2. Modular $p$-adic $L$-functions

In this section, we will do the exactly the same construction of $p$-adic $L$-functions for elliptic Hecke eigenforms in place of rational functions on $\mathbb{G}_m$.

\section{Elliptic modular forms}

Let $\Gamma_0(N) = \{(a \ b \ c \ d) \in SL_2(\mathbb{Z}) | c \equiv 0 \mod N\}$. This a subgroup of finite index in $SL_2(\mathbb{Z})$.

Exercise 2.1. Let $P^1(A)$ be the projective space of dimension 1 over a ring $A$. Prove $[SL_2(\mathbb{Z}) : \Gamma_0(N)] = [P^1(\mathbb{Z}/N\mathbb{Z})] = N\prod_{\ell|N}(1 + \frac{1}{\ell})$ if $N$ is square-free, where $\ell$ runs over all prime factors of $N$. Hint: Let $(a \ b \ c \ d) \in SL_2(\mathbb{Z})$ acts on $P^1(A)$ by $z \mapsto \frac{az+b}{cz+d}$ and show that this is a transitive action if $A = \mathbb{Z}/N\mathbb{Z}$ and the stabilizer of $\infty$ is $\Gamma_0(N)$.

We let $(a \ b \ c \ d) \in GL_2(\mathbb{C})$ acts on $P^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$ by $z \mapsto \frac{az+b}{cz+d}$ (by linear fractional transformation).

Exercise 2.2. Prove the following facts:

1. there are two orbits of the action of $GL_2(\mathbb{R})$ on $P^1(\mathbb{C})$: $P^1(\mathbb{R})$ and $\mathfrak{H} \cup \overline{\mathfrak{H}}$, where $\mathfrak{H} = \{z \in \mathbb{C} | \text{Im}(z) > 0\}$ and $\overline{\mathfrak{H}} = \{z \in \mathbb{C} | \text{Im}(z) < 0\}$.

2. the stabilizer of $i = \sqrt{-1}$ is the center times $SO_2(\mathbb{R}) = \{(\cos(\theta), \sin(\theta)) | \theta \in \mathbb{R}\}$.

3. $\gamma \in GL_2(\mathbb{R})$ with $\det(\gamma) < 0$ interchanges the upper half complex plane $\mathfrak{H}$ and lower half complex plane $\overline{\mathfrak{H}}$.

4. the upper half complex plane is isomorphic to $SL_2(\mathbb{R})/SO_2(\mathbb{R})$ by $SL_2(\mathbb{R}) \ni g \mapsto g(\sqrt{-1}) \in \mathfrak{H}$.

Then $Y_0(N) = \Gamma_0(N) \backslash \mathfrak{H}$ is an open Riemann surface with hole at cusps. In other words, $X_0(N) = \Gamma_0(N) \backslash (\mathfrak{H} \cup P^1(\mathbb{Q}))$ is a compact Riemann surface.

Exercise 2.3. Show that $SL_2(K)$ acts transitively on $P^1(K)$ for any field $K$ by linear fractional transformation. Hint: $(\frac{1}{0} \ a)(0) = a$.

Let $f : \mathfrak{H} \to \mathbb{C}$ be a holomorphic functions with $f(z+1) = f(z)$. Since $\mathfrak{H}/\mathbb{Z} \cong D = \{z \in \mathbb{C} \ | \ |z| < 1\}$ by $z \mapsto q = e^{2\pi i z}$, we may regard $f$ as a function of $q$ undefined at $q = 0 \Leftrightarrow z = i\infty$. Then the Laurent expansion of $f$ gives

$$f(z) = \sum_n a(n, f)q^n = \sum_n a(n, f)\exp(2\pi inz).$$

In particular, we may assume that $q$ is the coordinate of $X_0(N)$ around the infinity cusp $\infty$. We call $f$ is finite (resp. vanishing) at $\infty$ if $a(n, f) = 0$ if $n < 0$ (resp. if $n \leq 0$). By Exercise 2.3, we can bring any point $c \in P^1(\mathbb{Q})$ to $\infty$; so, the coordinate around the cusp $c$ is given by $q \circ \alpha$ for $\alpha \in SL_2(\mathbb{Q})$ with $\alpha(c) = \infty$.

Exercise 2.4. Show that the above $\alpha$ can be taken in $SL_2(\mathbb{Z})$. Hint: write $c = \frac{a}{b}$ as a reduced fraction; then, we can find $x, y \in \mathbb{Z}$ such that $ax - by = 1$.

We consider the space of holomorphic functions $f : \mathfrak{H} \to \mathbb{C}$ satisfying the following conditions:
Exercise 2.5. Define $f \mid (a \ b \ c \ d) (z) = f(z)(cz + d)^2$. Prove the following facts:

1. $(f \mid \alpha) \mid \beta = f((\alpha \beta)$ for $\alpha \in SL_2(\mathbb{R})$.
2. If $f$ satisfies (M1), $f \alpha$ satisfies (M1) replacing $\Gamma_0(N)$ by $\Gamma = \alpha^{-1}\Gamma_0(N)\alpha$.
3. If $\alpha \in SL_2(\mathbb{Z})$, show that $\Gamma$ contains $\Gamma(N) = \{ \gamma \in SL_2(\mathbb{Z}) | \gamma - 1 \in N M_2(\mathbb{Z}) \}$.

By (3) of the above exercise, for $\alpha \in SL_2(\mathbb{Z})$, we find $f(\alpha(z + N) = f(\alpha(z))$; thus, $f \mid \alpha$ has expansion $f \mid \alpha = \sum_n a(n, f(\alpha)q^n$. We call $f$ is finite (resp. vanishing) at the cusp $\alpha^{-1}(\infty)$ if $f \mid \alpha$ is finite (resp. vanishing) at $\infty$.

(M2) $f$ is finite at all cusps of $X_0(N)$.

We write $M_2(\Gamma_0(N))$ for the space of functions satisfying (M1–2). Replace (M2) by (S) $f$ is vanishing at all cusps of $X_0(N)$, we define subspace $S_2(\Gamma_0(N)) \subset M_2(\Gamma_0(N))$ by imposing (S). Element in $S_2(\Gamma_0(N))$ is called a holomorphic cusp form on $\Gamma_0(N)$ of weight 2.

2.2. Modular cohomology group. Take a holomorphic differential $\omega$ on $X_0(N)$. Then we pull back $\omega$ to $\mathfrak{H}$ and still write $\omega$. We can write $\omega = f(z)dz$ on $\mathfrak{H}$ because $\mathfrak{H}$ is simply connected.

Exercise 2.6. For $\alpha = (a \ b \ c \ d) \in SL_2(\mathbb{R})$, prove $\alpha^*dz = d((a \ b \ c \ d) = (cz + d)^{-2}dz$.

