mathbb{Z}_p)$:

$$X \cong \Lambda[\mu_{p-1}]^-/(L_p)$$

for the T-expansion L_p of the Kubota–Leopoldt p-adic L-function.