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Non-critical Values of Adjoint L-Functions for $\mathrm{SL}(2)$

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To Goro Shimura with admiration

ABSTRACT. For a given system $\lambda(T(\mathfrak{p}))$ of eigenvalues of Hecke operators acting on cohomological cusp forms on $\mathrm{GL}(2)$ over a number field F, we look into the adjoint square L-function $L(s, \mathrm{Ad}(\lambda) \otimes \alpha)$ twisted by a Hecke character α. If λ is associated to a 2-dimensional Galois representation φ, the adjoint square $\mathrm{Ad}(\varphi)$ is the three dimensional factor of $\varphi \otimes^t \varphi^{-1}$, whose L-function is given by $L(s, \mathrm{Ad}(\lambda))$. The L-value $L(1, \mathrm{Ad}(\lambda) \otimes \alpha)$ is critical if and only if F is totally real and the Hecke character α is totally even. We are interested in both critical and non-critical cases. When α is quadratic, a rationality result: $L(1, \mathrm{Ad}(\lambda) \otimes \alpha)/\Omega \in \mathbb{Q}(\lambda)$ is shown, where Ω is a canonical (topological) period of the base change lift of λ to the quadratic extension K/F associated to α, and $\mathbb{Q}(\lambda)$ is the number field generated by $\lambda(T(\mathfrak{p}))$ for all primes $\mathfrak{p}$. Alongside, we shall give an evidence for the divisibility of the L-value by the order of the Selmer group of $\mathrm{Ad}(\varphi) \otimes \alpha$. Towards the end, a period relation is given, as an application of our main result, when F is totally imaginary.

1. Introduction

If one has two canonical rational structures on a given complex vector space, one can define a period which is the determinant of the linear transformation bringing one rational structure to the other. This principle applied to cohomology groups on a projective variety V (defined over a number field F) yields the classical periods of F-rational differential forms on V. In this case, one rational structure is given by Betti cohomology, and another comes from algebraic de Rham cohomology. The two cohomology groups are put together into one vector space by the comparison isomorphism. This definition extends to motives, and the periods are conjectured to give canonical transcendental factors of the critical values of the motivic L-functions (a conjecture of Deligne; cf. [**Hi94**]).

Even if the manifold V is not algebraic, it is feasible to define periods in a similar way if V is modular, that is,

$$V = H(\mathbb{Q})\backslash H(\mathbb{A})/UZ_H(\mathbb{R})C_\infty,$$

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where $H_{/\mathbb{Q}}$ is a classical linear group with center Z_H, C_∞ is the standard maximal compact subgroup of $H(\mathbb{R})$, and U is an open compact subgroup of the finite part $H(\mathbb{A}^{(\infty)})$. For a locally constant sheaf L on V coming from a polynomial representation of H, we have a canonical rational structure on the cuspidal cohomology group $H^q_{\mathrm{cusp}}(V, L)$ coming from the rational structure of L. This cohomology group is often isomorphic to a product of several copies of a space S of cohomological cusp forms on $H(\mathbb{A})$. If one can specify this isomorphism in a canonical way as Eichler and Shimura did for elliptic modular forms [**Sh71**], we have another rational structure on the cohomology groups provided that the Fourier expansion of cusp forms gives a good rational structure on S. If $H = \mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}(2)_{/F}$, there is an optimal function W, in the Whittaker model of a cuspidal automorphic representation π, giving the standard L-function under the Mellin transform. The Fourier expansion with respect to W gives a rational structure on the space of cohomological cusp forms. Applying the above principle to H and $H \times H$, I proved rationality and integrality theorems for critical values of standard L and Rankin products in [**Hi94**]. This was extended to the so-called twisted tensor L-functions of π in [**Gh**] for imaginary quadratic F. The results for GL(2) described above could be generalized to GL(n). Anyway hereafter we assume that $H = \mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}(2)_{/F}$.

In the investigation in [**Hi94**] and [**Gh**], only the minimal degree cohomology group is used, and the minimal degree seems to yield rationality only for critical values. If H has a holomorphic structure yielding a Shimura variety V ($\iff$ F is totally real), the degree of non-trivial cuspidal cohomology is unique (that is $q = [F : \mathbb{Q}]$), which is the minimal degree I meant, and it is natural from the conjecture of Deligne that we can get results only for critical values. If F is not totally real, there are several values of q with non-trivial cohomology. However the space S of cohomological cusp forms is independent of q.

In this paper, we study the rational structures for the maximal degree and some middle degree cohomology groups, and we shall prove a rationality result of the adjoint L-value $L(1, \mathrm{Ad}(\pi) \otimes \alpha)$ for α with $\alpha^2 = 1$ relative to the period $\Omega(\widehat{\lambda})$ of a cohomology class of degree depending on K/F, where K/F is the quadratic extension of F associated to α (Corollaries 3.2 and 4.2 for $F = \mathbb{Q}$, Theorem 6.1 for totally real F and Theorems 7.1 and 8.1 for F with complex places). Here we write λ for the system of Hecke eigenvalues associated to π, that is, $L(s, \pi) = \sum_{\mathfrak{n}} \lambda(T(\mathfrak{n}))N(\mathfrak{n})^{-s}$, and we hereafter write $L(s, \mathrm{Ad}(\lambda))$ for $L(s, \mathrm{Ad}(\pi))$. This value $L(1, \mathrm{Ad}(\lambda) \otimes \alpha)$ is non-critical if either the character α is odd at some real places of F or F is not totally real. Thus in the non-critical case, the automorphic period is close to the Beilinson period [**RSS**], assuming his conjecture and the existence of a motive yielding the adjoint L-function. While in the critical case, our automorphic period should be equal to the Deligne period (see [**Hi94**, Section 1]). Moreover, when $F = \mathbb{Q}$ and α is a quadratic Dirichlet character, we shall prove, under some assumptions, that the L-value gives congruences between non-base-change forms on $\mathrm{GL}(2)_{/K}$ and the base change $\widehat{\lambda}$ of λ (cf. [**J**] and [**L**]) to the quadratic extension K as conjectured in [**DHI**] (see Theorem 5.2). This shows that the p-primary part of the value is close to the order of the Selmer group of $\mathrm{Ad}(\rho) \otimes \alpha$, ρ being the p-adic Galois representation of λ. As is shown by Wiles [**W**, Chapter 4] and [**TW**], the p-primary part of $L(1, \mathrm{Ad}(\lambda))/\Omega$ for many p gives the exact order of the Selmer group of $\mathrm{Ad}(\rho)$. Our result (Theorem 5.2) is a partial generalization of this non-abelian class number formula. Thus we expect the conjectures in [**DHI**] made originally for real cyclic extensions to hold even for imaginary cyclic extensions of $\mathbb{Q}$ (see Conjecture 5.1).

Actually the period $\Omega(\widehat{\lambda})$ giving the transcendental factor of the L-value is defined using the base change lift $\widehat{\lambda}$ to $G = \mathrm{Res}_{K/\mathbb{Q}}(\mathrm{GL}(2))$ for the quadratic extension K/F associated to α. Because of this, we need to assume

(cusp) $\widehat{\lambda}$ *remains cuspidal.*

This condition is equivalent to the condition that the λ-eigenspace in the space of cusp forms on $H(\mathbb{A})$ is orthogonal to any theta series associated to the norm form of the quadratic extension K (see [**L**, Lemma 11.3]). We assume this condition throughout the paper. When F is totally real, the definition of the automorphic period is a little more transparent than the other cases, because we are either in the minimal or maximal degree case where the multiplicity of $\widehat{\lambda}$ in the modular cohomology group for G is basically 1 up to a group action at archimedean places. If F has complex place, things are more complicated, and we need to use (H, λ) and $(G, \widehat{\lambda})$ at the same time (see Sections 7 and 8). In addition to this, the explicit description of the Eichler-Shimura map obtained in [**Hi94**] from a result of Harder [**Ha**] is different depending on the shape of K/F, although the general principle is the same as explained in Section 2.4. This is why we treat the imaginary quadratic case in Section 3, the real quadratic case in Section 4, the case of totally real F in Section 6 and general cases in Sections 7 and 8. It is an interesting problem to study relations among periods of λ of different degrees. We list some of them in Section 9, which follow easily from our main result. It is also interesting to know what type of L-values can be dealt with by looking into middle degrees.

It is not an isolated phenomenon that topological rational structure yields a canonical transcendental factor of an L-value. Starting from a number field K, we induce the trivial Galois character from K to $\mathbb{Q}$. Then $\mathrm{Ind}_{\mathbb{Q}}^{K}\,\mathrm{id} = \mathrm{id} \oplus \chi$ for an Artin Galois representation χ. The classical class number formula is written in terms of $L(1,\chi)$ whose main transcendental factor is the regulator of K. As is obvious from the definition, the regulator is the period of the maximal degree cohomology group of $F_{\mathbb{A}}^{1}/F^{\times}$ for the norm 1 ideles $F_{\mathbb{A}}^{1}$ normalized with respect to the L-function (see [**Hi89**, p. 90]). The fact that $\mathrm{Ind}_{\mathbb{Q}}^{K}\,\mathrm{id}$ contains the identity representation once is essentially used to identify the residue of the Dedekind zeta function $L(s, \mathrm{Ind}_{\mathbb{Q}}^{K}\,\mathrm{id})$ of K with the Artin L-value $L(1,\chi)$ as a product of the regulator and the class number. Computation of the residue tends to be easier than the computation of values. In our case, a similar phenomenon occurs. For the contragredient $\rho^{\vee}$, $\rho \otimes \rho^{\vee} \cong \mathrm{id} \oplus \mathrm{Ad}(\rho)$, and the residue formula of the Rankin product $L(s, \rho \otimes \rho^{\vee})$is essentially used to obtain the non-abelian class number formula for $L(1, \mathrm{Ad}(\rho))/\Omega$ [**Hi81**], [**Hi88b**] and [**Hi89**] (see also [**U**] for a generalization to imaginary quadratic K). The transcendental factor Ω is the period of the maximal degree cohomology group for $\mathrm{GL}(2) \times \mathrm{GL}(2)$ (see [**Hi81**] and Section 8 in the text). Exactly the same phenomenon happens also for $L(1, \mathrm{Ad}(\rho) \otimes \alpha)/\Omega(\widehat{\lambda})$, although the period may not be of maximal degree. The idea of proof is simple, which is summarized in Section 2.4, although the computation, in order to get an effective integral expression of the L-value, is a bit demanding. In the process of obtaining the integral expression, it is necessary to find a Hecke character of K with a prescribed restriction to F. The argument to find such a character is substantially shortened by a suggestion made by the referee of this paper, in particular, Lemma 2.1 is supplied by him along with a concise proof. Here I wish to thank the referee for the suggestion and his careful reading of the manuscript.

Holomorphy of the adjoint L-function was first dealt with by Shimura [**Sh75**] and then generalized to arbitrary π and F by Gelbart-Jacquet [**GeJ**] (see also [**Sh94**] for another integral expression). The rationality result was dealt with for critical values by Sturm for $F = \mathbb{Q}$ and Im for general totally real F [**St**] and [**I**]. Our proof is a cohomological interpretation of the Rankin method studied by Shimura and Asai [**As**], which is generalized to $\mathrm{GL}(n)$ by Flicker [**Fl**] and [**FlZ**]. In the course of the proof of the congruence theorem (Theorem 5.2), we need to use a non-vanishing result of twisted tensor L-functions, which follows from a more general result of Shahidi [**S81**], [**S88**].

Here is general notation. We write $F_{\mathbb{A}}$ for the adele ring of F. When $F = \mathbb{Q}$, we write $\mathbb{A}$ for that. The finite part of $\mathbb{A}$ is written as $\mathbb{A}^{(\infty)}$, and the infinite part of $F_{\mathbb{A}}$ is written as F_∞. As a subring of $\mathbb{A}^{(\infty)}$, we write $\widehat{\mathbb{Z}}$ for the product of the p-adic integer ring $\mathbb{Z}_p$ over all primes p. For the integer ring $\mathfrak{r}$ of F, we put $\widehat{\mathfrak{r}} = \mathfrak{r} \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$ as a subring of $F_{\mathbb{A}}^{(\infty)}$. For a number field X, we write I_X for the set of embeddings of X into $\mathbb{C}$. We write Σ_X for the set of archimedean places of X and decompose $\Sigma_X = \Sigma_X(\mathbb{R}) \sqcup \Sigma_X(\mathbb{C})$ for the set of all real places $\Sigma_X(\mathbb{R})$. For a number field denoted by F in the text, which is the base field, we drop the subscript "F" like I for I_F.

Contents

2. Idea of the proof, and preliminaries

In this section, we first describe how we can interpret analytic integration of cuspidal automorphic forms in terms of group and sheaf cohomology theory in an algebraic way (Sections 2.1–2.2). Then we define various modular L-functions, and we study multiplicative relations among the L-functions we defined (Section 2.3). This relation combined with an integral expression gives a key to our proof of rationality theorem (Section 2.4). At the end of this section, we describe Γ-factors of L-functions and criticality of L-values in terms of motives (Section 2.5).

2.1. Integration of cuspidal cohomology classes. Let G be a classical linear algebraic group defined over $\mathbb{Z}$. We consider an open compact subgroup S of $G(\widehat{\mathbb{Z}})$ of $G(\mathbb{A}^{(\infty)})$. We write $G(\mathbb{R})_+$ for the identity component of the Lie group

$G(\mathbb{R})$. We put $G(\mathbb{A})_+ = G(\mathbb{A}^{(\infty)}) \times G(\mathbb{R})_+$ and $G(\mathbb{Q})_+ = G(\mathbb{A})_+ \cap G(\mathbb{Q})$, where $\mathbb{A}^{(\infty)}$ is the finite part of the adele ring. Then we study the modular manifold associated to S

$$Y(S) = G(\mathbb{Q})_+ \backslash G(\mathbb{A})_+ / C_{\infty +} Z(\mathbb{R}) S$$

for the center $Z = Z_G$ of G and the maximal compact subgroup $C_{\infty +}$ of $G(\mathbb{R})_+$. Decompose $Y(S) = \sqcup_a Y_a$ into a finite disjoint union of connected components Y_a. Then we fix a and write $Y = Y_a$. Then $Y_a \cong \Gamma \backslash \mathfrak{Z}$ for a discrete arithmetic subgroup Γ of $G(\mathbb{Q})_+$ and the symmetric space $\mathfrak{Z} = G(\mathbb{R})_+ / Z_G(\mathbb{R}) C_{\infty +}$. When it is necessary to indicate the dependence on a, we write $\Gamma^{(a)}$ for Γ. Then we suppose that we have a coordinate system $(t, x_1, ..., x_{d-1})$ $(d = \dim(Y))$ of a coordinate neighborhood U_s around the cusp s of Y such that

$$U_s \cong (t_0, \infty) \times (\Gamma_s \backslash \mathbb{R}^{d-1})$$

(for an open interval (t_0, ∞), $t_0 > 0$) with compact quotient $\Gamma_s \backslash \mathbb{R}^{d-1}$ for a discrete subgroup Γ_s of Γ acting on $\mathbb{R}^{d-1}$. This is the case where G has a maximal $\mathbb{Q}$-split torus of rank 1, and the variable t is given by the variable of the unique $\mathbb{Q}$-split torus (of a Levi subgroup) in the minimal parabolic subgroup fixing the cusp s. Let L be a finite dimensional $\mathbb{R}$-vector space with an action of Γ. Let ω be a C^∞-closed p-form on $(t_0, \infty) \times (\mathbb{R}^{d-1})$ with values in L decreasing exponentially as $t \to \infty$. We suppose that $\gamma^* \omega = \gamma \omega$ for $\gamma \in \Gamma_s$. Here $\gamma\omega(x)$ is the image under the action $\gamma : L \to L$ applied to the value $\omega(x)$. We write

$$\omega = \sum_{i_1 < \ldots < i_p} \alpha_{i_1 \ldots i_p} dx_{i_1} \wedge \ldots \wedge dx_{i_p} + \sum_{j_1 < \ldots < j_{p-1}} \beta_{j_1 \ldots j_{p-1}} dt \wedge dx_{j_1} \wedge \ldots \wedge dx_{j_{p-1}}.$$