Since $\gamma^*\omega = \omega$ for all $\gamma \in \Gamma_0(N)$, we find

$$f(z)dz = \omega = \gamma^*\omega = f(\gamma(z))\gamma^*dz = f(\gamma(z))(cz + d)^{-2}dz$$

if $\gamma = (a \ b \ c \ d)$. Thus $f$ has to satisfy (M1). At infinity, since $dz = \frac{2\pi i dq}{q}$, $\omega$ with respect to the coordinate $q$ is finite at $\infty$, and hence $f(z)dz = 2\pi i \frac{dq}{q} \sum a(n, f)q^n$ is finite at $q = 0$. This implies $f$ has to be vanishing at $\infty$. Writing $H^0(X_0(N), \Omega_{X_0(N)/\mathbb{C}})$ for the space of holomorphic 1-forms on $X_0(N)$, we thus find

Proposition 2.7. We have a canonical isomorphism $S_2(\Gamma_0(N)) \cong H^0(X_0(N), \Omega_{X_0(N)/\mathbb{C}})$ sending $f$ to $f(z)dz$.

Let $C$ be the divisor on $X_0(N)$ which is the formal sum of all cusps. If we write $H^0(X_0(N), \Omega_{X_0(N)/\mathbb{C}}(-C))$ for the space of meromorphic 1-forms on $X_0(N)$ with at most simple poles at cusps, by the same argument, we have

Corollary 2.8. We have an isomorphism $M_2(\Gamma_0(N)) \cong H^0(X_0(N), \Omega_{X_0(N)/\mathbb{C}}(-C))$ sending $f$ to $f(z)dz$.

For any compact Riemann surface $X$, general theory of Riemann surface tells us $H^1(X, \mathbb{C}) \cong H^0(X, \Omega_X/\mathbb{C}) + H^0(X, \overline{\Omega}_X/\mathbb{C})$ (the Hodge decomposition, where $H^0(X, \Omega_X/\mathbb{C})$ is the space of holomorphic 1-forms on $X$ and $H^0(X, \overline{\Omega}_X/\mathbb{C})$ is the space of antiholomorphic 1-forms on $X$. Since $H^0(X, \overline{\Omega}_X/\mathbb{C})$ is the complex conjugate of $H^0(X, \Omega_X/\mathbb{C})$, we get
Proposition 2.9. We have a canonical isomorphism
\[ H^1(X_0(N), \mathbb{C}) \cong S_2(\Gamma_0(N)) \oplus \overline{S}_2(\Gamma_0(N)), \]
where \( \overline{S}_2(\Gamma_0(N)) \) is made up of complex conjugate \( \overline{f} \) for \( f \in S_2(\Gamma_0(N)) \). In particular, \( S_2(\Gamma_0(N)) \) is finite dimensional, and its dimension is given by the genus of \( X_0(N) \).

We add a small circle at each cusp of \( Y_0(N) \) and getting a different compactification \( \overline{Y}_0(N) \) of \( Y_0(N) \) from \( X_0(N) \). Taking the circle \( S \) around the cusp \( c \). Then \( \int_S \omega \) is essentially the residue of \( \omega \) and if we write \( \omega = f(z)dz \) it is given by \( a(0, 0|\alpha) \) for \( \alpha \in SL_2(\mathbb{Z}) \) taking the cusp to \( \infty \). Thus we get

Corollary 2.10. If we write \( g_0(N) \) for the genus of \( X_0(N) \) and \( c_0(N) \) for the number of cusps of \( X_0(N) \), the dimension of the space \( M_2(\Gamma_0(N)) \) is bounded by \( g_0(N) + c_0(N) \).

In fact, it is equal to \( g_0(N) + c_0(N) - 1 \)

The fact that the dimension is one less than \( g_0(N) + c_0(N) \) follows from the fact that \( M_2(SL_2(\mathbb{Z})) = 0 \) (or \( H^1(X_0(1), \Omega_{X_0(1)}/\mathbb{C}(-C)) = 0 \), because a punctured sphere is still simply connected).

By the de Rham theorem, we have the following duality given by integration:

Proposition 2.11. The space \( H_1(X_0(N), \mathbb{C}) \) and \( H_1(\overline{Y}_0(N), \partial \overline{Y}_0(N); \mathbb{C}) \) are dual to \( H^1(X_0(N), \mathbb{C}) \cong S_2(\Gamma_0(N)) \oplus \overline{S}_2(\Gamma_0(N)) \), and \( H_1(\overline{Y}_0(N), \mathbb{C}) \) is dual to \( M_2(\Gamma_0(N)) \oplus \overline{S}_2(\Gamma_0(N)) \).

2.3. Hecke operators. Let \( GL_2^+(\mathbb{R}) = \{ \alpha \in GL_2(\mathbb{R}) | \det(\alpha) > 0 \} \) and put \( GL_2^+(A) = GL_2^+(\mathbb{R}) \cap GL_2(A) \) for \( A \subset \mathbb{R} \). For \( \alpha = ( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} ) \in GL_2^+(\mathbb{R}) \) and a function \( f : \mathfrak{H} \to \mathbb{C} \), we define \( f(\alpha(z)) = \det(\alpha)f(\alpha(z))(cz + d)^{-2} \).

Exercise 2.12. Prove \( |f(\alpha)|\beta = f(|\alpha(\beta) \) for \( \alpha, \beta \in GL_2^+(\mathbb{R}) \).

Then \( f \in S_2(\Gamma_0(N)) \) (resp. \( f \in M_2(\Gamma_0(N)) \)) if and only if \( f \) vanishes (resp. finite) at all cusps of \( X_0(N) \) and \( f|\gamma = f \) for all \( \gamma \in \Gamma_0(N) \). Let \( \Gamma = \Gamma_0(N) \). For \( \alpha \in GL_2(\mathbb{R}) \) with \( \det(\alpha) > 0 \), if \( \Gamma \alpha \Gamma \) can be decomposed into a disjoint union of finite left cosets \( \Gamma \alpha \Gamma = \bigsqcup_{j=1}^h \Gamma \alpha_j \), we can think of the finite sum \( g = \sum_j f|\alpha_j \). If \( \gamma \in \Gamma \), then \( \alpha_j \gamma \in \Gamma \alpha_{\sigma(j)} \) for a unique index \( 1 \leq \sigma(j) \leq h \) and \( \sigma \) is a permutation of \( 1, 2, \ldots, h \). If further, \( f|\gamma = f \) for all \( \gamma \in \Gamma \), we have
\[
g|\gamma = \sum_j f|\alpha_j \gamma = \sum_j f|\gamma_j \alpha_{\sigma(j)} = \sum_j (f|\gamma_j)|\alpha_{\sigma(j)} = \sum_j f|\alpha_{\sigma(j)} = g.\]

Thus under the condition that \( f|\gamma = f \) for all \( \gamma \in \Gamma \), \( f \mapsto g \) is a linear operator only dependent on the double coset \( \Gamma \alpha \Gamma \); so, we write \( g = f|\Gamma \alpha \Gamma \). More generally, if we have a set \( T \subset GL_2^+(\mathbb{R}) \) such that \( \Gamma T \Gamma = T \) with finite \( |\Gamma \backslash T| \), we can define the operator \( [T] \) by \( f \mapsto \sum_j f|T_j \) if \( T = \bigsqcup_j \Gamma T_j \). We define
\[
\Delta_0(N) = \{ (\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in M_2(\mathbb{Z}) \cap GL_2^+(\mathbb{R}) | c \equiv 0 \mod N, \ a\mathbb{Z} + \mathbb{Z} = \mathbb{Z} \}.
\]

Exercise 2.13. Prove that \( \Gamma \Delta_0(N) \Gamma = \Delta_0(N) \) for \( \Gamma = \Gamma_0(N) \).
Lemma 2.14. Let $\Gamma = \Gamma_0(N)$.

