HOME WORK EXAM MATH 205C, 2009 SPRING

Solve three problems in the following list. The difficulty is indicated by the letter $AAA > AA > A$; so, if you solve completely a problem with higher rank, you have better chance of getting better course grade. The dead line of turning-in the homework is June 3 (the final day of teaching).

In the following problems, p always denotes a prime and $\mathbb{F}_p$ is the field with p elements..

AA1 (a) Compute $\mathrm{Ext}^1_{\mathbb{Z}[G]}(\mathbb{F}_p, \mathbb{F}_p)$ for a prime cyclic group G,
(b) For a nontrivial p-group G, if G acts on $\mathbb{F}_p$, the action has to be trivial,
(c) Show that $\mathrm{Ext}^1_{\mathbb{F}_p[G]}(\mathbb{F}_p, \mathbb{F}_p)$ cannot be trivial if G is a nontrivial p-group.

A1 Let K be a perfect field and $\overline{K}$ be an algebraic closure of K (here K is called perfect if all finite extensions of K inside $\overline{K}$ is separable). Let $G = \mathrm{Gal}(\overline{K}/K)$, and equip it with the topology whose system of neighborhoods of the identity is made up of subgroups $\mathrm{Gal}(\overline{K}/F)$ for all finite Galois extensions F/K inside $\overline{K}$. Show that G is a profinite group. Is G topologically finitely generated if K is a finite field?

AAA1 Let $\mathbb{F} = \overline{\mathbb{F}}_p$ be an algebraic closure of the finite field $\mathbb{F}_p$ with p elements for a prime p. Let $C = \mathbb{Z}$ on which $G = \mathrm{Gal}(\mathbb{F}/\mathbb{F}_p)$ acts trivially. Prove that (G, C) is a class formation. Is the reciprocity map in this case surjective? Hint: Prove $H^1_{ct}(G, \mathbb{Q}/\mathbb{Z}) \cong \mathbb{Q}/\mathbb{Z}$ and the (inverse of the) connection map $H^1_{ct}(G, \mathbb{Q}/\mathbb{Z}) \cong H^2_{ct}(G, \mathbb{Z})$ gives inv_G.

A2 Let G be a finite group and A be a commutative ring with identity. For any finitely generated $\mathbb{Z}[G]$-module X and any $A[G]$-module M, show $\mathrm{Hom}_{\mathbb{Z}[G]}(X, M) \cong \mathrm{Hom}_{A[G]}(X \otimes_{\mathbb{Z}} A, M)$. Hint: First prove this for a $\mathbb{Z}[G]$-free X module of finite rank, and then use a free presentation $\mathbb{Z}[G]^r \to \mathbb{Z}[G]^s \to X \to 0$.

AA2 Let C be a free $\mathbb{Z}[G]$-module of finite rank and M be a $\mathbb{Z}[G]$-module for a finite group G. Show that $M \otimes_{\mathbb{Z}[G]} C \cong \mathrm{Hom}_{\mathbb{Z}[G]}(\widehat{C}, M)$ by $\nu : m \otimes c \mapsto (\phi \mapsto \sum_{g \in G} \phi(gc)gm)$, where $\widehat{C} = \mathrm{Hom}_{\mathbb{Z}}(C, \mathbb{Z})$ on which $g \in G$ acts by $g\phi(c) = \phi(g^{-1}c)$. What happens if G is an infinite group?

A3 Show that the power series ring $\mathbb{Z}_p[[T_1, \ldots, T_n]]$ is a profinite ring ($0 < n \in \mathbb{Z}$).

AAA2 Let G be a profinite group, and $\mathcal{C}$ be the category of discrete G-modules (on which G acts continuously). Find an exact sequence $0 \to M \to N \to L \to 0$ of G-modules such that M and L are in $\mathcal{C}$ but N is not.

AA3 Show that $A[G]$ is not in $\mathcal{C}$ if G is an infinite profinite group, where $\mathcal{C}$ the category of discrete G-modules (on which G acts continuously).

AAA3 Let K/F be a finite Galois extension of a field F with Galois group $G = \mathrm{Gal}(K/F)$ (which could be of positive characteristic).
(a) Prove $H^q(G, K) = 0$ for all $q > 0$, where K is regarded as an F-vector space on which G acts naturally.
(b) Prove $H^1(G, K^\times) = 0$, where $K^\times$ is the multiplicative group of K regarded as a G-module.
(c) Find an example of $H^2(G, K^\times) \neq 0$.
(d) Prove for the separable closure F_s of F, $H^1_{ct}(\mathrm{Gal}(F_s/F), F_s^\times) = 0$.

I am happy to assist you in my office hours.

Date: May 20, 2009.