Since $d\omega = 0$,

$$\frac{\partial \alpha_{i_1 \ldots i_p}(t,x)}{\partial t} = \sum_{k=1}^{p} (-1)^{k-1} \frac{\partial \beta_{i_1 \ldots i_{k-1} i_{k+1} \ldots i_p}(t,x)}{\partial x_{i_k}}.$$

Then we have

$$\alpha_{i_1 \ldots i_p}(t,x) = \int_\infty^t \frac{\partial \alpha_{i_1 \ldots i_p}(t,x)}{\partial t} dt = \sum_{k=1}^{p} (-1)^{k-1} \int_\infty^t \frac{\partial \beta_{i_1 \ldots i_{k-1} i_{k+1} \ldots i_p}}{\partial x_{i_k}} dt.$$

Let $\theta = \sum_{j_1 < \ldots < j_{p-1}} (\int_\infty^t \beta_{j_1 \ldots j_{p-1}}(t,x) dt) dx_{j_1} \wedge \ldots \wedge dx_{j_{p-1}}$. Then $d\theta = \omega$. Note that θ is invariant under Γ_s, that is, $\gamma^* \theta = \gamma \theta$ for $\gamma \in \Gamma_s$ because t is invariant under the action of Γ_s. Thus for a given ω, we can find a canonical lift θ. Let C be the set of all cusps of $Y(S)$. We take an open neighborhood U_s for each $s \in C$ as above. Let us consider the quotient $\underline{L} = \Gamma \backslash (\mathfrak{Z} \times L)$ given by the action $\gamma(z, \lambda) = (\gamma x, \gamma \lambda)$. We write $\pi : \underline{L} \to Y$ for the projection. Then we consider the sheaf $\mathcal{L}$ made of locally constant sections of π. For each cuspidal closed differential p-form ω on Y which is a C^∞-section of $\mathcal{L} \otimes_{\mathbb{R}} \Omega^p$, we take $\theta(\omega|_{U_s})$ as above so that $d\theta(\omega|_{U_s}) = \omega|_{U_s}$. Then we take a C^∞-function $\phi : Y \to \mathbb{R}$ such that ϕ is identically 1 if $t > t_1$ with a $t_1 > t_0$ for every s, and outside $\cup_{s \in C} U_s$, ϕ is identically 0. Then we define $\theta_\phi(\omega) = \sum_{s \in C} \phi \theta(\omega|_{U_s})$. The form $\omega - d\theta_\phi(\omega)$ is compactly supported. The cohomology class $[\omega - d\theta_\phi(\omega)]$ in $H_c^p(Y, \mathcal{L})$ is independent of the choice of U_s and ϕ, because $\theta_\phi(\omega) - \theta_{\phi'}(\omega)$ is compactly supported for any other choice of ϕ'. For $\mathbb{Q}$-rank 1 case, we have a canonical choice of t given by the variable of the

$\mathbb{Q}$-split torus of G and $\{x_i\}$ coming from $\mathbb{Q}$-non-split torus and unipotent radical of the minimal parabolic subgroup fixing s. Thus we have a canonical section

$$i : H^p_{\mathrm{cusp}}(Y, \mathcal{L}) \to H^p_c(Y, \mathcal{L}),$$

which is compatible with Hecke operator action. The compatibility follows from the expression of the Hecke operators on boundary cohomology groups, for example for GL(2), the expression is given in [**Hi93b**, Section 3], and the uniqueness of $i([\omega]) = [\omega - d\theta_\phi(\omega)]$.

We now compactify Y adding the boundary $\infty \times (\Gamma_s \backslash \mathbb{R}^{d-1})$ to U_s for all cusps s. We write the compactification as $\overline{Y}$. Then $\overline{Y}$ is a manifold with boundary $\partial\overline{Y} = \sqcup_s \infty \times (\Gamma_s \backslash \mathbb{R}^{d-1})$. Let C be a C^∞-class p-cycle modulo $\partial\overline{Y}$. By our construction, $\omega - i(\omega) = d\theta_\phi(\omega)$ is rapidly decreasing towards cusps s. We assume that Y has finite volume with respect to the Haar measure on $G(\mathbb{R})$. Then if ω is rapidly decreasing towards cusps s (that is, exponentially decreasing with respect to t), $\int_C \omega$ converges. We see easily from the Stokes theorem

$$\text{(it1)} \qquad \int_C \omega = \int_C i(\omega).$$

If $p = \dim Y = d$, we see that $\mathrm{Tr} : H^d_c(Y, A) \cong A$ by the evaluation at d-relative cycle $\overline{Y}$ modulo $\partial\overline{Y}$. In particular, if $A = \mathbb{C}$,

$$\text{(it2)} \qquad \mathrm{Tr}([\omega]) = \int_Y \omega = \int_Y i(\omega).$$

This is usually stated for Y smooth (for example, if S is sufficiently small), but is valid always, because of the following reason. We take sufficiently small normal subgroup Γ' of Γ of finite index such that $Y' = \Gamma' \backslash \mathfrak{Z}$ is smooth. Then $H^d(Y, \mathbb{C}) = H^d(Y', \mathbb{C})^\Delta$ for $\Delta = \Gamma/\Gamma'$, and Tr with respect to Γ' induces that of Y. This shows the assertion $(it2)$ for Y' implies that for Y.

2.2. Modular cohomology groups. We summarize here the definition of modular cohomology groups and Hecke operator action on them. A detailed exposition can be found in [**Hi94**] and [**Hi88a**]. Let F be a number field with the integer ring $\mathfrak{r}$. We consider the torus $T = \mathrm{Res}_{\mathfrak{r}/\mathbb{Z}}\mathbf{G}_m$. We fix an algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ inside $\mathbb{C}$. The group of characters $X(T) = \mathrm{Hom}_{alg-gr}(T_{/\overline{\mathbb{Q}}}, \mathbf{G}_{m/\overline{\mathbb{Q}}})$ can be identified with the formal free module $\mathbb{Z}[I]$ generated by the set I of all field embeddings of F into $\overline{\mathbb{Q}}$. Since any $\sigma \in I$ induces an algebra homomorphism $\sigma : F \otimes_{\mathbb{Q}} A \to A$ for any $\overline{\mathbb{Q}}$-algebra A by $k \otimes a \mapsto \sigma(k)a$, for each $n = \sum_{\sigma \in I} n_\sigma \sigma$, n as an element of $X(T)$ takes $a \in T(A)$ to $a^n = \prod_\sigma \sigma(a)^{n_\sigma}$. Note that $T(\mathbb{A})$ for the adele ring $\mathbb{A}$ is the idele group $F^\times_{\mathbb{A}}$. Thus $T(\mathbb{A})/T(\mathbb{Q})$ is the idele class group. A Hecke character $\psi : T(\mathbb{A})/T(\mathbb{Q}) \to \mathbb{C}^\times$ is called arithmetic if it induces an element $\infty(\psi)$ in $X(T)$ on the identity connected component $T(\mathbb{R})_+$ of the archimedean part $T(\mathbb{R})$ of $T(\mathbb{A})$. This element $\infty(\psi) \in \mathbb{Z}[I]$ is called the infinity type of ψ. We write Ξ_F for the submodule of $\mathbb{Z}[I] = X(T)$ made of infinity types of arithmetic Hecke characters. For each arithmetic Hecke character ψ, the field generated by $\psi(x)$ for all $x \in T(\mathbb{A}^{(\infty)})$ is a finite extension $\mathbb{Q}(\psi)$ of $\mathbb{Q}$, which is either a totally real or a CM field. A Hecke character ω is called algebraic if $\omega(x) \in \overline{\mathbb{Q}}$ for all $x \in T(\mathbb{A}^{(\infty)})$. There are many algebraic characters which are not arithmetic (cf. [**Hi94**, p. 467]).

We put

$$H = \mathrm{Res}_{\mathfrak{r}/\mathbb{Z}}\mathrm{GL}(2)_{/\mathfrak{r}} \quad\text{and}\quad \mathfrak{H} = H(\mathbb{R})_+/C_{\infty+}Z_H(\mathbb{R}).$$

As S, we take

$$U_0(N) = U_{0,F}(N) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in H(\widehat{\mathbb{Z}}) \middle| c \in \widehat{N} \right\}$$

for an ideal N of $\mathfrak{r}$, where $\widehat{N} = N \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$. We write $Y_0(N) = Y_{0,F}(N)$ for $Y(S)$. For L, we take polynomial representations of H. Let F^{cl} be the Galois closure of $F/\mathbb{Q}$, and write $\mathfrak{r}^{cl}$ for the integer ring of F^{cl}. Writing I for the set of all embeddings of F into $\overline{\mathbb{Q}}$, for each $\mathfrak{r}^{cl}$-algebra A, each σ induces a projection $\sigma : H(A) \to \mathrm{GL}_2(A)$ which coincides with σ on $\mathfrak{r}$. In particular, $H(F^{cl}) \cong \prod_{\sigma \in I} \mathrm{GL}_2(F^{cl})$ via $a \mapsto (\sigma(a))_\sigma$. Then over F^{cl}, each irreducible polynomial representation of H is isomorphic to

$$\bigotimes_{\sigma \in I} \det(\sigma(x))^{v_\sigma} \mathrm{Sym}(\sigma(x))^{\otimes n_\sigma}$$

for integer tuples (v_σ) and (n_σ) with $n_\sigma \geq 0$. Here $\mathrm{Sym}(\sigma(\mathrm{x}))^{\otimes \mathrm{n}_\sigma}$ is the symmetric n_σ-th tensor matrix of $\sigma(x)$. Thus irreducible polynomial representations of H are classified by tuples (n, v) of $\mathbb{Z}[I]$, where $n = \sum_{\sigma \in I} n_\sigma \sigma$ and $v = \sum_\sigma v_\sigma \sigma$. We write κ for the pair (n, v) sometimes. We can concretely realize the above polynomial representation on an A-free module $L(\kappa; A)$ made of polynomials of $2[F : \mathbb{Q}]$ variables $(X_\sigma, Y_\sigma)_{\sigma \in I}$ with coefficients in A homogeneous of degree n_σ for each pair (X_σ, Y_σ). We let $\gamma \in H(A)$ act on $P \in L(\kappa; A)$ by

$$\gamma P(X_\sigma, Y_\sigma) = \det(\gamma)^v P((X_\sigma, Y_\sigma)^t \sigma(\gamma)^\iota),$$

where $\gamma^\iota = \det(\gamma)\gamma^{-1}$ and $\det(\gamma)^v = \prod_\sigma \det(\sigma(\gamma))^{v_\sigma}$.

By the approximation theorem, choosing a complete set R of representatives for the class group $Cl(S) = T(\mathbb{Q})_+ \backslash T(\mathbb{A}) / \det(S) T(\mathbb{R})$, we have

$$H(\mathbb{A})_+ = \bigsqcup_{a \in R} H(\mathbb{Q})_+ \left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right) S H(\mathbb{R})_+.$$

Thus $Y(S) = \sqcup_a Y_a$ and $Y_a = \Gamma^{(a)} \backslash \mathfrak{Z}$, where $\Gamma^{(a)} = tSH(\mathbb{R})t^{-1} \cap H(\mathbb{Q})_+$ for $t = \left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right)$. When $S = U_0(N)$, for a Hecke character ψ with $\psi_\infty(x) = x^{-n-2v}$ for all $x \in F_\infty^\times$ whose conductor is a factor of N, we twist a little the action of $\Gamma^{(a)}$ on $L(\kappa; A)$ as follows:

$$P(X_\sigma, Y_\sigma) \mapsto \psi_N(d) \det(\gamma)^v P((X_\sigma, Y_\sigma)^t \sigma(\gamma)^\iota)$$

if $\gamma = \left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right) \in \Gamma^{(a)}$, where ψ_N is the restriction of ψ to $\prod_{\mathfrak{p}|N} F_\mathfrak{p}^\times \subset T(\mathbb{A})$. We write this twisted module as $L(\kappa, \psi; A)$. To have a non-trivial sheaf, the action of $\Gamma^{(a)}$ has to factor through the fundamental group $\pi_1(Y)$ which is $\Gamma^{(a)}/\mathfrak{r}^\times$. Therefore, the center $\mathfrak{r}^\times$ of $\Gamma^{(a)}$ has to act trivially on $L(\kappa, \psi; A)$. The condition: $\psi_\infty(x) = x^{-n-2v}$ for all $x \in F_\infty^\times$ assures this. When ψ is trivial on $T(\widehat{\mathbb{Z}})$, $L(\kappa, \psi; A) = L(\kappa; A)$. Thus $L(\kappa, \psi; A)$ only depends on the restriction of ψ to units $T(\widehat{\mathbb{Z}}) = \widehat{\mathfrak{r}}^\times$.

We describe the cohomology groups

$$H^q(Y_0(N), \mathcal{L}(\kappa, \psi; A)), \quad H_c^q(Y_0(N), \mathcal{L}(\kappa, \psi; A))$$

and

$$H_{\mathrm{cusp}}^q(Y_0(N), \mathcal{L}(\kappa, \psi; A))$$

defined in [**Hi94**, Sections 3 and 5]. Let $\mathbb{Q}(\kappa)$ be the subfield of $\overline{\mathbb{Q}}$ fixed by the subgroup

$$\mathcal{G}(\kappa) = \left\{ \sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \middle| \kappa\sigma = (n\sigma, v\sigma) = \kappa \right\}.$$

Let $\mathbb{Q}(\psi)$ be a subfield of $\overline{\mathbb{Q}}$ generated by $\psi(x)$ for all $x \in T(\mathbb{A}^{(\infty)})$, which is a CM field finite over $\mathbb{Q}$. We write $\mathbb{Q}(\kappa,\psi)$ for the composite of $\mathbb{Q}(\kappa)$ and $\mathbb{Q}(\psi)$. Write

$$P(X_\sigma, Y_\sigma) = \sum_{0\le j\le n} a_j X^{n-j}Y^j \in L(\kappa,\psi,A),$$

where $X^{n-j}Y^j = \prod_\sigma X_\sigma^{n_\sigma - j_\sigma} Y_\sigma^{j_\sigma}$, $a_j \in A$, and $0 \le j \le n$ implies $0 \le j_\sigma \le n_\sigma$ for all $\sigma \in I$. Then we let $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ act on $L(\kappa,\psi;\overline{\mathbb{Q}})$ by

$$\sum_{0\le j\le n} a_j X^{n-j}Y^j \mapsto \sum_{0\le j\le n} a_j^\sigma X^{n\sigma - j\sigma}Y^{j\sigma}.$$

Then σ takes $L(\kappa,\psi;\overline{\mathbb{Q}})$ onto $L(\kappa\sigma,\psi^\sigma;\overline{\mathbb{Q}})$, where ψ^σ is the unique Hecke character such that $\psi^\sigma(x) = \psi(x)^\sigma$ for $T(\mathbb{A}^{(\infty)})$ and $\infty(\psi^\sigma) = \infty(\psi)\sigma$. Thus the $\Gamma^{(a)}$-module $L(\kappa,\psi;\mathbb{Q}(\kappa,\psi))$ is well defined, and for the integer ring $\mathbb{Z}(\kappa,\psi)$ of $\mathbb{Q}(\kappa,\psi)$,

$$tL(\kappa,\psi;\mathbb{Z}(\kappa,\psi)) = tL(\kappa,\psi;\mathbb{Z}(\kappa,\psi)\otimes_{\mathbb{Z}}\widehat{\mathbb{Z}}) \bigcap L(\kappa,\psi;\mathbb{Q}(\kappa,\psi))$$

is an $\mathbb{Z}(\kappa,\psi)$-lattice in $L(\kappa,\psi;\mathbb{Q}(\kappa,\psi))$ stable under $\Gamma^{(a)}$ (see the paragraph below (3.5) of [**Hi94**]), where $t = \left(\begin{smallmatrix} a & 0\\ 0 & 1\end{smallmatrix}\right)$, and the intersection is taken in

$$L(\kappa,\psi;\mathbb{A}^{(\infty)}(\kappa,\psi)) \quad \text{for} \quad \mathbb{A}^{(\infty)}(\kappa,\psi) = \mathbb{Q}(\kappa,\psi)\otimes_{\mathbb{Q}}\mathbb{A}^{(\infty)}.$$