1. If $\alpha \in M_2(\mathbb{Z})$ with positive determinant, $|\Gamma \setminus (\Gamma \alpha \Gamma)| < \infty$;
2. If $p$ is a prime,

$$
\Gamma \left( \begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix} \right) \Gamma = \left\{ \alpha \in \Delta_0(N) \mid \det(\alpha) = p \right\} = \begin{cases} 
\Gamma \left( \begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix} \right) \sqcup \bigcup_{j=0}^{p-1} \Gamma \left( \begin{smallmatrix} 1 & j \\ 0 & p \end{smallmatrix} \right) & \text{if } p \nmid N, \\
\bigcup_{j=0}^{p-1} \Gamma \left( \begin{smallmatrix} 1 & j \\ 0 & p \end{smallmatrix} \right) & \text{if } p|N.
\end{cases}
$$

3. for an integer $n > 0$,

$$
T_n := \left\{ \alpha \in \Delta_0(N) \mid \det(\alpha) = n \right\} = \bigcup_{a=b=0}^{d-1} \Gamma_0(N) \left( \begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix} \right) \quad (a > 0, ad = n, (a, N) = 1, a, b, d \in \mathbb{Z}),
$$

4. Write $T(n)$ for the operator corresponding to $T_n$. Then we get the following identity of Hecke operators for $f \in M_2(\Gamma_0(N))$:

$$
a(m, f|T(n)) = \sum_{0 < d|(m,n), (d,N)=1} d \cdot a\left(\frac{mn}{d^2}, f\right).
$$

5. $T(m)T(n) = T(n)T(m)$ for all integers $m$ and $n$.

Proof. Note that (1) and (2) are particular cases of (3). We only prove (2), (4) when $n = p$ for a prime $p$ and (5), leaving the other cases as an exercise (see [IAT] Proposition 3.36 and (3.5.10) for a detailed proof of (3) and (4)).

We first deal with (2). Since the argument in each case is essentially the same, we only deal with the case where $p \nmid N$ and $\Gamma = \Gamma_0(N)$. Take any $\gamma = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in M_2(\mathbb{Z})$ and $ad - bc = p$. If $c$ is divisible by $p$, then $ad$ is divisible by $p$; so, one of $a$ and $d$ has a factor $p$. We then have

$$
\gamma = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) = \left( \frac{a}{p} \frac{b}{d} \right) \left( \begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix} \right) \in \Gamma_0(N) \left( \begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix} \right)
$$

if $a$ is divisible by $p$. If $d$ is divisible by $p$ and $a$ is prime to $p$, choosing an integer $j$ with $0 \leq j \leq p - 1$ with $ja \equiv b \mod p$, we have $\gamma \left( \begin{smallmatrix} 1 & j \\ 0 & p \end{smallmatrix} \right)^{-1} \in GL_2(\mathbb{Z})$. If $c$ is not divisible by $p$ but $a$ is divisible by $p$, we can interchange $a$ and $c$ via multiplication by $\left( \begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix} \right)$ from the left-side. If $a$ and $c$ are not divisible by $p$, choosing an integer $j$ so that $ja \equiv -c \mod p$, we find that the lower left corner of $\left( \begin{smallmatrix} 1 & 0 \\ j & 1 \end{smallmatrix} \right)$ $\gamma$ is equal to $ja + c$ and is divisible by $p$. This finishes the proof of (2).

We now deal with (4) assuming $n = p$. By (2), we have

$$
(2.1) \quad f|T(p) = \begin{cases} 
p \cdot f(pz) + \sum_{j=0}^{p-1} f \left( \frac{z+j}{p} \right) & \text{if } p \nmid N, \\
\sum_{j=0}^{p-1} f \left( \frac{z+j}{p} \right) & \text{if } p|N.
\end{cases}
$$

Writing $f = \sum_{n=1}^{\infty} a(n, f)q^n$ for $q = e(z)$, we find

$$
a(m, f|T(p)) = a(mp, f) + p \cdot a\left(\frac{m}{p}, f\right).
$$
Here we put \(a(r, f) = 0\) unless \(r\) is a non-negative integer.

The formula of Lemma 2.14 (4) is symmetric with respect to \(m\) and \(n\); so, we conclude \(T(m)T(n) = T(n)T(m)\). This proves (5). \(\square\)

**Exercise 2.15.** Give a detailed proof of the above lemma.

The following exercise is more difficult:

**Exercise 2.16.** Let \(\Gamma = SL_2(\mathbb{Z})\). Prove that \(|\Gamma \setminus (\Gamma \alpha \Gamma)| < \infty\) for \(\alpha \in GL_2(\mathbb{R})\) if and only if then \(\alpha \in M_2(\mathbb{Q})\) modulo real scalar matrices.

Write \(\pi : \mathbb{H} \to Y_0(N) = \Gamma \setminus \mathbb{H}\) for the quotient map.

**Lemma 2.17.** If \(\Gamma \alpha \Gamma = \bigsqcup_{j=1}^h \Gamma \alpha_j\) for \(\Gamma = \Gamma_0(N)\) and \(\alpha \in GL_2(\mathbb{R})\), then for a chain \(c \in C_1(\mathbb{H}; A)\) with \(\partial(c) = \pi(\partial(c)) = 0\), \(\partial(\pi(\sum_{j=1}^h \alpha_j(c))) = 0\).

**Proof.** If \(\pi(\partial c)\) in \(Y_0(N)\) vanishes, writing \(\partial c = \sum_z a_z[z]\) for points \(z\) in \(\mathbb{H}\), we may assume that \(a_\gamma + a_{\gamma(z)} = 0\) for some \(\gamma \in \Gamma\). If \(\Gamma \alpha \Gamma = \bigsqcup_{j=1}^h \Gamma \alpha_j\), then \(\alpha_j \gamma = \gamma_j \alpha_{\sigma(j)}\) and \(\sum_j (a_{\alpha_j(z)} + a_{\alpha_j(z)}) = \sum_j (a_{\alpha_j(z)} + a_{\gamma_j \alpha_{\sigma(j)}(z)}) = 0\). This shows that \(\pi(\partial(\sum_j \alpha_j(c))) = 0\), which finishes the proof. \(\square\)

Obviously, for any 2-chain \(c\), \(\pi(\sum_j \partial \alpha_j(c)) = \pi(\sum_j \partial \alpha_j(c)))\), and therefore, the operator \(c \mapsto \sum_j \partial \alpha_j(c)\) preserves boundaries and cycles. In this way, the Hecke operator \([\Gamma \alpha \Gamma]\) acts on \(H_1(Y_0(N), A)\), and hence on \(H^1(Y_0(N), A)\) by the definition of cohomology group. On \(H^1_{DR}(Y_0(N), \mathbb{C})\), the action of \([\Gamma \alpha \Gamma]\) is given by \([\omega] \mapsto [\sum_{j=1}^h \alpha_j^* \omega]\).

Since \(z \mapsto \alpha_j(z)\) takes cusps to cusps, we can show similarly that Hecke operators act on \(H_1(X_0(N), A)\) and \(H_1(Y_0(N), \partial \overline{Y}_0(N), A)\). Since we can verify \(\alpha_j^* f(z)dz = (f(\alpha)dz by the chain rule, the Eichler-Shimura isomorphism \(S_2(\Gamma_0(N)) \oplus \overline{S}_2(\Gamma_0(N)) \cong H^1(X_0(N), \mathbb{C})\) is equivariant under Hecke operators.

**2.4. Duality.** Let \(A \subset \mathbb{C}\) be a subring, and define

\[
S_2(\Gamma_0(N), A) = \{ f \in S_2(\Gamma_0(N)) \mid a(n, f) \in A \}.
\]

By definition, \(S_2(\Gamma_0(N), \mathbb{C}) = S_2(\Gamma_0(N))\). By Lemma 2.14, \(T(n)\) preserves the \(A\)-submodule \(S_2(\Gamma_0(N), A)\) of \(S_2(\Gamma_0(N))\). Define

\[
h(N, A) = A[T(n)|n = 1, 2, \ldots] \subset \text{End}_A(S_2(\Gamma_0(N), A)),
\]

and call \(h(N, A)\) the Hecke algebra on \(\Gamma_0(N)\). By Lemma 2.14 (5), \(h(N, A)\) is a commutative \(A\)-algebra.

We define an \(A\)-bilinear pairing

\[
\langle \cdot, \cdot \rangle : h(N, A) \times S_2(\Gamma_0(N), A) \to A
\]

by \(\langle h, f \rangle = a(1, f|h)\).
Proposition 2.18.  (1) We have the following canonical isomorphism:

\[ \text{Hom}_A(S_2(\Gamma_0(N), A), A) \cong h(N, A) \quad \text{and} \quad \text{Hom}_A(h(N, A), A) \cong S_2(\Gamma_0(N), A), \]

and the latter is given by sending an \( A \)-linear form \( \phi : h(N, A) \to A \) to the \( q \)-expansion \( \sum_{n=1}^{\infty} \phi(T(n))q^n \).

(2) (Shimura) We have

\[ S_2(\Gamma_0(N), A) = S_2(\Gamma_0(N), \mathbb{Z}) \otimes A \quad \text{and} \quad h(N, A) = h(N, \mathbb{Z}) \otimes A. \]

Proof. We start with proving the result for a subfield \( A \) of \( \mathbb{C} \). Since \( h(N, A) \) and \( S_2(\Gamma_0(N), A) \) are both finite dimensional, we only need to show the non-degeneracy of the pairing. By Lemma 2.14 (4), we find \( (T(n), f) = a(n, f) \); so, if \( (h, f) = 0 \) for all \( n \), we find \( f = 0 \). If \( (h, f) = 0 \) for all \( f \), we find

\[ 0 = (h, f[T(n)]) = a(1, f[T(n)]h = a(1, f[hT(n)]) = (T(n), f[h] = a(n, f[h]). \]

Thus \( f[h = 0 \) for all \( f \), implying \( h = 0 \) as an operator.

We have the Poincaré duality pairing \( \langle \cdot, \cdot \rangle : H^1(X_0(N), A) \times H^1(X_0(N), A) \to A \) which is a perfect pairing. Define \( \Theta : H^1(X_0(N), A) \otimes_A H^1(X_0(N), A) \to S_2(\Gamma_0(N), A) \) by \( \Theta(\xi \otimes \eta) = \sum_{n=1}^{\infty} (\xi, \eta|T(n))q^n \). Indeed, \( h \mapsto (\xi|\eta|h) \) is an \( A \)-linear form on \( h(N, A) \), and by the result already proven, we have \( \sum_{n=1}^{\infty} (\xi, \eta|T(n))q^n \in S_2(\Gamma_0(N), A) \) if \( A = \mathbb{C} \) is a field. By Proposition 2.9, \( \Theta \) is surjective if \( A = \mathbb{C} \). Indeed, by the self-duality of \( H^1(X_0(N), \mathbb{C}) \), the projection \( H^1(X_0(N), A) \to S_2(\Gamma_0(N)) \) induces a \( T(n) \)-equivariant inclusion \( h(N, \mathbb{C}) = \text{Hom}_{\mathbb{C}}(S_2(\Gamma_0(N), \mathbb{C}) \to H^1(X_0(N), \mathbb{C}) \), and thus any linear form on \( h(N, \mathbb{C}) \) is a linear combination of \( h \mapsto (\xi|\eta, h) \). This is equivalent to the surjectivity of \( \Theta \) over \( \mathbb{C} \).

Since \( H^1(X_0(N), \mathbb{Z}) \otimes A = H^1(X_0(N), A) \), the image under \( \Theta \) of \( H^1(X_0(N), \mathbb{Z}) \otimes H^1(X_0(N), \mathbb{Z}) \) spans \( S_2(\Gamma_0(N), \mathbb{C}) \). Thus \( S_2(\Gamma_0(N), \mathbb{Z}) \) span \( S_2(\Gamma_0(N), \mathbb{C}) \). This shows

\[ S_2(\Gamma_0(N), \mathbb{Z}) \otimes_{\mathbb{Z}} A = S_2(\Gamma_0(N), A), \]

and therefore

\[ S_2(\Gamma_0(N), \mathbb{Z}) \otimes_{\mathbb{Z}} A = S_2(\Gamma_0(N), A). \]

for any ring \( A \). In particular, \( h(N, A) \) is a subalgebra of \( \text{End}_{\mathbb{C}}(S_2(\Gamma_0(N))) \) generated over \( A \) by \( T(n) \) for all \( n \). Then by definition \( h(N, A) = h(N, \mathbb{Z}) \otimes_{\mathbb{Z}} A \) for any subring \( A \subset \mathbb{C} \).

As for \( A = \mathbb{Z} \), we only need to show that \( \phi \mapsto \sum_{n=1}^{\infty} \phi(T(n))q^n \) is well defined and is surjective onto \( S_2(\Gamma_0(N), \mathbb{Z}) \) from \( h(N, \mathbb{Z}) \), because this is the case if we extend scalar to \( A = \mathbb{Q} \). The cusp form \( f \in S_2(\Gamma_0(N), A) \) corresponding to \( \phi \) satisfies \( (h, f) = \phi(h) \); so, \( a(n, f) = (T(n), f) = \phi(T(n)) \). Thus \( f = \sum_{n=1}^{\infty} \phi(T(n))q^n \in S_2(\Gamma_0(N), A) \). However

\[ f \in S_2(\Gamma_0(N), \mathbb{Z}) \iff \phi \in \text{Hom}(h(N, \mathbb{Z}), \mathbb{Z}), \]

because \( h(N, \mathbb{Z}) \) is generated by \( T(n) \) over \( \mathbb{Z} \). This is enough to conclude surjectivity.