Thus writing $tL(A) = tL(\kappa,\psi;A)$ for $tL(\kappa,\psi;\mathbb{Z}(\kappa,\psi))\otimes_{\mathbb{Z}(\kappa,\psi)} A$, we have the covering $\bigsqcup_a \underline{tL}(A) \to Y_0(N)$. We write $\mathcal{L}(\kappa,\psi;A)$ for the sheaf of locally constant sections of this covering. Then, if $Y_0(N)$ is smooth, the cohomology groups H^q and H^q_c are defined in the usual manner, and the cuspidal cohomology group $H^q_{\mathrm{cusp}}(Y_0(N),\mathcal{L}(\kappa,\psi;\mathbb{C}))$ is defined to be the subspace of $H^q(Y_0(N),\mathcal{L}(\kappa,\psi;\mathbb{C}))$ spanned by cuspidal harmonic forms [**Hi94**, Section 2]. When $Y_0(N)$ is not smooth, we take a normal subgroup $S \subset U_0(N)$ and define $H^q_{\mathrm{cusp}}(Y_0(N),\mathcal{L}(\kappa,\psi;\mathbb{C}))$ by the subspace of $H^q_{\mathrm{cusp}}(Y(S),\mathcal{L}(\kappa,\psi;\mathbb{C}))$ fixed by $U_0(N)/S$. Similarly, we define $H^q_c(Y_0(N),\mathcal{L}(\kappa,\psi;\mathbb{C}))$ for non-smooth $Y_0(N)$. For any $\mathbb{Z}(\kappa,\psi)$-subalgebra A of $\mathbb{C}$, we define $H^q_{\mathrm{cusp}}(Y_0(N),\mathcal{L}(\kappa,\psi;A))$ by the intersection of $H^q_{\mathrm{cusp}}(Y_0(N),\mathcal{L}(\kappa,\psi;\mathbb{C}))$ with the natural image of $H^q_c(Y_0(N),\mathcal{L}(\kappa,\psi;A))$. Anyway, as seen in 2.1, we have a canonical section

$$i : H^q_{\mathrm{cusp}}(Y_0(N),\mathcal{L}(\kappa,\psi;\mathbb{C})) \hookrightarrow H^q_c(Y_0(N),\mathcal{L}(\kappa,\psi;\mathbb{C})),$$

and

$$H^q_{\mathrm{cusp}}(Y_0(N),\mathcal{L}(\kappa,\psi;\mathbb{C})) \cong H^q_{\mathrm{cusp}}(Y_0(N),\mathcal{L}(\kappa,\psi;\mathbb{Z}(\kappa,\psi)))\otimes_{\mathbb{Z}(\kappa,\psi)}\mathbb{C}.$$

The cohomology groups $H^q_{\mathrm{cusp}}(Y_0(N),\mathcal{L}(\kappa,\psi;A))$ have a natural action of Hecke operators $T(\mathfrak{n})$ for integral ideals $\mathfrak{n}$ of $\mathfrak{r}$ and the action of the center $Z(\mathbb{A})$ [**Hi94**, Section 4] as long as either A is a $\mathbb{Q}(\kappa,\psi)$-algebra or $v \ge 0$ ($\iff$ $v_\sigma \ge 0$ for all σ), and i is equivariant under Hecke operators. When A is not a $\mathbb{Q}$-algebra and $v \not\ge 0$, we need to modify $T(\mathfrak{n})$ as in [**Hi94**, Section 4] to preserve integrality. By a result of Harder, the cuspidal cohomology group is trivial if one of the following conditions is satisfied (i) $q < r_1(F)+r_2(F)$; (ii) $q > r_1(F)+2r_2(F)$, and (iii) $n \ne nc$ for complex conjugation c (cf. [**Ha**], [**Hi94**, Section 2]), where $r_1(F)$ (resp. $r_2(F)$) is the number of real (resp. complex) places of F.

We assume that $H^q_{\mathrm{cusp}}(Y_0(N),\mathcal{L}(\kappa,\psi;\mathbb{C})) \ne 0$. Since we have an action of the center $Z_H(\mathbb{A}^{(\infty)}) = T(\mathbb{A}^{(\infty)})$, we can decompose $H^q_{\mathrm{cusp}}(Y_0(N),\mathcal{L}(\kappa,\psi;\mathbb{C}))$ into a product of eigenspaces under this action:

$$H^q_{\mathrm{cusp}}(Y_0(N), \mathcal{L}(\kappa, \psi; \mathbb{C})) = \bigoplus_{\psi'} H^q_{\mathrm{cusp}}(Y_0(N), \mathcal{L}(\kappa, \psi; \mathbb{C}))[\psi'],$$

where ψ' runs over arithmetic Hecke characters such that $\psi' = \psi$ on $T(\widehat{\mathbb{Z}})$ and $\infty(\psi') = \infty(\psi)$. We write $h_\kappa(N, \psi; A)_{/F}$ for the A-subalgebra of

$$\mathrm{End}_{\mathbb{C}}(H^q_{\mathrm{cusp}}(Y_0(N), \mathcal{L}(\kappa, \psi; \mathbb{C}))[\psi])$$

generated by $T(\mathfrak{n})$ for all integral ideals $\mathfrak{n}$. Again by Harder [**Ha**], the algebra $h_\kappa(N, \psi; A)$ is independent of q. Let $\lambda : h_\kappa(N, \psi; \mathbb{C}) \to \mathbb{C}$ be an algebra homomorphism. Then λ-eigenspace is non-trivial in the cohomology group. Its dimension depends on q and F. To describe this, write $\Sigma = \Sigma_F$ (resp. $\Sigma(\mathbb{R}) = \Sigma_F(\mathbb{R}), \Sigma(\mathbb{C}) = \Sigma_F(\mathbb{C})$) for the set of archimedean (resp. real, complex) places of F. We identify $\Sigma(\mathbb{C})$ with a subset of complex embeddings in I so that each place is induced by the corresponding embedding. Note that

$$\mathfrak{H} \cong \prod_{\sigma \in \Sigma(\mathbb{R})} \mathcal{H}_\sigma \times \prod_{\tau \in \Sigma(\mathbb{C})} \mathcal{H}_\tau,$$

where $\mathcal{H}_\sigma$ is a upper half complex plane on which $\sigma(\gamma) \in \mathrm{GL}_2(\mathbb{R})$ $(\sigma \in \Sigma(\mathbb{R}))$ acts through a linear fractional transformation, and

$$\mathcal{H}_\tau = \left\{ \begin{pmatrix} x & -y \\ y & \overline{x} \end{pmatrix} \middle| x \in \mathbb{C},\ 0 < y \in \mathbb{R} \right\},$$

on which $\tau(\gamma) \in \mathrm{GL}_2(\mathbb{C})$ $(\tau \in \Sigma(\mathbb{C}))$ acts as in [**Hi94**, 2.2]. In [**Hi94**, Section 3 (M1-4)], we defined a space of cohomological cusp forms $S_{\kappa,J}(N; \psi)_{/F}$ for each subset J of $\Sigma(\mathbb{R})$. Actually, we need to assume that functions in $S_{\kappa,J}(N; \psi)_{/F}$ are rapidly decreasing towards cusps, which follows from the cuspidal condition [**Hi94**, M4] if F is different from $\mathbb{Q}$ or imaginary quadratic fields. In [**Hi94**], this condition is implicitly assumed when F is $\mathbb{Q}$ or imaginary quadratic field (see $(m'3)$ of [**Hi94**, p. 460]). An element $f \in S_{\kappa,J}(N; \psi)$ corresponds to a real analytic modular form on $\mathfrak{H}$ holomorphic on the copy $\mathcal{H}_\sigma$ at $\sigma \in J$ and anti-holomorphic at $\sigma \in \Sigma(\mathbb{R}) - J$ (see the remark in [**Hi94**, p. 60] after $(m'3)$). This space, when $F = \mathbb{Q}$, is isomorphic to the classical space of elliptic cusp forms. More precisely, $S_{\kappa,I}(N, \psi)$ is isomorphic to the space $S_k(\Gamma_0(N), \psi_N)$ of holomorphic cusp forms of weight $k = n + 2$ with Neben character ψ_N, which is the restriction of ψ to $\widehat{\mathbb{Z}}^\times$ regarded as a Dirichlet character. The isomorphism is given by $f \mapsto \phi(x + iy) = y^{v-1} f\left(\left(\begin{smallmatrix} y & x \\ 0 & 1 \end{smallmatrix}\right)\right)$. Thus the space itself does not depends on v, but the Hecke operator $T(n) = T_v(n)$ depends on v in the following way: When $v = 0$, $T_0(n)$ is the classical Hecke operator acting on $S_k(\Gamma_0(N), \psi_N)$ defined by Hecke. Then $T_v(n) = n^v T_0(n)$. In other words, by pulling back classical cusp forms in $S_k(\Gamma_0(N), \psi_N)$ to $H(\mathbb{A})$, we get the space $S_{(n,0),I}(N, \psi)$ and

$$S_{(n,v),I}(N, \psi) = \left\{ f(x) |\det(x)|_{\mathbb{A}}^{-v} \middle| f \in S_{(n,0),I}(N, \psi) \right\}.$$

When $F \neq \mathbb{Q}$, $S_{(n,v),J}(N, \psi)$ does not necessarily have such a simple relation to $S_{(n,0),J}(N, \psi)$.

In [**Hi94**, Sections 2-3], we described a very explicit map

$$\delta_{J,J'} : S_{\kappa,J}(N; \psi) \hookrightarrow H^q_{\mathrm{cusp}}(Y_0(N), \mathcal{L}(\kappa, \psi; \mathbb{C}))[\psi]$$

indexed by $J \subset \Sigma(\mathbb{R})$ and $J' \subset \Sigma(\mathbb{C})$. The linear map $\delta_{J,J'}$ is Hecke equivariant and takes a cohomological cusp form f to a differential form holomorphic of degree

1 in the variables of $\mathcal{H}_\sigma$ for $\sigma \in J$, anti-holomorphic of degree 1 for $\sigma \in \Sigma(\mathbb{R}) - J$, harmonic of degree 1 for $\sigma \in J'$ and harmonic of degree 2 for $\Sigma(\mathbb{C}) - J'$. Thus the total degree of the differential form is $r_1 - |J'| + 2r_2$. We will recall the explicit form of $\delta_{J,J'}$ later in our computation in specific cases. Then, for $d = [F : \mathbb{Q}]$,

$$\delta = \oplus_{J,J'}\delta_{J,J'} : \bigoplus_{J,J':\#(J')=d-q} S_{\kappa,J}(N,\psi) \cong H^q_{\mathrm{cusp}}(Y_0(N), \mathcal{L}(\kappa,\psi;\mathbb{C}))[\psi].$$

Since the λ-eigenspace of $S_{\kappa,J}(N,\psi)$ is 1-dimensional, this completely determines the dimension of the cohomological eigenspace. Since $T(\mathfrak{n})$ leaves stable the cohomology group, $\lambda(T(\mathfrak{n}))$ is an algebraic number in a fixed finite extension. We write $\mathbb{Q}(\lambda)$ for the subfield of $\overline{\mathbb{Q}}$ generated by $T(\mathfrak{n})$ for all $\mathfrak{n}$, which is a CM field or a totally real field containing $\mathbb{Q}(\kappa,\psi)$.

For a standard Whittaker function $W = W_\kappa : T(\mathbb{R})_+ \to L((n^*,0);\mathbb{C})$ with $n^* = \sum_{\sigma\in\Sigma(\mathbb{C})}(n_\sigma + n_{\sigma c} + 2)\sigma$, $f \in S_{\kappa,J}(N,\psi)$ has a Fourier expansion of the following form [**Hi94**, Section 6]:

$$\text{(F)} \qquad f\left(\begin{pmatrix} y & x \\ 0 & 1\end{pmatrix}\right) = |y|_{F_\mathbb{A}} \sum_{\xi\in F^\times} \mathbf{a}(\xi y\mathfrak{d}; f)W(\xi y_\infty)\mathbf{e}_F(\xi x),$$

where ξ runs over all elements in F with $\xi^\sigma > 0$ for $\sigma \in J$ and $\xi^\sigma < 0$ for $\sigma \in \Sigma(\mathbb{R}) - J$, $y \in T(\mathbb{A})$ with $y_\infty \in T(\mathbb{R})_+$, $\mathfrak{n} \mapsto \mathbf{a}(\mathfrak{n}; f)$ is a function with values in $\mathbb{C}$ supported by the set of integral ideals, $\mathfrak{d} = \mathfrak{d}_F$ is the different of $F/\mathbb{Q}$, and $\mathbf{e}_F : F_\mathbb{A}/F \to \mathbb{C}$ is the standard additive character with $\mathbf{e}_F(x_\infty) = \exp(2\pi\sqrt{-1}\sum_{\sigma\in I} x^\sigma)$. The function W is the optimal element in the Whittaker model at archimedean places, whose Mellin transform gives the exact Γ-factor of the standard L-function. Its explicit form will be recalled later. This Fourier expansion determines the cusp form uniquely. If $f|T(\mathfrak{n}) = \lambda(T(\mathfrak{n}))f$ with $\mathbf{a}(\mathfrak{r}; f) = 1$, then $\mathbf{a}(\mathfrak{n}; f) = \lambda(T(\mathfrak{n}))$. Thus the eigenspace of λ is one dimensional (Multiplicity 1) (cf. [**Hi94**, Theorem 6.4]). For any automorphism $\sigma \in \mathrm{Aut}(\mathbb{C})$, we always have f^σ in $S_{\kappa\sigma,J}(N,\psi^\sigma)$ such that $\mathbf{a}(\mathfrak{n}; f^\sigma) = \mathbf{a}(\mathfrak{n}; f)^\sigma$ by the Hecke equivariance of δ and the rational structure of the cuspidal cohomology groups. In particular, $\lambda^\sigma(T(\mathfrak{n})) = \lambda(T(\mathfrak{n}))^\sigma$ gives an algebra homomorphism of $h_{\kappa\sigma}(N,\psi^\sigma;\mathbb{C})$ into $\mathbb{C}$. Sometimes, we call λ a system of Hecke eigenvalues.