Since \( h(N, A) = h(N, \mathbb{Z}) \otimes A \) and \( S_2(\Gamma_0(N), \mathbb{Z}) \otimes_{\mathbb{Z}} A = S_2(\Gamma_0(N), A) \), the duality over \( \mathbb{Z} \) implies that over \( A \). \( \square \)

Corollary 2.19. We have the following assertions.
(1) For any \( \mathbb{C} \)-algebra homomorphism \( \lambda : h(N, \mathbb{C}) \to \mathbb{C} \), \( \lambda(h(N, \mathbb{Z})) \) is in the integer ring of an algebraic number field. In other words, \( \lambda(T(n)) \) for all \( n \) generates an algebraic number field \( \mathbb{Q}(\lambda) \) over \( \mathbb{Q} \) and \( \lambda(T(n)) \) is an algebraic integer.

(2) For any \( \mathbb{Z} \)-algebra homomorphism \( \lambda : h(N, \mathbb{Z}) \to \mathbb{Q}(\lambda) \),
\[
S_2(\Gamma_0(N), \mathbb{Q}(\lambda))[\lambda] = \{ f \in S_2(\Gamma_0(N), \mathbb{Q}(\lambda)) | f|T(n) = \lambda(T(n))f \text{ for all } n \}
\]
is one dimensional and is generated by \( \sum_{n=1}^{\infty} \lambda(T(n))q^n \).

(3) For any \( \mathbb{Z} \)-algebra homomorphism \( \lambda : h(N, \mathbb{Z}) \to \mathbb{Q}(\lambda) \),
\[
H^1(X_0(N), \mathbb{Q}(\lambda))[\lambda] = \{ c \in H^1(X_0(N), \mathbb{Q}(\lambda)) | c|T(n) = \lambda(T(n))c \text{ for all } n \}
\]
is two dimensional, and is isomorphic to
\[
S_2(\Gamma_0(N), \mathbb{Q}(\lambda))[\lambda] \oplus S_2(\Gamma_0(N), \mathbb{Q}(\lambda))[\overline{\lambda}].
\]

Proof. Since \( h(N, \mathbb{Z}) \) is of finite rank over \( \mathbb{Z} \), \( R = \lambda(h(N, \mathbb{Z})) \) has finite rank \( d \) over \( \mathbb{Z} \). Then the characteristic polynomial \( P(X) \) of multiplication by \( r \in R \) (regarding \( R \cong \mathbb{Z}^d \)) is satisfied by \( r \), that is, \( P(r) = 0 \). Since \( P(X) \in \mathbb{Z}[X] \), \( r \) is an algebraic integer. Then \( R \otimes_{\mathbb{Z}} \mathbb{Q} \) is a finite extension \( \mathbb{Q}(\lambda) \) of degree \( d \) over \( \mathbb{Q} \).

Let \( K \) be a field. For any finite dimensional commutative \( K \)-algebra \( A \), a \( K \)-algebra homomorphism \( \lambda : A \to K \) gives rise to a generator of \( \lambda \)-eigenspace of the line dual \( \text{Hom}_K(A, K) \). Applying this fact to \( \text{Hom}_K(h(N, K), K) \equiv S_2(\Gamma_0(N), K) \) for \( K = \mathbb{Q}(\lambda) \), we get the second assertion.

The third assertion then follows from Proposition 2.9.

Let \( \varepsilon = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \). Define \( \varepsilon(z) = -\bar{z} \). Then \( \varepsilon \) takes \( \mathfrak{h} \) to \( \overline{\mathfrak{h}} \). Define the action of \( GL^+_2(\mathbb{R}) \varepsilon \) on \( \mathfrak{h} \) by \( \gamma \varepsilon(z) = \gamma(-\bar{z}) \). Since \( GL_2(\mathbb{R}) = GL^+_2(\mathbb{R}) \cup \varepsilon GL^+_2(\mathbb{R}) \), we have well defined map \( \gamma : \mathfrak{h} \to \mathfrak{h} \) for all \( \gamma \in GL_2(\mathbb{R}) \).

Exercise 2.20. Prove the following fact.

1. The above action is an action of the group \( GL_2(\mathbb{R}) \) on \( \mathfrak{h} \). In other words, \( \alpha(\beta(z)) = (\alpha\beta)(z) \) for \( \alpha, \beta \in GL_2(\mathbb{R}) \).

2. We have \( \mathfrak{h} \cong GL_2(\mathbb{R})/\mathbb{R}^xO_2(\mathbb{R}) \) by \( g \mapsto g(\sqrt{-1}) \).

Since \( \varepsilon^2 = 1 \) and \( \varepsilon T_n \varepsilon^{-1} = T_n \), the action of \( \varepsilon \) commutes with Hecke operators. Thus if \( A \) is a ring in which \( 2 \) is invertible, we have a decomposition \( H^1(X_0(N), A) \) into direct sum of \( \pm 1 \) eigenspaces of \( \varepsilon \). We write \( H^+_1 \) or \( H^-_1 \) for the eigenspace.

Proposition 2.21. Let \( A \subset \mathbb{C} \) be a PID. If \( \lambda : h(N, \mathbb{Z}) \to A \) be an algebra homomorphism. Then the cohomology and homology groups \( H^1_+(X_0(N), A)[\lambda], H^1_-(X_0(N), A)[\lambda] \) and \( H^+_1(\overline{\Gamma}_0(N), \partial Y_0(N), A)[\lambda] \) are free of rink 1 over \( A \).

Proof. Since \( \varepsilon^*f = f(\varepsilon(z)) \in \overline{S}_2(\Gamma_0(N), \mathbb{C}) \) if \( f \in S_2(\Gamma_0(N), \mathbb{C}) \), \( \pi_{\pm}(f) = \frac{f \pm f^*}{2} \neq 0 \) if \( f \neq 0 \). Moreover \( \pi_{\pm} \) induces an isomorphism of \( S_2(\Gamma_0(N), \mathbb{C})[\lambda] \) onto \( S_2(\Gamma_0(N), \mathbb{C}) \oplus \overline{S}_2(\Gamma_0(N), \mathbb{C}))^{\pm}[\lambda] \), where the superscript \( \pm \) indicates the \( \pm \) eigenspace for \( \varepsilon \). Since \( S_2(\Gamma_0(N), \mathbb{C})[\lambda] \) is one dimensional, we conclude that \( (S_2(\Gamma_0(N), \mathbb{C}) \oplus \overline{S}_2(\Gamma_0(N), \mathbb{C}))^{\pm}[\lambda] \)
Lemma 2.23. Since \( n > 0 \), \( X \) is a continuous function on the compact space bounded by a positive constant.


\[
\text{Modular Hecke } L\text{-functions. Let } \lambda : h(N, \mathbb{Z}) \to \mathbb{C} \text{ be an algebra homomorphism. Then we define } L(s, \lambda) = \sum_{n=1}^{\infty} \lambda(T(n)) n^{-s}, \text{ which is the modular Hecke } L\text{-function of } \lambda. \text{ We now prove that this Dirichlet series converges absolutely if } \Re(s) > 2. \text{ We start with}
\]

Lemma 2.22. If \( f \in S_2(\Gamma_0(N)) \), then \( |f(x + iy)| \leq Cy^{-1} \) for a constant independent of \( x \) and \( y \).

Proof. Let \( \alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{R}) \). By definition,
\[
\alpha \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} \alpha(z) & \alpha(\overline{z}) \\ 1 & 1 \end{pmatrix} \begin{pmatrix} cz + d \\ 0 \end{pmatrix}.
\]
Taking the determinant of this, we get \( \det(\alpha) \Im(z) = \Im(\alpha(z))|cz + d|^2 \). Thus \( g(z) = |f(z)\Im(z)| \) factors through \( X_0(N) \). Since \( f(z) \) vanishes at cusps, \( g(z) \) also; so, \( g(z) \) is a continuous function on the compact space \( X_0(N) \). Thus the positive function \( g(z) \) is bounded by a positive constant \( C' : |g(z)| \leq C' \), which proves the lemma.

Lemma 2.23. There exists a constant \( B > 0 \) such that \( |\lambda(T(n))| \leq B \cdot n \) for all integers \( n > 0 \).

Proof. Since \( f = f_\lambda = \sum_{n=1}^{\infty} \lambda(T(n)) q^n \) is a cusp form in \( S_2(\Gamma_0(N)) \) with \( f_\lambda|T(n) = \lambda(T(n)) f_\lambda \), picking any \( f \in S_2(\Gamma_0(N)) \), we need to prove \( |a(n, f)| \leq Bn \) for all \( n \).

Since \( a(n, f) = (2\pi i)^{-1} \int_{|q|=r} f(q) q^{-n-1} dq \) for any \( r > 0 \) by the residue formula, taking \( r = \exp(-1/n) \) \( \Leftrightarrow \Im(z) = \frac{1}{2\pi n} \), by Lemma 2.22, we get \( |a(n, f)| \leq 2Ce\pi n \). Thus \( B = 2Ce\pi \).

Exercise 2.24. Prove, by a standard argument, \( L(s, \lambda) \) converges absolutely if \( \Re(s) > 2 \).

Exercise 2.25. Let \( \tau = \begin{pmatrix} 0 & 1 \\ N & 0 \end{pmatrix} \). Prove the following facts:

1. \( \tau \Gamma_0(N) \tau^{-1} = \Gamma_0(N) \).
2. If \( f \in S_2(\Gamma_0(N)) \), then \( g = f|\tau \in S_2(\Gamma_0(N)) \).

Lemma 2.26. If \( f \in S_2(\Gamma_0(N)) \), then \( L(s, f) = \sum_{n=1}^{\infty} a(n, f)n^{-s} \) converges absolutely if \( \Re(s) > 2 \) and can be continued analytically to a holomorphic function on the whole complex plane.
Proof. By the same computation as in (1.3), we have
\[ \int_0^\infty f(iy)y^{s-1}dy = (2\pi)^{-s}\Gamma(s)L(s, f) \]
if \( \text{Re}(s) > 2 \). However the integral \( \int_0^\infty f(iy)y^{s-1}dy \) is convergent for all \( s \in \mathbb{C} \), because for any power \( y^s \), \( \lim_{y \to -\infty} f(iy)y^{s-1} = \lim_{y \to 0} f(iy)y^{s-1} = 0 \) (by the \( q \)-expansion). This gives us the analytic continuation of \( L(s, f) \) to the whole complex plane. \( \square \)

By the above expression, we have
\[ L(1, \lambda) = -\int_0^\infty (2\pi i)f_\lambda(z)dz \]
Since \( e^s(f(z)dz) = -f(-\bar{z})d\bar{z} \), we have
\[ L(1, \lambda) = -\int_0^\infty e^s((2\pi i)f_\lambda(z)dz) \]

2.6. Rationality of Hecke \( L \)-values. Let \( A \subset \mathbb{C} \) be a PID and \( \lambda : h(N, \mathbb{Z}) \rightarrow A \) be an algebra homomorphism. Define \( \omega_\pm(\lambda) = \frac{1}{2}((2\pi i)f_\lambda(z)dz \pm e^s((2\pi i)f_\lambda(z)dz)) \) and \( \omega_\pm(f) = \frac{1}{2}((2\pi i)f(z)dz \pm e^s((2\pi i)f(z)dz)) \) for \( f \in S_2(\Gamma_0(N)) \). Let \( \delta_\pm(\lambda) \) be a generator of \( H^1_\pm(X_0(N), A)[\lambda] \over A; \) so,
\[ H^1_\pm(X_0(N), A)[\lambda] = A\delta_\pm(\lambda). \]
Then by Proposition 2.21, we have \( [\omega_\pm(\lambda)] = \Omega_\pm(\lambda; A)\delta_\pm(\lambda) \). We call \( \Omega_\pm(\lambda; A) \) the \( \pm \) period of \( \lambda \). Let \( \gamma_a \in H^1(\mathcal{Y}_0(N), \partial\mathcal{Y}_0(N), \mathbb{Z}) \) for \( a \in \mathbb{Q} \) be the relative 1-cycle represented by vertical line in \( \mathcal{Y} \) passing through \( a \in \mathbb{Q} \).

Lemma 2.27. We have \( \frac{L(1, \lambda)}{\Omega_\pm(\lambda; A)} \in M^{-1}A \) for a positive integer \( M \) only dependent on \( N \). Moreover \( \int_{\gamma_a} \delta_\pm(\lambda) \in M^{-1}A \) for all \( a \in \mathbb{Q} \).

Proof. We have a natural map \( \iota : H^1(X_0(N), A) \rightarrow H^1(\mathcal{Y}_0(N), \partial\mathcal{Y}_0(N), A) \), and after tensoring \( \mathbb{C} \), \( \iota \) becomes an isomorphism by Proposition 2.11, \( \text{Coker}(\iota) \) is finite of order \( M \). Since the vertical line \( \gamma_a \) passing through \( a \in \mathbb{Q} \) is a 1-cycle in \( H^1(\mathcal{Y}_0(N), \partial\mathcal{Y}_0(N), A) \), we have \( M\gamma_a \in H^1(X_0(N), A) \). Thus by (2.3) and (2.4), we get
\[ \frac{M\int_{\gamma_a} \omega_\pm(\lambda)}{\Omega_\pm(\lambda; A)} = \int_{M\gamma_a} \delta_\pm(\lambda) \in A. \]
The same argument also applies to \( \gamma_a \). This finishes the proof. \( \square \)

Since \( (1, 0, 1)(z) = z + 1, \gamma_a = \gamma_{a+1} \). Thus the cycle \( \gamma_a \) only depends \( a \in \mathbb{Q}/\mathbb{Z} \). Let \( \chi : (\mathbb{Z}/m\mathbb{Z}) \rightarrow A \) be a primitive Dirichlet character. Consider \( \gamma(\chi) = \sum_{u \mod m} \chi^1(u)\gamma_{\frac{u}{m}} \in H^1(\mathcal{Y}_0(N), \partial\mathcal{Y}_0(N), A) \).

Lemma 2.28. We have \( e(\gamma(\chi)) = \chi(-1)\gamma(\chi) \).
Proof. Note that \( \varepsilon(\gamma_a) = \gamma_{-a} \), and from this, we have
\[
\varepsilon(\gamma(\chi)) = \sum_{u \mod m} \chi^{-1}(u)\varepsilon(\gamma_{a}/m) = \sum_{u \mod m} \chi^{-1}(u)(\gamma_{-a}/m)^{u - u} = \chi(-1)\gamma(\chi).
\]

We consider \( f|R_\chi(z) = \sum_{u \mod m} \chi^{-1}(u)f(z + \frac{u}{m}) \). The following exercise is a bit difficult.