2.3. Modular L-functions. We fix a system of Hecke eigenvalues

$$\lambda : h_\kappa(N,\psi;\mathbb{C}) \to \mathbb{C}$$

and define several L-functions of λ we will study. The standard L-function of λ twisted by a Hecke character $\eta : T(\mathbb{A})/T(\mathbb{Q}) \to \mathbb{C}^\times$ is given by

$$\begin{aligned} L(s,\lambda\otimes\eta) &= \sum_{\mathfrak{n}\subset\mathfrak{r}} \eta(\mathfrak{n})\lambda(T(\mathfrak{n}))N_{F/\mathbb{Q}}(\mathfrak{n})^{-s} \\ &= \prod_{\mathfrak{p}} \left\{(1-\alpha_\mathfrak{p}\eta(\mathfrak{p})N_{F/\mathbb{Q}}(\mathfrak{p})^{-s})(1-\beta_\mathfrak{p}\eta(\mathfrak{p})N_{F/\mathbb{Q}}(\mathfrak{p})^{-s})\right\}^{-1}, \end{aligned}$$

which is continued to an entire function on the whole complex s-plane and has a functional equation if λ is primitive (that is, λ gives eigenvalues of a primitive form of conductor N; cf. [**Mi**]). When η is arithmetic with infinity type $-w \in \mathbb{Z}[I]$, $\lambda\otimes\eta : T(\mathfrak{n}) \mapsto \eta(\mathfrak{n})\lambda(T(\mathfrak{n}))$ for $\mathfrak{n}$ prime to the conductor $C = C(\eta)$ of η gives an algebra homomorphism of $h_{\kappa'}(N\cap C^2,\psi\eta^2;\mathbb{C})$ into $\mathbb{C}$ for $\kappa' = \kappa + (0,w)$. Thus we can view $L(s, \lambda \otimes \eta)$ as the standard L-function of $\lambda \otimes \eta$. The adjoint L-function is given by

$$L(s, \mathrm{Ad}(\lambda) \otimes \eta) = \prod_{\mathfrak{p}} \left\{ \left(1 - \frac{\alpha_{\mathfrak{p}} \beta_{\mathfrak{p}}^{-1} \eta(\mathfrak{p})}{N_{F/\mathbb{Q}}(\mathfrak{p})^s}\right) \left(1 - \frac{\eta(\mathfrak{p})}{N_{F/\mathbb{Q}}(\mathfrak{p})^s}\right) \left(1 - \frac{\alpha_{\mathfrak{p}}^{-1} \beta_{\mathfrak{p}} \eta(\mathfrak{p})}{N_{F/\mathbb{Q}}(\mathfrak{p})^s}\right) \right\}^{-1},$$

which again has a meromorphic continuation to the whole complex s-plane [**Sh75**], [**Sh94**] and [**GeJ**]. It has a functional equation after adding finitely many Euler factors if necessary. For another system μ of Hecke eigenvalues, we define the Rankin product of λ and μ. Writing the Euler $\mathfrak{p}$ factor of $L(s, \mu)$ as $(1 - \alpha'_{\mathfrak{p}}/N(\mathfrak{p})^s)^{-1}(1 - \beta'_{\mathfrak{p}}/N(\mathfrak{p})^s)^{-1}$, we put

$$L(s, \lambda \otimes \mu) = \prod_{\mathfrak{p}} \left\{ \left(1 - \frac{\alpha_{\mathfrak{p}} \alpha'_{\mathfrak{p}}}{N_{F/\mathbb{Q}}(\mathfrak{p})^s}\right) \left(1 - \frac{\alpha_{\mathfrak{p}} \beta'_{\mathfrak{p}}}{N_{F/\mathbb{Q}}(\mathfrak{p})^s}\right) \cdot \left(1 - \frac{\beta_{\mathfrak{p}} \alpha'_{\mathfrak{p}}}{N_{F/\mathbb{Q}}(\mathfrak{p})^s}\right) \left(1 - \frac{\beta_{\mathfrak{p}} \beta'_{\mathfrak{p}}}{N_{F/\mathbb{Q}}(\mathfrak{p})^s}\right) \right\}^{-1},$$

For each primitive Hecke eigensystem $\lambda : h_\kappa(N, \psi; \mathbb{C}) \to \mathbb{C}$, it is a well known conjecture that there exists a compatible system of $\mathfrak{l}$-adic representations $\rho = \rho(\lambda)$ of $\mathrm{Gal}(\overline{F}/F)$ with coefficients in $\mathbb{Q}(\lambda)$ such that $L(s, \rho \otimes \eta) = L(s, \lambda \otimes \eta)$ (cf. [**Hi94**, Section 1]), where $\overline{F}$ is the algebraic closure of F. This conjecture is known for totally real F (see [**BR**]). If such ρ exists, $L(s, \lambda \otimes \mu) = L(s, \rho(\lambda) \otimes \rho(\mu))$ and $L(s, \mathrm{Ad}(\lambda)) = L(s, \mathrm{Ad}(\rho))$, where $\mathrm{Ad}(\rho)$ is a three dimensional representation fitting into the following exact sequence:

$$0 \to \mathrm{Ad}(\rho) \to \rho \otimes \rho^\vee \to \mathrm{id} \to 0$$

for the contragredient $\rho^\vee$ of ρ and the identity Galois character id.

We take a semi-simple quadratic extension K/F. We allow $K = F \oplus F$. We write α for the quadratic character of $T(\mathbb{A})/T(\mathbb{Q})$ associated to K/F if K is a field. If $K = F \oplus F$, we simply put $\alpha = \mathrm{id}$ for the identity character id. We write $\mathcal{R}$ for the integral closure of $\mathfrak{r}$ in K. We put $G = \mathrm{Res}_{\mathcal{R}/\mathbb{Z}} \mathrm{GL}(2)_{/\mathcal{R}}$. If K is not a field, we simply agree to put $G = H \times H$. We consider the cohomological modular forms on G. Thus if $G = H \times H$ and $N = C \oplus C \subset \mathcal{R}$, then

$$S_{(\kappa,\mu)}(N, (\psi, \psi')) = S_\kappa(C, \psi) \otimes_{\mathbb{C}} S_\mu(C, \psi'),$$

where $f \otimes g(x, x') = f(x)g(x')$ on $G(\mathbb{A})$. We define $Y_{0,K}(N)$ as above for G if K is a field, and otherwise, $Y_{0,K}(N) = Y_{0,F}(C) \times Y_{0,F}(C)$. We write I_K for the set of all non-trivial algebra homomorphisms of K into $\overline{\mathbb{Q}}$. Thus if $K = F \oplus F$, $I_K \cong I \sqcup I$ canonically. Let $\mu : h_\kappa(N, \chi; \mathbb{C})_{/K} \to \mathbb{C}$ be a system of Hecke eigenvalues for $\kappa \in \mathbb{Z}[I_K]$. Suppose that K is a field and the compatible system $\rho = \rho(\mu)$ exists. Extending a non-trivial automorphism of K/F to $\sigma \in \mathrm{Gal}(\overline{F}/F)$, we put $\rho^\sigma(g) = \rho(\sigma g \sigma^{-1})$. Then $\Psi = \rho \otimes \rho^\sigma$ is equivalent to Ψ^σ, and it extends to $\mathrm{Gal}(\overline{F}/F)$ in two ways. Realizing Ψ on $V \otimes V$ for a two dimensional vector space V on which ρ acts, one of the two extensions, writing Ψ_+, satisfies $\Psi_+(\sigma)(x \otimes y) = y \otimes \rho(\sigma^2)x$ and the other is given by $\Psi_-(\sigma)(x \otimes y) = -y \otimes \rho(\sigma^2)x$ (cf. [**Gh**, 5.1]). We write the L-function of Ψ_+ twisted by a Hecke character η of F as $L(s, (\mu \otimes \mu_\sigma)_+ \otimes \eta)$. Then $L(s, \Psi_- \otimes \eta) = L(s, \Psi_+ \otimes \alpha\eta) = L(s, (\mu \otimes \mu_\sigma)_+ \otimes \alpha\eta)$. See [**Gh**, 5.1] for an explicit description of Euler factors of $L(s, (\mu \otimes \mu^\sigma)_+)$. Although it is assumed in

[**Gh**] that K is an imaginary quadratic field, the computation (and the description) is the same for general K/F. We have

$$L(s,\mu\otimes\mu_\sigma)=L(s,(\mu\otimes\mu_\sigma)_+)L(s,(\mu\otimes\mu_\sigma)_+\otimes\alpha),$$

where μ_σ is a system of Hecke eigenvalues given by $\mu_\sigma(T(\mathfrak{n}))=\mu(T(\mathfrak{n}^\sigma))$. When $K=F\oplus F$, we just put

$$L(s,(\mu\otimes\mu_\sigma)_+)=L(s,\lambda\otimes\lambda)$$

if μ is given by $\lambda\otimes\lambda'$ taking $(T(\mathfrak{n}),T(\mathfrak{m}))$ to $\lambda(T(\mathfrak{n}))\lambda'(T(\mathfrak{m}))$ for two systems of Hecke eigenvalues λ and λ' of H. Then we have

$$\text{(R1)}\qquad L(s+1,(\mu\otimes\mu_\sigma)_+\otimes\eta)=L(2s,\chi_F\eta^2)L_{K/F}(s,\mu,\eta),$$

where χ_F is the restriction of χ to $F_{\mathbb{A}}^\times$ and

$$L_{K/F}(s,\mu,\eta)=\sum_{\mathfrak{n}\subset\mathfrak{r}}\eta(\mathfrak{n})\mu(T(\mathfrak{n}))N_{F/\mathbb{Q}}(\mathfrak{n})^{-s-1}$$

for a Hecke character η of F. Here $\mathfrak{n}$ runs over all ideals of $\mathfrak{r}$ (not $\mathcal{R}$) extended to $\mathcal{R}$.

2.4. Idea of the proof. There is a way to get an integral expression of

$$L(1,\mathrm{Ad}(\lambda)\otimes\alpha)$$

for a Hecke character α with $\alpha^2=1$. Here we summarize the idea, and in the following sections, we shall give details of computation. This integral expression gives a key to prove the rationality theorem. Let $\mu=\widehat{\lambda}$ for the base change lift of λ of H to G. Here we need to invoke the assumption that $\widehat{\lambda}$ remains cuspidal. When K is a field, $\widehat{\lambda}$ is given by Jacquet [**J**] so that $L(s,\rho(\lambda)|_K\otimes\eta)=L(s,\widehat{\lambda}\otimes\eta)$ for all arithmetic Hecke character η of K, where $\rho|_K$ is the restriction of ρ to $\mathrm{Gal}(\overline{F}/K)$. If $G=H\times H$, we simply put $\widehat{\lambda}=\lambda\otimes\lambda$. Then $\chi=\psi\circ N_{K/F}$ and $\chi_F=\psi^2$. We write $\widehat{\kappa}=(\widehat{n},\widehat{v})\in\mathbb{Z}[I_K]^2$ for the weight of $\widehat{\lambda}$. We have the restriction map $\mathrm{Res}_F^K:\mathbb{Z}[I_K]\to\mathbb{Z}[I]$ which takes σ to its restriction to F, where F is embedded into $F\oplus F$ diagonally if K is not a field. Writing $\mathrm{Inf}_F^K:\mathbb{Z}[I]\to\mathbb{Z}[I_K]$ for the inflation map: $\mathrm{Inf}(\sigma)=\sum_{\tau\in I_K,\tau|_F=\sigma}\tau$, we see that $\widehat{n}=\mathrm{Inf}_F^K(n)$. Then we get, again looking into Euler factorization

$$\begin{aligned}L_{K/F}(s,\widehat{\lambda},\eta)L(2s,(\psi\eta)^2)&=L(s+1,(\mu\otimes\mu_\sigma)_+\otimes\eta)\\&=L(s,\alpha\psi\eta)L(s,\mathrm{Ad}(\lambda)\otimes\psi\eta)\end{aligned}$$

up to finite Euler factors. Let N_0 be the least common multiple of the conductor of η and $N\cap\mathfrak{r}$. Since discrepancy of Euler factors can only occur for primes dividing N_0, we get the following exact identity:

$$\text{(R2)}\qquad L_{N_0}(2s,(\psi\eta)^2)L_{K/F,N_0}(s,\widehat{\lambda},\eta)=L_{N_0}(s,\alpha\psi\eta)L_{N_0}(s,\mathrm{Ad}(\lambda)\otimes\psi\eta),$$

where we write $L_{N_0}(s)$ for the L-function obtained from the original $L(s)$ by removing Euler factors for primes $\mathfrak{p}$ dividing N_0. Thus assuming $\alpha\psi\eta=\mathrm{id}$, $L_{N_0}(s,\alpha\psi\eta)$ has a simple pole at $s=1$. Thus we get the following residue formula:

$$\text{(Res1)}\qquad\begin{aligned}&\{\mathrm{Res}_{s=1}\zeta_{F,N_0}(s)\}\,L_{N_0}(1,\mathrm{Ad}(\lambda)\otimes\alpha)\\&=\mathrm{Res}_{s=1}\left\{\zeta_{F,N_0}(2s)L_{K/F,N_0}(s,\widehat{\lambda},\alpha\psi^{-1})\right\}.\end{aligned}$$

Thus we need to compute the residue of the right-hand side of ($Res1$). Keeping the assumption that $\alpha\psi\eta = \mathrm{id}$, we use for that purpose a pull back integration of $\delta(g)$ over $Y_{0,F}(N')$ (for a suitable N') to get an integral expression of $L(s+1,(\mu\otimes\mu_\sigma)_+\otimes\eta)$, where g is a suitable cohomological modular form having Hecke eigenvalues related to $\widehat{\lambda}$. To describe g, we need to express $\eta = \alpha\psi^{-1} = \varphi_F\omega$ for the restriction φ_F of an arithmetic Hecke character φ of K to F and a Hecke character ω of conductor 1. We will show later that this is possible in most cases. We write $w = -\infty(\varphi)$ and C for the conductor of φ. We put $N' = N\cap C^2\cap\mathfrak{r}$. Then up to finitely many Euler factors

$$\begin{aligned}\zeta_F(s)L(s,\mathrm{Ad}(\lambda)\otimes\alpha) &= L(s+1,(\widehat{\lambda}\otimes\widehat{\lambda}_\sigma)_+\otimes\eta)\\ &= L(s+1,((\widehat{\lambda}\otimes\varphi)\otimes(\widehat{\lambda}\otimes\varphi)_\sigma)_+\otimes\omega).\end{aligned}$$

We choose suitable $J\subset\Sigma_K(\mathbb{R})$ and $J'\subset\Sigma_K(\mathbb{C})$ so that

$$g\in S_{\widehat{\kappa}+(0,w),J}(N\cap C^2,\chi\varphi^2)[\widehat{\lambda}\otimes\varphi]$$

and $\delta_{J,J'}(g)$ gives a cohomology class in $H^q_{\mathrm{cusp}}(Y_{0,K}(N\cap C^2),\mathcal{L}(\widehat{\kappa}+(0,w),\chi\varphi^2;\mathbb{C}))$ for $q=\dim Y_{0,F}(N')$. The cusp form g is the image of λ-eigenvector f under the twisting operator $R(\varphi)$ and the (cohomological) rationality of $\delta(f)$ and $\delta(f|R(\varphi))$ are equal (see [**Hi94**, 6.8] and the proof of Theorem 8.1). Under the assumption: $\alpha\psi\eta=\mathrm{id}$, we have a non-trivial sheaf morphism

$$\pi:\mathcal{L}(\widehat{\kappa}+(0,w),\chi\varphi^2;\mathbb{C})|_{Y_{0,F}(N')}\to\mathcal{L}(0,\omega^{-2};\mathbb{C}).$$

Thus for a suitable Eisenstein series $E(s)$ giving a global section of $\mathcal{L}(0,\omega^2;\mathbb{C})$, we can prove by a Rankin convolution method that the integral

$$\int_{Y_{0,F}(N')}\pi(\delta_{J,J'}(g))E(s)$$

gives

$$L(s+1,((\widehat{\lambda}\otimes\varphi)\otimes(\widehat{\lambda}\otimes\varphi)_\sigma)_+\otimes\omega)$$

up to the canonical Γ-factor and a constant. Actually,

$$E(s)=E(x,s)=\omega(\det(x))E_0(x,s)$$

as a function of $x\in H(\mathbb{A})$, where $E_0(x,s)$ is the weight 0 Eisenstein series attached to the trivial character of the Borel subgroup of H. As is well known, $E_0(x,s)$ has a simple pole at $s=1$ with constant residue equal to $\mathrm{Res}_{s=1}\zeta_F(s)$ up to rational numbers (see Appendix). Then comparing the residue of the two sides of ($Res1$), we finally for $C'=C(\varphi)\cap\mathfrak{r}$,

$$\text{(Res2)}\qquad L_{C'}(1,\mathrm{Ad}(\lambda)\otimes\alpha)=C_0\int_{Y_{0,F}(N')}\pi(\delta_{J,J'}(g))\det(\omega(x)),$$

where the constant C_0 is the product of the Gauss sum $G(\psi\alpha)$ and a power of π up to rational numbers. Thus if the $\widehat{\lambda}$-eigenspace $H^q_{\mathrm{cusp}}(Y_{0,K}(N),\mathcal{L}(\widehat{\kappa},\chi;\mathbb{C}))[\widehat{\lambda}]$ is one dimensional, we see that $\delta(f)=\Omega(\widehat{\lambda})\xi$ for a $\mathbb{Q}(\widehat{\lambda})$-rational cohomology class ξ, and we know the algebraicity of $L_{N'}(1,\mathrm{Ad}(\lambda)\otimes\alpha)$ up to $\Omega(\widehat{\lambda})$ and a power of π. This happens when K is a CM quadratic extension of totally real F. Let $\Sigma_K(\mathbb{R})$ be the set of real places of K. When F is totally real, we can further decompose the cohomology group through the action of $C_\infty/C_{\infty+}=\{\pm1\}^{\Sigma_K(\mathbb{R})}$ into a direct sum of one dimensional pieces, and hence we can still prove the algebraicity of the L-value. For general K/F, in place of $H^q_{\mathrm{cusp}}(Y_{0,K}(N),\mathcal{L}(\widehat{\kappa},\chi;\mathbb{C}))$, we can use

$H_c^q(Y_{0,F}(N'), \mathbb{C}) \cong \mathbb{C}$, which is one dimensional. Therefore similarly we can define the period $\Omega(\widehat{\lambda})$ and get the algebraicity, although the definition of the transcendental factor Ω is not so transparent as the case where F is totally real. This is one of the reasons why we have divided our argument according to the shape of K/F at the archimedean places as described in the introduction.