**Exercise 2.29.** Let \( N' \) be the LCM of \( N \) and \( m^2 \). Prove \( (f|R_\chi)|\gamma = \chi^2(a)f|R_\chi \) for \( \gamma = (\alpha \beta) \in \Gamma_0(N) \).

Note that
\[
\int_{\gamma(\chi)} \omega_\pm(\lambda) = \int_{\gamma_0} \omega_\pm(f|R_\chi).
\]

Then we have
\[
(2.5) \quad f|R_\chi(z) = \sum_{u \mod m} \chi^{-1}(u)f(z + \frac{u}{m}) = \sum_{n=1}^{\infty} a(n, f)q^n \sum_{u} \chi^{-1}(u)\exp\left(\frac{2\pi i nu}{m}\right) = G(\chi^{-1})\sum_{n=1}^{\infty} a(n, f)\chi(n)q^n.
\]

Then by the same argument proving Lemma 2.27 applied to \( \gamma(\chi) \), we get

**Proposition 2.30.** Let \( \chi \) be a primitive Dirichlet character modulo \( m \) with values in a PID \( A \subset \mathbb{C} \). Then there exists a constant \( M \) only depending on \( N \) such that
\[
\prod \left( 1 - \lambda(T(\ell))\chi(\ell)^{-s} + \chi(\ell)^2\ell^{-1-2s} \right) = \prod \left( 1 - \frac{\alpha_\ell\chi(\ell)}{\ell^s}(1 - \frac{\beta_\ell\chi(\ell)}{\ell^s}) \right)^{-1},
\]
where \( \alpha_\ell \) and \( \beta_\ell \) are two roots of \( X^2 - \lambda(T(\ell))X + \ell = 0 \).

**Exercise 2.31.** If \( f \in S_2(\Gamma_0(N)) \), prove that \( f(pz) \in S_2(\Gamma_0(Np)) \).

We consider an algebra homomorphism \( \lambda : h(N, \mathbb{Z}) \rightarrow \mathbb{C} \). Then we have a Hecke eigenform \( f = \sum_{n=1}^{\infty} \lambda(T(\ell))q^n \in S_2(\Gamma_0(N)) \) with \( f|T(n) = \lambda(T(n))f \). The L-function for a Dirichlet character \( \chi \) modulo \( M \)
\[
L(s, \lambda \otimes \chi) = \sum_{n=1}^{\infty} \lambda(T(n))\chi(n)n^{-s}
\]
has the following Euler product:

**Exercise 2.32.**

1. Prove \( T(m)T(n) = \sum_{d | \gcd(m, n)} d \cdot T(mn/d^2) \) by Lemma 2.14 (4).
2. Prove the above Euler factorization of \( L(s, \lambda \otimes \chi) \).

**Lemma 2.33.** Let \( \alpha = \alpha_p \) and \( \beta = \beta_p \), and put \( f_\alpha(z) = f(z) - \beta f(pz) \). Then we have \( f_\alpha \in S_2(\Gamma_0(Np)) \) and \( f_\alpha|U(p) = \alpha f_\alpha \) and \( f_\alpha|T(n) = \lambda(T(n))f_\alpha \) for all \( n > 0 \) prime to \( p \), where \( U(p) \) is the Hecke operator \( T(p) \) acting on \( S_2(\Gamma_0(Np)) \).
Proof. By Lemma 2.14 (4), we have
\[ a(m, f|T(n)) = \sum_{0 < d | (m, n), (d, N) = 1} d \cdot a\left(\frac{mn}{d^2}, f\right). \]
From this, it is easy to see that \( T(m)T(n) = T(mn) \) if \( m \mathbb{Z} + n \mathbb{Z} = \mathbb{Z} \). Note that \( a(n, f) = \lambda(T(n)) \) and hence \( a(mn, f) = a(m, f)a(n, f) \) if \( m \mathbb{Z} + n \mathbb{Z} = \mathbb{Z} \). Since \( f_\alpha = \sum_{m=1}^{\infty} a(m, f)q^m - \beta \sum_{m=1}^{\infty} a(m, f)q^{mp} \), we see \( a(m, f_\alpha) = a(m, f) \) if \( p \nmid m \). In particular, if \( p \nmid m \) and \( p \nmid n \), we have
\[ a(m, f_\alpha|T(n)) = \sum_{0 < d | (m, n), (d, Np) = 1} d \cdot a\left(\frac{mn}{d^2}, f_\alpha\right) \]
\[ = \sum_{0 < d | (m, n), (d, N) = 1} d \cdot a\left(\frac{mn}{d^2}, f\right) = \lambda(T(n))a(m, f) = \lambda(T(n))a(m, f_\alpha). \]
If \( m = m_0p \), we have \( a(m, f_\alpha) = a(m, f) - \beta \cdot a(m_0, f) \). Thus if \( p | m \) and \( p \nmid n \), we have
\[ a(m, f_\alpha|T(n)) = \sum_{0 < d | (m, n), (d, Np) = 1} d \cdot a\left(\frac{mn}{d^2}, f_\alpha\right) \]
\[ = \sum_{0 < d | (m, n), (d, N) = 1} d \cdot \left(a\left(\frac{mn}{d^2}, f\right) - \beta \cdot a\left(\frac{m_0n}{d^2}, f\right)\right) \]
\[ = \lambda(T(n))(a(m, f) - \beta \cdot a(m_0, f) = \lambda(T(n))a(m, f_\alpha). \]
This shows that \( f_\alpha|T(n) = \lambda(T(n))f_\alpha \) for \( n \) prime to \( p \).

Now we have \( a(m, f_\alpha|U(p)) = a(mp, f_\alpha) = a(mp, f) - \beta \cdot a(m, f) \). On the other hand, \( (\alpha + \beta)a(m, f) = a(m, f|T(p)) = p \cdot a\left(\frac{m}{p}, f\right) + a(mp, f) = (\alpha \beta) \cdot a\left(\frac{m}{p}, f\right) + a(mp, f) \).

This shows
\[ a(m, f_\alpha|U(p)) = (\alpha + \beta)a(m, f) - (\alpha \beta) \cdot a\left(\frac{m}{p}, f\right) - \beta \cdot a(m, f) = \alpha \cdot a(m, f_\alpha). \]
This finishes the proof. \( \square \)

**Corollary 2.34.** Let the notation be as in Lemma 2.33. If \( p \nmid N \) and \( \lambda : h(N, \mathbb{Z}) \to \overline{\mathbb{Q}} \) is an algebra homomorphism, we have an algebra homomorphism \( \lambda_\alpha : h(pN, \mathbb{Z}) \to \overline{\mathbb{Q}} \) such that \( \lambda_\alpha(U(p)) = \alpha \) and \( \lambda_\alpha(T(n)) = \lambda(T(n)) \) if \( p \nmid n \). Moreover, we have \( L(s, \lambda_\alpha \otimes \chi) = (1 - \beta \chi(p)p^{-s})L(s, \lambda \otimes \chi) \).