In this paper, we only study the pull back integration of degree $q = \dim Y_{0,F}(N')$ cohomology class. Presumably the same process for degree $q' < q$ would yield another integral expression of $L_{K/F}(s, \mu, \eta)$. In this case, the Eisenstein series $E(s)$ has to be replaced by an Eisenstein differential form. When K is an imaginary quadratic field, Ghate [**Gh**] treated the case of minimal q', that is, $q' = 1$ (while $q = 2$) and obtained a rationality result for critical values of $L(s, (\mu \otimes \mu)_+)$, which implies rationality of some critical values of $L(s, \mathrm{Ad}(\lambda) \otimes \eta)$. Since our result covers the non-critical value $L(1, \mathrm{Ad}(\lambda) \otimes \alpha)$ in this case, the two results are disjoint. For a general quadratic extension, there are several values of q' between maximal q and the minimal one. It is an interesting problem to study the integral for intermediate values q'.

Here, we study extensibility of $\psi' = \alpha\psi^{-1}$ to a Hecke character φ of K up to Hecke characters of F of conductor 1. Here is a general lemma supplied by the referee of this paper:

LEMMA 2.1. *Let K/F be a Galois extension. Let J'' be the set of archimedean places of F which ramify in K. A Hecke character χ of finite order of $F_{\mathbb{A}}^\times$ extends to a Hecke character of finite order of K if and only if $\chi_\sigma = 1$ for all $\sigma \in J''$.*

PROOF. Here is a proof which is a version of the proof supplied by the referee. For a multiplicative $\mathrm{Gal}(K/F)$-module A, we write $H^r(A)$ (resp. $\widehat{H}^r(A)$) for the group cohomology group $H^r(\mathrm{Gal}(K/F), A)$ (resp. the Tate cohomology group $\widehat{H}^r(\mathrm{Gal}(K/F), A)$). Thus $\widehat{H}^r(A)$ is defined also for negative r, $\widehat{H}^r(A) = H^r(A)$ for $r > 0$ and $\widehat{H}^0(A) = H^0(A)/N_{K/F}A$. Let $C_X = X_{\mathbb{A}}^\times / X^\times$ for a number field X be the idele class group. Write D_X for the identity component of C_X. Thus by the Artin reciprocity map, C_X/D_X is isomorphic to the Galois group of the maximal abelian extension of X. Note that C_F and $C_F/C_F \cap D_K$ are closed subgroups of C_K and C_K/D_K, respectively. Thus we see

1. A character $\chi : C_F \to \mathbb{C}^\times$ is of finite order $\iff$ χ is trivial on D_F;
2. χ extends to a finite order character of C_K $\iff$ χ is trivial on $C_F \cap D_K$.

By Hilbert's theorem 90 applied to $K^\times$, we have $H^0(C_K) = C_F$, and hence $H^0(D_K) = C_F \cap D_K$. For each complex place $\sigma \in \Sigma_K(\mathbb{C})$, we write $T_\sigma = \{z \in K_\sigma = \mathbb{C} \big| |z|_\sigma = 1\}$ and D'_K for the image of $K^\times \prod_{\sigma \in \Sigma_K(\mathbb{C})} T_\sigma$ in C_K. By [**ArT**, Theorem 4, p. 91], $\widehat{H}^r(D'_K) = \widehat{H}^r(D_K)$ for all r. Applying this to $r = 0$, we get

$$H^0(D_K) = H^0(D'_K) N_{K/F}(D_K)$$

in D_K. Let J be the subset of Σ_K made of places above J''. Then writing $\{\pm 1\}_\sigma$ for the subgroup of order 2 in T_σ, we consider the image D''_K of $K^\times \prod_{\sigma \in J} \{\pm 1\}_\sigma$. Then by [**ArT**, Theorem 5], (or its proof) on page 92, $H^0(D_K) = D''_K D_F$, which shows the assertion. □

By this lemma, our question of the extensibility is reduced to the study of the infinity type of the characters in question. Let Ξ_X be the set of all infinity types of arithmetic Hecke characters of a number field X. We write Ξ for Ξ_F. We have a typical element $\mathbf{1}_X = \sum_{\sigma \in I_X} \sigma \in \Xi_X$. We write $\mathbf{1}$ for $\mathbf{1}_F$. If a number field X contains a CM field, we write X_{CM} for the largest CM subfield in X. Then

$$\Xi = \begin{cases} \mathrm{Inf}_{F/F_{CM}}(\Xi_{F_{CM}}) & \text{if } F \text{ contains a CM field} \\ \mathbb{Z}\mathbf{1} & \text{if } F \text{ does not contains any CM field.} \end{cases}$$

First suppose that F contains a CM field. If $K_{CM} \neq F_{CM}$, K_{CM} and F are linearly disjoint over F_{CM}, and hence,

$$\mathrm{Res}_{K/F}\Xi_K = \mathrm{Res}_{K/F}\mathrm{Inf}_{K/K_{CM}}(\Xi_{K_{CM}}) = \mathrm{Inf}_{F/F_{CM}}\mathrm{Res}_{K_{CM}/F_{CM}}(\Xi_{K_{CM}}).$$

Note that $\mathrm{Res}_{K_{CM}/F_{CM}}(\Xi_{K_{CM}}) + \mathbb{Z}\mathbf{1}_{F_{CM}} = \Xi_{F_{CM}}$, which follows from the fact that for any CM field X, $\{\tau - \tau c\}_{\tau \in I_X}$ and $1_X = \sum_{\tau \in I_X} \tau$ generate Ξ_X. Similarly, if K contains a CM field but F does not, $\mathrm{Res}_F^K \Xi_K = \Xi$. Thus we have

$$(\mathrm{Res}_{K/F}\Xi_K) + \mathbb{Z}\mathbf{1} = \Xi$$

if one of the following three conditions is satisfied: (i) $K_{CM} \neq F_{CM}$, (ii) F does not contains any CM field, and (iii) K does not contains any CM field. If K contains a CM field and $K_{CM} = F_{CM}$, we have

$$\mathrm{Res}_{K/F}\Xi_K = 2\Xi.$$

Suppose that K contains a CM field L and either $K_{CM} \neq F_{CM}$ or F does not contains a CM field. Let Φ_L be a CM type of L and $\Phi = \mathrm{Inf}_L^K \Phi_L$. Then $\Phi \in \Xi_K$. Choose any Hecke character ξ of infinity type Φ, and put $\beta = \xi_F | \, |_{F_{\mathbb{A}}}^{-1}$. Then we have $\beta_\sigma = \alpha_\sigma$ for all $\sigma \in J''$. If $\infty(\psi) \in \mathrm{Res}_F^K \Xi_K$, then we take a Hecke character φ' of K with $\mathrm{Res}_F^K \infty(\varphi') = \infty(\psi)$ and $\varphi'(x_\infty) = x_\infty^{\infty(\varphi')}$ for all $x_\infty \in K_\infty^\times$. Then $\psi' \varphi'_F = \alpha \psi^{-1} \varphi'_F$ is of finite order. Since $\psi(x_\infty) = x_\infty^{-n-2v}$, we see $(\psi^{-1}\varphi'_F)_\infty = 1$. Thus $(\alpha\psi^{-1}\varphi'_F)_\infty = \alpha_\infty$ and hence, by Lemma 2.1, we can find a finite order Hecke character φ'' of K such that $\alpha\psi^{-1}\varphi'_F\beta = \varphi''_F$. Thus $\alpha\psi^{-1} = \varphi_F\omega$ for $\varphi = (\varphi')^{-1}\varphi''\xi^{-1}$ and $\omega = | \, |_{F_{\mathbb{A}}}$. If $\infty(\psi) \notin \mathrm{Res}_F^K \Xi_K$, then $\infty(\psi^{-1} | \, |_{F_{\mathbb{A}}}^{-1}) \in \mathrm{Res}_F^K \Xi_K$. In this case, choosing an arithmetic φ' such that $\varphi'_F(x_\infty) = x_\infty^{n+2v-\mathbf{1}}$ for all $x \in F_\infty^\times$. Then $(\alpha\psi^{-1} | \, |_{F_{\mathbb{A}}}^{-1} (\varphi'_F)^{-1})_\sigma = 1$ for all $\sigma \in J''$ and hence, by Lemma 2.1, we can find a finite order character φ'' of K such that $\alpha\psi^{-1} | \, |_{F_{\mathbb{A}}}^{-1} (\varphi'_F)^{-1} = \varphi''_F$. Thus again we get $\alpha\psi^{-1} = \varphi_F\omega$ for $\varphi = \varphi'\varphi''$ and $\omega = | \, |_{F_{\mathbb{A}}}$.

Now suppose that F does not contains any CM field and $J'' \neq \emptyset$. Then $\mathrm{Res}_F^K \Xi_K = \mathbb{Z}\mathbf{1} + \Xi$ and $n + 2v = m\mathbf{1}$ for an integer m. We see that m is even if and only if $n + 2v \in \mathrm{Res}_F^K \Xi_K$. Suppose that m is odd. Then $(\alpha\psi^{-1} | \, |_{F_{\mathbb{A}}}^{-m})_\sigma = 1$ for all $\sigma \in J''$. Thus by Lemma 2.1, we can find a finite order Hecke character φ' such that $\alpha\psi^{-1} | \, |_{F_{\mathbb{A}}}^{-m} = \varphi'_F$. Thus $\alpha\psi^{-1} = \varphi_F\omega$ with $\varphi = \varphi'$ and $\omega = | \, |_{F_{\mathbb{A}}}^m$. If K contains a CM field and m is even, we can argue in the same way, replacing m by $m - 1$ and requiring $\infty(\varphi') = \Phi$.

Now suppose only that $\infty(\psi) \in \mathrm{Res}_F^K \Xi_K$. Thus by the above argument, if there exists a finite order character ω of conductor 1 such that $\alpha_\infty = \omega_\infty$, we can find an arithmetic character φ of K such that $\alpha\psi^{-1} = \varphi_F\omega$. In particular, if α_∞ is trivial, then we can find φ and ω.

Note that $\infty(\psi) = -n - 2v$ and $n = nc$ by the unitarity of cuspidal automorphic representations if $H^q_{\mathrm{cusp}}(Y_0(N), \mathcal{L}(\kappa, \psi; \mathbb{C})) \neq 0$. Thus we see that if $x \in K_{\infty+}^\times$, then $x^{\widehat{n}} \in F_{\infty+}^\times$ for the identity connected component $X_{\infty+}^\times$ of $X_\infty^\times$. Thus we have a unique positive square root $x^{\widehat{n}/2}$ of $x^{\widehat{n}}$, and there exists a Hecke character φ' of K such that $\varphi'_\infty(x_\infty) = x_\infty^{\widehat{n}/2 + \widehat{v}}$ for all $x_\infty \in K_{\infty+}^\times$. Then $\varepsilon = \psi\varphi'_F$ is a finite order character. In particular, if $J'' = \emptyset$ (for example, if F is totally imaginary), we can find a finite order Hecke character φ'' with $\alpha\varepsilon = \varphi''_F$, and hence $\alpha\psi^{-1} = \varphi_F$ for $\varphi = \varphi'(\varphi'')^{-1}$. We record what we have proven as follows:

LEMMA 2.2. (o) *Suppose that* $\infty(\psi) \in \mathrm{Res}^K_F \Xi_K$. *If there exists a finite order character* ω *of conductor* 1 *such that* $\alpha_\infty = \omega_\infty$, *we can find an arithmetic character* φ *of* K *such that* $\alpha\psi^{-1} = \varphi_F\omega$. *In particular, if* α_∞ *is trivial, then we can find* φ *with* $\alpha\psi^{-1} = \varphi_F$.

(i) *Suppose that* K *contains a CM field. Suppose either that* $K_{CM} \neq F_{CM}$ *or that* F *does not contains a CM field. Then* $\Xi = \mathbb{Z}\mathbf{1} + \mathrm{Res}^K_F \Xi_K$. *Moreover we can find an arithmetic Hecke character* φ *of* K *such that* $\varphi_F\psi|\ |_{F_\mathbb{A}} = \alpha$.

(ii) *Suppose that* F *does not contain a CM field and there is a real place of* F *which extends to a complex place of* K. *Then* $\Xi = \mathbb{Z}\mathbf{1} + \mathrm{Res}^K_F \Xi_K$. *Write* $n + 2v = m\mathbf{1}$ *for* $m \in \mathbb{Z}$. *Then we can choose* φ *and* ω, *up to finite order characters of conductor* 1, *so that* $\alpha^{-1}\psi = \varphi_F\omega$ *and*

$$\omega = \begin{cases} |\ |^m_{F_\mathbb{A}} & \text{if } m \text{ is odd,} \\ |\ |^{m-1}_{F_\mathbb{A}} & \text{if } m \text{ is even and } K \text{ contains a CM field,} \end{cases}$$

$$\infty(\varphi) = \begin{cases} 0 & \text{if } m \text{ is odd,} \\ \Phi \text{ for a CM type } \Phi & \text{if } m \text{ is even and } K \text{ contains a CM field.} \end{cases}$$

(iii) *If* F *is totally imaginary, then we can find an algebraic Hecke character* φ *of* K *such that* $\varphi_F\psi = \alpha$ *and* $\infty(\varphi) = 1/2(\widehat{n} + 2\widehat{v})$.

2.5. Motivic interpretation and criticality. We already mentioned the conjecture associating λ to a compatible system $\rho(\lambda)$ of $\mathfrak{l}$-adic Galois representations. As is well known, we can state a stronger version of the conjecture in terms of pure motives of dimension 2. This is a special case of a generalization by Langlands of the Shimura-Taniyama conjecture, because H^1 of an elliptic curve is a rank 2 motive. A precise statement can be found in [**Hi94**, Conjecture 0.1]. The point of the conjecture is that there exists a finite extension $E/\mathbb{Q}(\lambda)$, which is either totally real or a CM field, and a pure simple rank 2 motive $M(\lambda)$ with coefficients in E such that $L(s, M(\lambda) \otimes \eta) = L(s, \lambda \otimes \eta)$. In particular, the system $\rho(\lambda)$ is obtained from the étale realization of $M(\lambda)$, and the Hodge type of $M(\lambda) \otimes_{F,\sigma} \mathbb{C}$ is given by

$$(n_\sigma + 1 + v_\sigma, v_{\sigma c}), \quad (v_\sigma, n_\sigma + 1 + v_{\sigma c})$$

for complex conjugation c of $\mathbb{C}$. Thus $w = n_\sigma + 1 + v_\sigma + v_{\sigma c}$ is the weight of $M(\lambda)$ and is independent of σ, which is known without supposing the conjecture. Then the Galois representation $\mathrm{Ad}(\rho(\lambda))$ corresponds to a rank 3 motive $\mathrm{Ad}(M(\lambda))$ sitting inside $M(\lambda) \otimes M(\lambda)^\vee$ for the dual $M(\lambda)^\vee$ of $M(\lambda)$. The functional equation of $L(s, \mathrm{Ad}(\lambda))$ has the Γ-factor exactly equal to that predicted by the theory of motives [**GeJ**]. To describe the Γ-factor of $L(s, \mathrm{Ad}(\lambda) \otimes \eta)$, we write $\eta_\sigma(-1) = -(-1)^{\nu_\sigma}$ with $\nu_\sigma \in \{0, 1\}$ for the restriction η_σ of η to F_σ for each $\sigma \in \Sigma_F(\mathbb{R})$. Then the Γ-factor of $L(s, \mathrm{Ad}(\lambda) \otimes \eta)$ coincides with that of $L(s, \mathrm{Ad}(M(\lambda)) \otimes \eta)$ and is given by, for a finite order character η

$$(\Gamma) \quad \Gamma(s, \mathrm{Ad}(\lambda) \otimes \eta) = \prod_{\sigma \in I} \Gamma_\mathbb{C}(s + n_\sigma + 1) \times \prod_{\sigma \in \Sigma(\mathbb{C})} \Gamma_\mathbb{C}(s)^2 \times \prod_{\sigma \in \Sigma(\mathbb{R})} \Gamma_\mathbb{R}(s + \nu_\sigma),$$

where $\Gamma_\mathbb{C}(s) = (2\pi)^{-s}\Gamma(s)$ and $\Gamma_\mathbb{R}(s) = \pi^{-s/2}\Gamma(s/2)$. Since $\mathrm{Ad}(M)$ is self dual, the functional equation gives the reflection: $s \mapsto 1 - s$, which is known to be true for $L(s, \mathrm{Ad}(\lambda) \otimes \eta)$ with finite order η. A motivic L-value $L(m, M)$ at an integer m

is called critical if the value of the Γ-factor at m and at its reflection point (of the functional equation) is finite. Therefore $L(1, \mathrm{Ad}(\lambda) \otimes \eta)$ is critical if and only if F is totally real and $\nu_\sigma = 1$ for all $\sigma \in I$ (that is, η is totally even). Moreover we have

$$\Gamma(1, \mathrm{Ad}(\lambda) \otimes \eta) = \frac{\Gamma_F(k)}{(2\pi)^{k+2\Sigma(\mathbb{C})}\pi^{\Sigma_+(\eta)}},$$
$$\mathrm{ord}_{s=0}\Gamma(s, \mathrm{Ad}(\lambda) \otimes \eta) = \#\Sigma_-(\eta) + 2r_2(F),$$

where $\mathrm{ord}_{s=0}$ is the order of the pole at $s = 0$; $\Sigma_-(\eta) = \{\sigma|\nu_\sigma = 0\}$; $\Sigma_+(\eta) = \{\sigma|\nu_\sigma = 1\}$; $k = \sum_{\sigma\in I}(n_\sigma + 2)\sigma$; $\Gamma_F(k) = \prod_\sigma \Gamma(k_\sigma)$; $x^k = \prod_\sigma x^{k_\sigma}$; and for a subset X of I, $x^X = x^{\#X}$. The number $\mathrm{ord}_{s=0}\Gamma(s, \mathrm{Ad}(\lambda)\otimes\eta)$ is a good measure to know how the value $L(1, \mathrm{Ad}(\lambda) \otimes \eta)$ is far from being critical. It is interesting that the period $\Omega(\widehat{\lambda})$ defined independent of $\mathrm{ord}_{s=0}\Gamma(s, \mathrm{Ad}(\lambda)\otimes\eta)$ in purely automorphic way gives the Beilinson and Deligne period if one admits the various conjectures including the above one and the Deligne-Beilinson conjecture for motivic L-values [**RSS**]. Moreover, we can prove a very close relation of the L-value $L(1, \mathrm{Ad}(\lambda) \otimes \alpha)$ to the congruence of cohomological cusp forms and hence, presumably, to the Selmer group of $\mathrm{Ad}(\lambda) \otimes \alpha$ if K is quadratic over $\mathbb{Q}$ (see Section 5). This is striking because the definition of $\Omega(\widehat{\lambda})$ is topological (and automorphic) and has nothing to do, in an apparent way, with the algebro-geometric property of $\mathrm{Ad}(M(\lambda)) \otimes \alpha$. Thus the quantity $\Omega(\widehat{\lambda})$ should be very close to the volume at infinity of the Tamagawa measure (of Bloch and Kato) of $\mathrm{Ad}(M(\lambda)) \otimes \alpha$ divided by the order of $H^0(\mathbb{Q}, (\mathrm{Ad}(M(\lambda)) \otimes \alpha) \otimes \mathbb{Q}/\mathbb{Z})$ as in [**BK**, (5.15.1)].

3. Imaginary quadratic case

Let K be an imaginary quadratic field with discriminant $-D$ ($D > 0$). Thus the different $\mathfrak{d}$ of $K/\mathbb{Q}$ is generated by $\sqrt{-D}$. We write σ for the unique non-trivial automorphism of K. Then $H = \mathrm{GL}(2)_{/\mathbb{Q}}$ and $G = \mathrm{Res}_{K/\mathbb{Q}}\mathrm{GL}(2)_{/K}$. We write $S_{\widehat{\kappa}}(N, \chi)_{/K}$ for the space $S_{\widehat{\kappa},\emptyset}(N, \chi)$ of cohomological cusp forms on $G(\mathbb{A})$ of weight $\widehat{\kappa}$. Let $\psi : \mathbb{Q}^\times\backslash\mathbb{A}^\times \to \mathbb{C}^\times$ be a Hecke character such that $\psi_\infty(x) = x^{-n-2v}$ for all $x \in \mathbb{R}^\times$. We put $\chi = \psi \circ N_{K/\mathbb{Q}}$ and consider the isomorphism for $\widehat{\kappa} = (\widehat{n}, \widehat{v})$:

$$\delta = \delta_{\emptyset,\emptyset} : S_{\widehat{\kappa}}(N, \chi)_{/K} \cong H^2_{\mathrm{cusp}}(Y_{0,K}(N), \mathcal{L}(\widehat{\kappa}, \chi; \mathbb{C}))[\chi].$$

We write $I = I_\mathbb{Q} = \{\tau\}$ and use the same symbol τ to indicate a fixed extension of τ to K. For complex conjugation c of $\mathbb{C}$, $\sigma\tau = \tau c$. Then χ is an arithmetic Hecke character whose ∞-type is $-\widehat{n} - 2\widehat{v} = -n_\tau\tau - n_\tau\tau c - 2v_\tau\tau - 2v_\tau\tau c$ for $0 \le \widehat{n} = n_\tau\tau + n_\tau\tau c \in \mathbb{Z}[I_K]$ and $\widehat{v} = v_\tau\tau + v_\tau\tau c \in \mathbb{Z}[I_K]$. Let $f \in S_{\widehat{\kappa}}(N, \chi)$ be a cusp form, and write $f^{(a)} \in S_{\widehat{\kappa}}(\Gamma^{(a)}, \chi)$ for the classical cusp form corresponding to f (see [**Hi94**, 3.5]). In other words, we consider $f^{(a)}(w) = f\left(\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right) w\right)$ for $w \in \mathrm{SL}_2(\mathbb{C}) \subset G(\mathbb{R})$ and $a \in R \subset T(\mathbb{A}^{(\infty)})$ (see 2.2). The Fourier expansion of $f^{(1)}$ is given as follows (cf. [**Hi94**, (6.1)]):

$$f^{(1)}(w) = |y|^{n_\tau}|y|_{K_\mathbb{A}} \left\{ \sum_{\xi\in K^\times} \mathbf{a}(\xi y\mathfrak{d}, f)\xi^{-\widehat{v}}W(\xi y)\mathbf{e}_K(\xi x) \right\},$$

where $w = y^{-1/2}\left(\begin{smallmatrix} y & x \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{C})$ and

$$W(y) = \sum_{0\le\alpha\le n^*} \binom{n^*}{\alpha} \left(\frac{y}{\sqrt{-1}|y|}\right)^{n_\tau+1-\alpha} K_{\alpha-n_\tau-1}(4\pi|y|)S^{n^*-\alpha}T^\alpha \quad (n^* = 2n_\tau + 2)$$

with values in $L((n^*,0);\mathbb{C})=\sum_\alpha \mathbb{C}S^{n^*-\alpha}T^\alpha$. We now write down $\delta(f^{(1)})$ following the process described in [**Hi94**, Section 2.5]. For that, we write $f^{(1)}=\sum_{0\le\alpha\le n^*} f_\alpha\binom{n_\tau^*}{\alpha}S^{n^*-\alpha}T^\alpha$ and

$$\begin{aligned}&(X_\tau V-Y_\tau U)^{n_\tau}(X_{\tau c}U+Y_{\tau c}V)^{n_\tau}(AV-BU)^2\\ &\quad=\sum_{0\le j_\tau\le n}\{(-1)^{j_\tau}\binom{n_\tau}{j_\tau}X_\tau^{n_\tau-j_\tau}Y_\tau^{j_\tau}V^{n_\tau-j_\tau}U^{j_\tau}\times\sum_{0\le j_{\tau c}\le n}\binom{n_\tau}{j_{\tau c}}\\ &\quad\quad X_{\tau c}^{n_\tau-j_{\tau c}}Y_{\tau c}^{j_{\tau c}}U^{n_\tau-j_{\tau c}}V^{j_{\tau c}}\}\times(A^2V^2-2ABUV+B^2U^2)\\ &\quad=\sum_{0\le j\le\widehat{n}}(-1)^{j_\tau}\binom{\widehat{n}}{j}X^{\widehat{n}-j}Y^j\{V^{n_\tau-j_\tau+j_{\tau c}+2}U^{n_\tau+j_\tau-j_{\tau c}}A^2\\ &\quad\quad-2V^{n_\tau-j_\tau+j_{\tau c}+1}U^{n_\tau+j_\tau-j_{\tau c}+1}AB+V^{n_\tau-j_\tau+j_{\tau c}}U^{n_\tau+j_\tau-j_{\tau c}+2}B^2\},\end{aligned}$$

where $\binom{\widehat{n}}{j}=\binom{n_\tau}{j_\tau}\binom{n_\tau}{j_{\tau c}}$. Then replacing $U^\alpha V^{n^*-\alpha}$ by $(-1)^{n^*-\alpha}f_\alpha^{(1)}$, (A,B) by $(y^{-1/2}A,y^{-1/2}B)$, and (A^2,AB,B^2) by $y^{-1}(dy\wedge dx,-2dx\wedge d\overline{x},dy\wedge d\overline{x})$, we have

$$(\delta)\qquad \begin{aligned}\delta(f^{(1)})=w\sum_{0\le j\le\widehat{n}}&(-1)^{n_\tau-j_{\tau c}}\binom{\widehat{n}}{j}X^{\widehat{n}-j}Y^j\{f_{n_\tau+j_\tau-j_{\tau c}}y^{-2}dy\wedge dx\\ &-f_{n_\tau+j_\tau-j_{\tau c}+1}y^{-2}dx\wedge d\overline{x}+f_{n_\tau+j_\tau-j_{\tau c}+2}y^{-2}dy\wedge d\overline{x}\},\end{aligned}$$

where w acts on $(X_\tau,Y_\tau,X_{\tau c},Y_{\tau c})$ as

$$(X_\tau,Y_\tau,X_{\tau c},Y_{\tau c})\mapsto(X_\tau,Y_\tau,X_{\tau c},Y_{\tau c})\begin{pmatrix}{}^tw^\iota&0\\0&{}^t\overline{w}^\iota\end{pmatrix}.$$

Now we restrict $\delta(f^{(1)})$ to the upper half complex plane $\mathfrak{H}$ from $\mathfrak{Z}=\mathcal{H}_\tau$ and write the pull back as $\delta(f^{(1)})|_{\mathbb{Q}}$. Then we have, for w as above

$$\begin{aligned}&\delta(f^{(1)})|_{\mathbb{Q}}=\\ &\quad(-1)^{n_\tau}w\sum_{0\le j\le\widehat{n}}(-1)^{j_{\tau c}}\binom{\widehat{n}}{j}X^{\widehat{n}-j}Y^j\{f_{n_\tau+j_\tau-j_{\tau c}}+f_{n_\tau+j_\tau-j_{\tau c}+2}\}y^{-2}dy\wedge dx.\end{aligned}$$

This differential form has values in

$$L(\widehat{\kappa},\chi;\mathbb{C})|_{\mathfrak{H}}\cong L(\kappa,\chi_F;\mathbb{C})\otimes L(\kappa;\mathbb{C}).$$

Noting that $\chi_F=\psi^2$, we know that

$$L(\kappa,\chi_F;A)\otimes L(\kappa;A)\ \cong\oplus_{0\le i\le n_\tau}L((2n_\tau-2i,2v_\tau+i),\psi^2;A)$$

for any $\mathbb{Q}(\kappa,\psi)$-algebra A [**Hi94**, (11.2a)]. In order to write down the projection to $L((0,2v_\tau+n_\tau),\chi_F;A)$, we introduce a differential operator:

$$\nabla=\frac{\partial^2}{\partial X_{\tau c}\partial Y_\tau}-\frac{\partial^2}{\partial X_\tau\partial Y_{\tau c}}=\frac{\partial^2}{\partial X_{\sigma\tau}\partial Y_\tau}-\frac{\partial^2}{\partial X_\tau\partial Y_{\sigma\tau}}.$$

Following the expression of the projection in [**Hi94**, p. 498], it is given by $(n_\tau!)^{-2}\nabla^{n_\tau}$. Thus we need to compute $(\nabla^{n_\tau}\delta(f^{(1)})|_{\mathbb{Q}})$. We see

$$\nabla^m=\sum_{0\le k\le m}(-1)^{m-k}\binom{m}{k}\left(\frac{\partial^2}{\partial X_{\sigma\tau}\partial Y_\tau}\right)^k\left(\frac{\partial^2}{\partial X_\tau\partial Y_{\sigma\tau}}\right)^{m-k}$$

and $k!^{-1}(\partial/\partial T)^k T^m = \binom{m}{k} T^{m-k}$. The differential operator ∇ decreases the degree by $(1,1)$ either in $(X_\tau, Y_{\sigma\tau})$ or in $(X_{\sigma\tau}, Y_\tau)$. Thus applying ∇ to a monomial n_τ times, we see

$$\nabla^{n_\tau} X^{n-j} Y^j = 0 \quad \text{unless} \quad n_\tau = j_\tau + j_{\sigma\tau}.$$

If $n_\tau = j_\tau + j_{\sigma\tau}$, we get

$$(n_\tau!)^{-2} \nabla^{n_\tau} X_\tau^{n_\tau - j_\tau} Y_\tau^{j_\tau} X_{\sigma\tau}^{n_{\sigma\tau} - j_{\sigma\tau}} Y_{\sigma\tau}^{j_{\sigma\tau}} = (-1)^{j_\tau} \frac{(j_\tau! j_{\sigma\tau}!)^2}{(n_\tau!)^2} \binom{n_\tau}{j_\tau} = (-1)^{j_\tau} \binom{n_\tau}{j_\tau}^{-1}.$$

Thus noting the fact that

$$\nabla^{n_\tau}(wP) = \nabla^{n_\tau} P \quad \text{if } \det(w) = 1,$$

and writing j for j_τ, we have

$$\begin{aligned}
&(n_\tau!)^{-2} (\nabla^{n_\tau} \delta(f^{(1)})|_{\mathbb{Q}}) \\
&\quad = \sum_{0 \le j \le n_\tau} \binom{n_\tau}{j}^2 (n_\tau!)^{-2} \nabla^{n_\tau} X^{n_\tau - j} Y_\tau^j X_{\tau c}^j Y_{\tau c}^{n_\tau - j} \{ f_{2j} + f_{2j+2} \} y^{-2} dy \wedge dx \\
&\quad = \sum_{0 \le j \le n_\tau} \binom{n_\tau}{j} (f_{2j} + f_{2j+2}) y^{-2} dy \wedge dx,
\end{aligned}$$

which is the explicit form of the pull back differential form we have written as $\pi(\delta_{J,J'}(f))$ in Section 2.4.