**Exercise 2.35.** Give a detailed proof of the above corollary.

2.8. **Elliptic modular \( p \)-adic measure.** Take an algebra homomorphism of the Hecke algebra \( \lambda : h(pN, \mathbb{Z}) \to \overline{\mathbb{Q}} \). Then we have \( f_\lambda = \sum_{n=1}^{\infty} S_2(\Gamma_0(Np)) \) with \( f_\lambda|T(n) = \lambda(T(n))f_\lambda \). We write the Hecke operator \( T(p) \) on \( S_2(\Gamma_0(Np)) \) as \( U(p) \); so, we have
\[ f|U(p)(z) = \frac{1}{p} \sum_{j=0}^{p-1} f\left(\frac{z + j}{p}\right). \]
Thus the action of $U(p)$ is exactly the same as in the case of $G_m$. We suppose $a = \alpha_p = \lambda(U(p))$ is a $p$-adic unit in $\mathbb{Q}_p(\lambda)$. Such $\lambda$ and $f_\lambda$ are called $p$-ordinary. We have an $A$-linear map: $H^1(\overline{\mathcal{Y}_0}(Np), \partial \overline{\mathcal{Y}_0}(Np), A) \to A$ given by $\omega \mapsto \int_{\gamma_\omega} \omega$. Then we consider a map

$$c : p^{-\infty} \mathbb{Z} = \bigcup_{i=1}^{\infty} p^{-i} \mathbb{Z} \to \text{Hom}_A(H^1(\overline{\mathcal{Y}_0}(Np), \partial \overline{\mathcal{Y}_0}(Np), A), K)$$

given by $c(x)(\omega) = \int_{\gamma_\omega} \omega$.

For $\omega \in H^1(\overline{\mathcal{Y}_0}(Np), \partial \overline{\mathcal{Y}_0}(Np), A)$, we write $c_\omega(r) = \int_{\gamma_r} \omega$. Then $c_\omega(r+1) = c_\omega(r)$ by definition, and $c_\omega$ factors through $\mathbb{Q}_p/\mathbb{Z}_p = p^{-\infty} \mathbb{Z}/\mathbb{Z}$. Supposing $\omega|U(p) = a\omega$ with $|a|_p = 1$, we define a distribution $\varphi_\omega$ on $\mathbb{Z}_p$ by

$$\varphi_\omega(z + p^m \mathbb{Z}_p) = a^{-m} c_\omega\left(\frac{z}{p^m}\right) \text{ for } z = 1, 2, \ldots \text{ prime to } p.$$  

This is well defined because $c_\omega(r+1) = c_\omega(r)$. We take the multiplicative group $G = \mathbb{Z}_p^\times$ and fix an isomorphism $G \cong \mu \times \mathbb{Z}_p$ for a finite group $\mu$, where $\mathbb{Z}_p$ in the right-hand-side is an additive group. Then the multiplicative subgroup $G_1 = 1 + p\mathbb{Z}_p$ corresponds to the additive group $p\mathbb{Z}_p$. To show that $\varphi_\omega$ actually gives a distribution, we need to check the distribution relation (1.8). We compute

$$\sum_{j=1}^{p} c_\omega\left(\frac{x+j}{p}\right) = \sum_j c\left(\frac{x+j}{p}\right)(\omega) = c(x)(\omega|U(p)) = a \cdot c_\omega(x).$$

This shows

$$\sum_{j=1}^{p} \varphi_\omega(x + jp^m + p^{m+1}\mathbb{Z}_p) = \varphi_\omega(x + p^m\mathbb{Z}_p).$$

The general distribution relation (1.8) then follows from the iteration of this relation. By a similar argument, we see that

$$|\varphi_\omega(z + p^m\mathbb{Z}_p)|_p = |a^{-m} c_\omega\left(\frac{z}{p^m}\right)|_p = |c\left(\frac{z}{p^m}\right)(\omega)|_p \leq |\omega|_p,$$

where $|\omega|_p = \text{Sup}_x |c(x)(\omega)|_p$ with $x$ running over $p^{-\infty}\mathbb{Z}$. Thus $\varphi_\omega$ is bounded (by the proof of Lemma 2.27) and, by Proposition 1.29, we have a unique measure $\varphi_\omega$ extending the distribution $\varphi_\omega$. Now we compute $\int_G \phi d\varphi_\omega(x)$. To do this, we may assume that $|\omega|_p \leq 1$ by multiplying by a constant if necessary. For $\phi \in C(G/G_m; A)$, we have

$$\int_G \phi d\varphi_\omega = a^{-m} \sum_{z=1}^{p^m} \phi(z) c_\omega\left(\frac{z}{p^m}\right).$$

Let $N > 1$ be a positive integer prime to $p$. We take $\omega = \delta_+(\lambda)$ for each algebra homomorphism $\lambda : h(Np, \mathbb{Z}) \to \overline{\mathbb{Q}}$. Then we write $\varphi_\omega$ as $\varphi = \varphi_\lambda$ and compute for any primitive character $\phi$ of $(\mathbb{Z}/p\mathbb{Z})^\times$ the integral $\int_G \phi d\varphi_\lambda$. Note that $\omega|U(p) = a\omega$. We
write $\alpha_x : \mathcal{Y}_0(Np) \to \mathcal{Y}_0(Np)$ be then translation $\alpha_x(z) = z + x$ for $x \in \mathbb{R}$. We see that, if $\phi \neq 1$,

$$
\int_G \phi d\varphi_\lambda = a^{-r} \sum_{x \in (\mathbb{Z}/p^r\mathbb{Z})^\times} \phi(x)c\left(\frac{x}{p^r}\right)(\omega) \\
= a^{-r} \int_{\gamma_0} \sum_{x \in (\mathbb{Z}/p^r\mathbb{Z})^\times} \phi(x)\alpha_x^* x/p^r \omega \\
= a^{-r} \int_{\gamma_0} \omega |R_{\phi^{-1}} \\
= a^{-r} \frac{G(\phi)L(1, \lambda \otimes \phi^{-1})}{\Omega_{\phi^{-1}}(\lambda; A)} 
$$

(2.9)

We have basically proved the following theorem of Mazur:

**Theorem 2.36.** Let $p$ be a prime and $N$ be a positive integer prime to $p$. Let $A = \{ x \in \mathbb{Q}(\lambda) \mid |x|_p \leq 1 \}$ be the discrete valuation ring in $\mathbb{Q}(\lambda)$ for a $p$-adic valuation $|\cdot|_p$ of $\mathbb{Q}(\lambda)$. For each algebra homomorphism $\lambda : h(Np, \mathbb{Z}) \to A$ with $|\lambda(U(p))|_p = 1$, we have a unique $p$-adic measure $\varphi_\lambda$ on $\mathbb{Z}_p^\times$ such that for all finite order characters $\phi$ of $\mathbb{Z}_p^\times$ and $1 \leq j \in \mathbb{Z}$, we have

$$
\int \phi(z) d\varphi_\lambda = \lambda(U(p))^{-r} \frac{G(\phi)L(1, \lambda \otimes \phi^{-1})}{\Omega_{\phi^{-1}}(\lambda; A)}. 
$$

We leave you to formulate the corresponding $p$-adic $L$-functions. See [LFE] Chapter 6 for more details of these facts.