Now we start computing the integral $\int_{Y_{0,\mathbb{Q}}(N')} \pi(\delta_{J,J'}(f))$. We put

$$\Phi = H(\mathbb{Q}) \cap U_0(N') H(\mathbb{R})_+ = \Gamma_0(N')$$

for $H = \mathrm{GL}(2)_{/\mathbb{Q}}$ and $N' = N \cap \mathbb{Q}$. Then $Y_{0,\mathbb{Q}}(N') = \Phi \backslash \mathfrak{H}$. We now define

$$\Phi_\infty = \left\{ \pm \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \mid m \in \mathbb{Z} \right\},$$

which is the stabilizer of the cusp ∞ in Φ. Then we look at

$$\int_{\Phi_\infty \backslash \mathfrak{H}} (n_\tau!)^{-2} \left(\nabla^{n_\tau} \delta(f^{(1)})|_{\mathbb{Q}} \right) y^s = \sum_{0 \le j \le n_\tau} \binom{n_\tau}{j} \int_0^\infty \int_0^1 (f_{2j} + f_{2j+2}) dx y^{s-2} dy.$$

Now we assume that $f|T(\mathfrak{n}) = \mu(T(\mathfrak{n}))f$ for a system of Hecke eigenvalues μ of G and f is normalized so that $\mathbf{a}(\mathfrak{n}, f) = \mu(T(\mathfrak{n}))$. We compute

$$\begin{aligned}\int_0^\infty \int_0^1 f_{2j} dx y^{s-2} dy &= \sum_{\xi \in K^\times} \mu(T(\xi\mathfrak{d}))\xi^{-\widehat{v}} \left(\frac{\xi}{i|\xi|}\right)^{n_\tau+1-2j} \\ &\quad \cdot \int_0^\infty y^{n_\tau+s} K_{2j-n_\tau-1}(4\pi|\xi y|) \int_0^1 \mathbf{e}(\mathrm{Tr}(\xi)x) dx dy \\ &= D^{v_\tau} \sum_{0\neq m\in\mathbb{Z}} \mu(T(m)) m^{-2v} \mathrm{sgn}(m)^{n_\tau+1-2j} \\ &\quad \cdot \int_0^\infty y^{n_\tau+s} K_{2j-n_\tau-1}(4\pi D^{-1/2}|my|) dy \\ &= D^{v_\tau}(1+(-1)^{n_\tau+1+2v}) \sum_{m=1}^\infty \mu(T(m)) m^{-2v} \\ &\quad \cdot \int_0^\infty y^{n_\tau+s} K_{2j-n_\tau-1}(4\pi D^{-1/2}|my|) dy \\ &= \frac{2^{n_\tau+s-1}(1+(-1)^{n_\tau+1+2v})D^{v_\tau}}{(4\pi D^{-1/2})^{n_\tau+s+1}} \Gamma\left(\frac{s+2j}{2}\right) \\ &\quad \cdot \Gamma\left(\frac{2n_\tau+s+2-2j}{2}\right) L_{K/\mathbb{Q}}(s,\mu,\omega),\end{aligned}$$

where $\omega(a) = |a|_{\mathbb{A}}^{n_\tau+2v_\tau}$ and thus $\omega((\ell)) = |\ell|^{-n_\tau-2v_\tau}$ (see Lemma 2.2 (ii)). Similarly we get

$$\begin{aligned}\int_0^\infty \int_0^1 f_{2j+2} dx y^{s-2} dy = &\frac{2^{n_\tau+s-1}(1+(-1)^{n_\tau+1+2v})D^{v_\tau}}{(4\pi D^{-1/2})^{n_\tau+s+1}} \Gamma\left(\frac{s+2j+2}{2}\right) \\ &\cdot \Gamma\left(\frac{2n_\tau+s-2j}{2}\right) L_{K/\mathbb{Q}}(s,\mu,\omega).\end{aligned}$$

We see from $\Gamma(s+1) = s\Gamma(s)$ that

$$\begin{aligned}\Gamma\left(\frac{s+2j}{2}\right)&\Gamma\left(\frac{2n_\tau+s+2-2j}{2}\right) + \Gamma\left(\frac{s+2j+2}{2}\right)\Gamma\left(\frac{2n_\tau+s-2j}{2}\right) \\ &= \frac{2n_\tau+s-2j}{2}\Gamma\left(\frac{s+2j}{2}\right)\Gamma\left(\frac{2n_\tau+s-2j}{2}\right) \\ &\quad + \frac{s+2j}{2}\Gamma\left(\frac{s+2j}{2}\right)\Gamma\left(\frac{2n_\tau+s-2j}{2}\right) \\ &= (n_\tau+s)\Gamma\left(\frac{s+2j}{2}\right)\Gamma\left(\frac{2n_\tau+s-2j}{2}\right).\end{aligned}$$

Note that by [**Hi94**, p. 505]

$$\sum_{j=0}^m \binom{m}{j} \Gamma\left(\frac{s+2j}{2}\right)\Gamma\left(\frac{2m+s-2j}{2}\right) = \frac{\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s}{2}\right)\Gamma(s+m)}{\Gamma(s)}.$$

This shows

$$\int_{\Phi_\infty\backslash\mathfrak{H}} (n_\tau!)^{-2}(\nabla^{n_\tau}\delta(f^{(1)})|_{\mathbb{Q}})y^s$$
$$= \frac{(1+(-1)^{n_\tau+1+2v})2^{n_\tau+s-1}D^{v_\tau}}{(4\pi D^{-1/2})^{n_\tau+s+1}} \frac{\Gamma(\frac{s}{2})\Gamma(\frac{s}{2})\Gamma(s+n_\tau+1)}{\Gamma(s)} L_{K/\mathbb{Q}}(s,\mu,\omega),$$

because (cf. (Γ) in 2.5)

$$(n_\tau+s)\frac{\Gamma(\frac{s}{2})\Gamma(\frac{s}{2})\Gamma(s+n_\tau)}{\Gamma(s)} = \frac{\Gamma(\frac{s}{2})\Gamma(\frac{s}{2})\Gamma(s+n_\tau+1)}{\Gamma(s)}.$$

We now apply convolution method and rewrite the above integral as an integral of the product of $(n_\tau!)^{-2}(\nabla^{n_\tau}\delta(f^{(1)})|_{\mathbb{Q}})$ and an Eisenstein series. There are two ways of associating a Dirichlet character to a Hecke character ξ of $\mathbb{Q}$. Let C be a positive multiple of the conductor $C(\xi)$ of ξ. Restricting ξ to $\prod_{\ell|C}\mathbb{Z}_\ell^\times \subset \mathbb{A}^\times$, we get a character ξ_C of $(\mathbb{Z}/C\mathbb{Z})^\times$. On the other hand, for each prime ℓ prime to C, we consider $\ell^{(C)} \in \mathbb{A}^\times$ which is equal to 1 at all places dividing C and coincides with ℓ outside C. Then $\xi^* : \ell \mapsto \xi(\ell^{(C)})$ induces a Dirichlet character $\xi^* : (\mathbb{Z}/C\mathbb{Z})^\times \to \mathbb{C}^\times$. Since $\xi(\ell)=1$, $\xi^* = \xi_C^{-1}$. We note that $L(s-\infty(\xi),\xi) = \sum_{n=1}^\infty \xi^*(n)n^{-s} = L(s,\xi^*)$ identifying $\Xi_{\mathbb{Q}}$ with $\mathbb{Z}$ via $\tau \mapsto 1$. Let N' be a positive integer generating $N\cap\mathbb{Z}$ and $\psi_{N'}^2 = (\chi_{\mathbb{Q}})_{N'}$ be the restriction of χ to $\widehat{\mathbb{Z}}^\times$. We extend $\psi_{N'}$ to Φ so that $\psi_{N'} : \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \mapsto \psi_{N'}(d)$. We have

$$\int_{\Phi_\infty\backslash\mathfrak{H}} (n_\tau!)^{-2}(\nabla^{n_\tau}\delta(f^{(1)})|_{\mathbb{Q}})y^s = \int_{\Phi\backslash\mathfrak{H}} \sum_{\gamma\in\Phi_\infty\backslash\Phi} \{(n_\tau!)^{-2}(\nabla^{n_\tau}\delta(f^{(1)})|_{\mathbb{Q}})y^s\}\circ\gamma$$
$$= \int_{\Phi\backslash\mathfrak{H}} (n_\tau!)^{-2}(\nabla^{n_\tau}\delta(f^{(1)})|_{\mathbb{Q}})y^s \sum_{\gamma\in\Phi_\infty\backslash\Phi} \psi_{N'}^2(\gamma)|j(\gamma,z)|^{-2s}.$$

Thus writing

$$E(s) = E(s,\psi_{N'}^2) = L_{N'}(2s,(\psi^*)^2)y^s \sum_{\gamma\in\Phi_\infty\backslash\Phi} \psi_{N'}^2(\gamma)|j(\gamma,z)|^{-2s},$$

we get

$$\int_{\Phi\backslash\mathfrak{H}} (n_\tau!)^{-2}(\nabla^{n_\tau}\delta(f^{(1)})|_{\mathbb{Q}})E(s)$$
$$= \frac{(1+(-1)^{n_\tau+1+2v})2^{n_\tau+s-1}D^{v_\tau}}{(4\pi D^{-1/2})^{n_\tau+s+1}} \frac{\Gamma(\frac{s}{2})\Gamma(\frac{s}{2})\Gamma(s+n_\tau+1)}{\Gamma(s)}$$
$$\cdot L(s+1,(\mu\otimes\mu_\sigma)_+\otimes\omega).$$

This follows from $(R1)$ because $\omega = |\ |_{\mathbb{A}}^m$ for $m = n_\tau + 2v_\tau$ and

$$L_{N'}(2s,(\psi^*)^2) = L_{N'}(2s,(\psi\omega)^2) = L_{N'}(2s,\chi_{\mathbb{Q}}\omega^2).$$

On the other hand, we have

$$E(z;s,\psi_{N'}^2) = \frac{1}{2}y^s \sum_{(m,n)\in\mathbb{Z}^2-(0,0)} \psi_{N'}^2(n)|mNz+n|^{-2s}.$$

As will be seen in Appendix (see also [**Hi93a**, Chapter 9, p. 293(1)]), the Fourier expansion of the Eisenstein series is given by

$$E\left(-\frac{1}{N'z}; s, \psi_{N'}^2\right) = 2^{1-2s}N'^{-s}(2\pi y)^{1-s}\frac{\Gamma(2s-1)L_{N'}(2s-1,(\psi^*)^2)}{\Gamma(s)^2}$$
$$+ \text{ Fourier expansion with coefficients in entire functions of } s.$$

This shows that

$$\mathrm{Res}_{s=1}E(z;s,\psi_{N'}^2) = 2^{-1}\pi N'^{-2}\phi(N')\delta_{\psi_{N'}^2,id},$$

where $\delta_{\varepsilon,\varepsilon'}$ is 1 or 0 according as $\varepsilon=\varepsilon'$ or not, and ϕ is the Euler function. From this we get, if $n+1+2v$ is even,

(Res3) $$\mathrm{Res}_{s=1}L(s+1,(\mu\otimes\mu_\sigma)_+\otimes\omega)$$
$$= \frac{(2\pi D^{-1/2})^{n+2}\phi(N')}{N'^2 D^{v_\tau}\Gamma(n_\tau+2)}\delta_{\psi_{N'}^2,id}\int_{\Phi\backslash\mathfrak{H}}(n_\tau!)^{-2}(\nabla^{n_\tau}\delta(f^{(1)}))|_{\mathbb{Q}}.$$

Let $\varphi : K^\times\backslash K_{\mathbb{A}}^\times \to \mathbb{C}^\times$ be a Hecke character of infinity type $-w$ with conductor $C = C(\varphi)$. We replace f in the above formula by the twist $g = f|R(\varphi) \in S_{\widehat{\kappa}+(0,w)}(N\cap C^2,\chi\varphi^2)$ defined in [**Hi94**, p. 480] whose Fourier coefficients are given by $\mathbf{a}(\mathfrak{n}, f|R(\varphi)) = G(\varphi)\mathbf{a}(\mathfrak{n},f)\varphi(\mathfrak{n})$. Here $G(\varphi)$ is the Gauss sum given by

$$G(\varphi)=\varphi(d)^{-1}\sum_{u\in R}\varphi_C(u)\mathbf{e}_K(d^{-1}u),$$

where R is a complete set of representatives of $C^{-1}/\mathcal{R}$ in $\prod_{\mathfrak{p}|C}K_{\mathfrak{p}}$ and $d\in K_{\mathbb{A}}^\times$ with $d^{(\mathfrak{d})}=1$ such that $d\mathcal{R}$ is the different $\mathfrak{d}=\mathfrak{d}_K$ of $K/\mathbb{Q}$. We write N' for the positive generator of $N\cap C^2\cap\mathbb{Z}$. Then we have from [**Hi94**, (6.8)], writing $E(s)=E(s,(\psi\varphi)_{N'}^2)$,

$$\int_{\Phi\backslash\mathfrak{H}}(n_\tau!)^{-2}(\nabla^{n_\tau}\delta(g^{(1)})|_{\mathbb{Q}})E(s)$$
$$= \frac{(1+(-1)^{n+1+2v+\mathrm{Res}(w)})2^{n+s-1}D^{v_\tau}\sqrt{-D}^{w}G(\varphi)}{(4\pi D^{-1/2})^{n_\tau+s+1}}$$
$$\cdot\frac{\Gamma(\frac{s}{2})\Gamma(\frac{s}{2})\Gamma(s+n_\tau+1)}{\Gamma(s)}L_{C'}(s+1,((\mu\otimes\varphi)\otimes(\mu\otimes\varphi)_\sigma)_+\otimes\omega),$$

where $C'=C\cap\mathbb{Z}$, and $\omega(x)=|x|_{\mathbb{A}}^{n_\tau+2v_\tau+\mathrm{Res}(w)}$ writing $\mathrm{Res}(w)$ for $\mathrm{Res}_{\mathbb{Q}}^K w\in\mathbb{Z}$. If $\psi\varphi_{\mathbb{Q}}\omega=\alpha$ and $\omega=|\ |_{\mathbb{A}}^{n_\tau+2v_\tau+\mathrm{Res}(w)}$, then $1+(-1)^{n+1+2v+\mathrm{Res}(w)}=2$, and the above integral does not vanish trivially (it could vanish by a different reason). Now as in (Res2) in 2.4, we get

PROPOSITION 3.1. *Let $F=\mathbb{Q}$ and $K=\mathbb{Q}(\sqrt{-D})$ be an imaginary quadratic field with discriminant $-D$ and $\kappa=(n_\tau\tau,v_\tau\tau)\in\mathbb{Z}[I]$ with $I=\{\tau\}$. Let φ be an arithmetic Hecke character with $\infty(\varphi)=-w\in\mathbb{Z}[I_K]$ and the conductor C. Let $\chi=\psi\circ N_{K/\mathbb{Q}}$ for an arithmetic Hecke character ψ with $\psi_\infty(x)=x^{-n_\tau-2v_\tau}$ for all $x\in\mathbb{A}_\infty^\times=\mathbb{R}^\times$. Suppose that $\alpha=\varphi_{\mathbb{Q}}\omega\psi$ with $\omega(x)=|x|_{\mathbb{A}}^{n_\tau+2v_\tau+\mathrm{Res}(w)}$ and $\alpha=(\frac{-D}{})$. If a primitive system $\mu: h_{\widehat{\kappa}}(N,\chi;\mathbb{C})_{/K}\to\mathbb{C}$ is a base change lift of $\lambda: h_\kappa(N_0,\psi;\mathbb{C})_{/\mathbb{Q}}\to\mathbb{C}$, then we have*

$$\int_{Y_{0,\mathbb{Q}}(N')}(n_\tau!)^{-2}(\nabla^{n_\tau}\delta(g^{(1)})|_{\mathbb{Q}}) = C_0^{-1}L_{C'}(1,\mathrm{Ad}(\lambda)\otimes\alpha),$$

where $g = f|R(\varphi)$ for the normalized primitive form f with $f|T(\mathfrak{n}) = \mu(T(\mathfrak{n}))f$ for all $\mathfrak{n} \subset \mathcal{R}$, N' is the positive generator of $N \cap C^2 \cap \mathbb{Z}$, $C' = C \cap \mathbb{Z}$ and

$$C_0 = (2\pi D^{-1/2})^{n_\tau+2} D^{-v} \sqrt{-D}^{-w} \{G(\varphi)\Gamma(n_\tau+2)\}^{-1} \phi(N') {N'}^{-2}$$
$$= (\chi\varphi)_\infty(\sqrt{-D}) D^{-1} \{G(\varphi)\Gamma(1, \mathrm{Ad}(\lambda)\otimes\alpha)\}^{-1} \phi(N') {N'}^{-2}.$$

If $\mu : h_{\widehat{\kappa}}(N,\chi;\mathbb{C})_{/K} \to \mathbb{C}$ is not a twist of any base change lift, then

$$\int_{Y_{0,\mathbb{Q}}(N')} (n_\tau!)^{-2} (\nabla^{n_\tau} \delta(g^{(1)})|_{\mathbb{Q}}) = 0.$$

PROOF. We only need to show the vanishing when μ is not a base change lift from H. We follow the argument in [**HaLR**, 3.12] (and [**Fl**, Section 5]). For that, it is sufficient to show that $L(s, ((\mu\otimes\varphi)\otimes(\mu\otimes\varphi)_\sigma)_\pm \otimes \omega)$ is holomorphic at $s = 2$. Since $\mu\otimes\varphi$ is not a base change lift, we may rewrite μ for $\mu\otimes\varphi$. Then we see, writing $\kappa' = \widehat{\kappa}+(0,w) = (n', v')$ for the weight of μ, $L(s, (\mu\otimes\mu_\sigma)_+\otimes\omega) = L(s+m, (\mu\otimes\mu_\sigma)_+)$ for $m = n_\tau + 2v_\tau + \mathrm{Res}(w)$. Note that $m\mathbf{1}_K = n' + 2v' + n'c + 2v'c$. Looking into Euler factorization, we get

$$L(s, (\mu\otimes\mu_\sigma)_+) L(s, (\mu\otimes\mu_\sigma)_-) = L(s, \mu\otimes\mu_\sigma)$$

with the notation of 2.3. Since $\mu \neq \mu_\sigma$ for non-base change μ, $L(s, \mu\otimes\mu_\sigma)$ is an entire function of s ([**J**], [**U**] and [**GeJ**]). We show that $L(s, (\mu\otimes\mu)_-)$ is entire. For that, we recall a result in [**Gh**]. Let $\phi : h_{\kappa''}(M,\xi;\mathbb{C}) \to \mathbb{C}$ be a system of Hecke eigenvalues of weight $\kappa'' = (n'', v'')$ for G. Then writing

$$\delta' = \delta_{\emptyset,J} : S_{\kappa''}(M,\xi) \cong H^1_{\mathrm{cusp}}(Y_{0,K}(M), \mathcal{L}(\kappa'',\xi;\mathbb{C}))[\xi]$$

for $J = \{\tau\}$, we have

$$\int_{Y_{0,\mathbb{Q}}(M')} [\delta'(h) \wedge E_2(s)]$$
$$= C_1 \Gamma(s + n''_\tau + 2) \Gamma\left(\frac{s+2}{2}\right)^2 \Gamma(s+2) L(n''_\tau + \mathrm{Res}(v'') + s, (\phi\otimes\phi_\sigma)_+),$$

if $n''_\tau \equiv \mathrm{Res}(v'') \bmod 2$ [**Gh**, 7.3(21) and 7.4], where h satisfies $h|T(\mathfrak{n}) = \phi(T(\mathfrak{n}))h$. Here $E_2(s)$ is a degree 1 Eisenstein differential form of weight 2 which is finite at $s = 2$ and $C_1 \neq 0$. We choose φ' such that $\varphi'_F((m)) = \alpha(m) m^{\mathrm{Res}(w')}$ for all $m \in \mathbb{Z}$. Then $\mathrm{Res}(w')$ is odd, and for $\phi = \mu\otimes\varphi'$,

$$L(n''_\tau + \mathrm{Res}(v'') + s, (\phi\otimes\phi_\sigma)_+) = L(n'_\tau + \mathrm{Res}(v') + s, (\mu\otimes\mu_\sigma)_-).$$

We can check the condition $n''_\tau \equiv \mathrm{Res}(v'') \bmod 2$ using the fact that $\mathrm{Res}(w')$ is odd and $\psi\varphi_{\mathbb{Q}}\omega = \alpha$. This shows the holomorphy of $L(s, (\mu\otimes\mu_\sigma)_-)$. On the other hand, by a result of Shahidi [**S88**, Theorem 5.1] and [**S81**, p. 564] (see also [**Fl**, Section 5] and [**FlZ**]), $L(s, (\mu\otimes\mu_\sigma)_\pm)$ does not vanish at $m+2$. Then the result follows from the holomorphy of $L(s, \mu\otimes\mu_\sigma)$. □

Let A be a Dedekind domain in $\mathbb{C}$ containing the values $\mu(T(\mathfrak{n}))$ for ideals $\mathfrak{n} \subset \mathcal{R}$. Then it is easy to see that A contains $\chi(\mathfrak{n})$ for ideals $\mathfrak{n} \subset \mathcal{R}$. We further suppose that $H^2_{\mathrm{cusp}}(Y_{0,K}(N), \mathcal{L}(\widehat{\kappa},\chi;A))[\mu] = A\xi(\mu)$. This condition holds for discrete valuation rings A and the integer ring A of sufficiently large finite extension of $\mathbb{Q}(\mu)$. We define $\Omega_2(\mu^\rho; A)$ by

$$\delta_2(f^\rho) = \Omega_2(\mu^\rho; A)\xi(\mu)^\rho,$$

where $\xi(\mu)^\rho$ is the Galois conjugate of $\xi(\mu)$ under the cohomological rational structure, and similarly, f^ρ is the Galois conjugate of f under the rational structure with respect to Fourier expansion. This is the definition of modular (second) periods in imaginary quadratic case. We put $\underline{\Omega}_2(\mu;\mathbb{Q}(\mu)) = (\Omega_2(\mu^\rho;\mathbb{Q}(\mu^\rho)))_\rho$ as an element of $\mathbb{Q}(\mu)\otimes_{\mathbb{Q}}\mathbb{C} = \prod_\rho \mathbb{C}$, where ρ runs over all embeddings of $\mathbb{Q}(\mu)$ into $\mathbb{C}$. Since $R(\varphi)$ sends $H^2_{\text{cusp}}(Y_{0,K}(N),\mathcal{L}(\widehat{\kappa},\chi;A))[\mu]$ into

$$H^2_{\text{cusp}}(Y_{0,K}(N\cap C^2),\mathcal{L}(\widehat{\kappa}+(0,w),\chi\varphi^2;A))[\mu\otimes\varphi]$$

as seen in the proof of Theorem 8.1 of [**Hi94**],

$$\mathbf{G}(\varphi)\underline{\Omega}_2(\mu\otimes\varphi;\mathbb{Q}(\mu\otimes\varphi)) = \underline{\Omega}_2(\mu;\mathbb{Q}(\mu))$$

in $\mathbb{Q}(\mu\otimes\varphi)\otimes_{\mathbb{Q}}\mathbb{C}$ up to factors in $\mathbb{Q}(\mu\otimes\varphi)^\times$, where $\mathbf{G}(\varphi) = (G(\varphi^\rho))_\rho \in \mathbb{Q}(\varphi)\otimes\mathbb{C}$.

For each $\rho\in\mathrm{Aut}(\mathbb{C}/\mathbb{Q})$, we take $m(\rho)\in\widehat{\mathbb{Z}}^\times$ such that the Artin symbol $[m(\rho),\mathbb{Q}]$ coincides with ρ on the maximal abelian extension of $\mathbb{Q}$. We now look into the ratio: $G(\varphi)/G(\alpha\psi^{-1})$. We conclude from $\alpha\psi^{-1} = \varphi_{\mathbb{Q}}\omega$ with ω of conductor 1 that

$$\left(\frac{G(\varphi)}{G(\alpha\psi^{-1})}\right)^\rho = \frac{\varphi(m(\rho))G(\varphi^\rho)}{\alpha\psi(m(\rho))G((\alpha\psi^{-1})^\rho)} = \frac{G(\varphi^\rho)}{G(\alpha(\psi^\rho)^{-1})}.$$

Hence $G(\varphi)/G(\alpha\psi^{-1})\in\mathbb{Q}(\varphi)$. We then put

$$L(\lambda^\rho) = \frac{(\chi\varphi)^{(\infty)}(\sqrt{-D})D\Gamma(1,\mathrm{Ad}(\lambda)\otimes\alpha)L_{C'}(1,\mathrm{Ad}(\lambda^\rho)\otimes\alpha)}{G(\alpha\psi^\rho)\Omega_2(\widehat{\lambda}^\rho;A)}.$$

Now assume that A is a valuation ring in $\mathbb{Q}(\lambda)$. In the process of proving Proposition 3.1, the only point where we might get a denominator in the L-value is through the maps: $R(\varphi)$ and $(n_\tau!)^{-2}\nabla^{n_\tau}$. Thus if $\varphi = \mathrm{id}$ and the residual characteristic of the valuation ring A is prime to $n_\tau!$, we get an A-integral value. Then from Proposition 3.1, we conclude

COROLLARY 3.2. *If $A = \mathbb{Q}(\lambda)$, we have, for all $\rho\in\mathrm{Aut}(\mathbb{C})$,*

$$L(\lambda)^\rho = L(\lambda^\rho).$$

This shows that $\{L(\lambda^\rho)\}_\rho\in\mathbb{Q}(\lambda)\otimes 1$ in $\mathbb{Q}(\lambda)\otimes_{\mathbb{Q}}\mathbb{C}$. Moreover, if A is a discrete valuation ring in $\mathbb{Q}(\lambda)$ with residual characteristic prime to $6(n!)$ and if $\varphi = \mathrm{id}$, then for the positive generator N_0 of $N\cap\mathbb{Z}$,

$${N_0}^2\phi(N_0)^{-1}\frac{(\chi)^{(\infty)}(\sqrt{-D})D\Gamma(1,\mathrm{Ad}(\lambda)\otimes\alpha)L(1,\mathrm{Ad}(\lambda)\otimes\alpha)}{\Omega_2(\widehat{\lambda};A)}\in A.$$

4. Real quadratic case

We assume that $K = \mathbb{Q}(\sqrt{D})$ is a real quadratic field with discriminant $D > 0$. Let $I_K = \{\tau,\sigma\tau\}$ be the set of real embeddings of K for the generator σ of $\mathrm{Gal}(K/\mathbb{Q})$. Let $\varphi: K^\times\backslash K^\times_{\mathbb{A}}\to\mathbb{C}^\times$ be a Hecke character of infinity type $-\widehat{w} = -w\tau - w\sigma\tau$ with conductor C. Put $\omega = |\ |_{\mathbb{A}}^{n+2v+2w}$ identifying $\mathbb{Z}[I]\cong\mathbb{Z}$ by $n\tau\leftrightarrow n$. We keep the notation for λ introduced in Section 3. Thus $\chi = \psi\circ N_{K/\mathbb{Q}}$ and $\psi_\infty(x) = x^{-n-2v}$ for $x\in\mathbb{A}^\times_\infty$. Let J be a subset of I_K with $|J| = 1$. We consider a normalized primitive form $f\in S_{k,J}(N,\chi)$ with $f|T(\mathfrak{n}) = \mu(T(\mathfrak{n}))f$ for all integral ideals $\mathfrak{n}$, where $\kappa = (\widehat{n}, \widehat{v})$ with $\widehat{n} = n\sigma\tau + n\tau$ and $\widehat{v} = v\sigma\tau + v\tau$. We may assume that $J = \{\tau\}$. Then, for $y^{-1/2}\left(\begin{smallmatrix} y & x \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{SL}_2(F_\infty)$, we see

$$\begin{aligned} &f^{(1)}\left(y^{-1/2}\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\right) \\ &= y^{\widehat{n}/2}|y|_{K_{\mathbb{A}}}\left\{\sum_{\xi\in K^\times, \xi^{\sigma\tau}<0, \xi^\tau>0} \mu(T(\xi\mathfrak{d}))|\xi|^{-\widehat{v}}\exp(-2\pi(|\xi^{\sigma\tau}y_{\sigma\tau}| + |\xi^\tau y_\tau|))\mathbf{e}(\xi x)\right\}, \end{aligned}$$

$$\begin{aligned} \delta\left(f^{(1)}\right) &= \delta_J\left(f^{(1)}\right) \\ &= (y^{\widehat{n}/2}|y|_{K_{\mathbb{A}}})^{-1} f^{(1)}(-1)^n (X_\tau - z_\tau Y_\tau)^n (X_{\sigma\tau} - \overline{z}_{\sigma\tau} Y_{\sigma\tau})^n dz_\tau \wedge d\overline{z}_{\sigma\tau}. \end{aligned}$$

This shows, as in the same manner in the previous section,

$$\begin{aligned} &(n!)^{-2}(\nabla^n\delta(f^{(1)})|_{\mathbb{Q}}) \\ &= (\overline{z} - z)^n\left\{\sum_{\xi^{\sigma\tau}<0, \xi^\tau>0} \mu(T(\xi\mathfrak{d}))|\xi|^{-\widehat{v}}\exp(-2\pi(|\xi^\tau| + |\xi^{\sigma\tau}|)y)\mathbf{e}_K(\mathrm{Tr}(\xi)x)\right\} dz \wedge d\overline{z}, \end{aligned}$$

where the summation is taken over $\xi \in K^\times$ with $\xi^{\sigma\tau} < 0$ and $\xi^\tau > 0$. Replacing f by $g = f|R(\varphi)$, we get

$$\begin{aligned} &(n!)^{-2}(\nabla^n\delta(g^{(1)})|_{\mathbb{Q}}) \\ &= G(\varphi)(-2iy)^n\left\{\sum_\xi \mu(T(\xi\mathfrak{d}))\frac{\varphi(\xi\mathfrak{d})}{|\xi|^{\widehat{v}+\widehat{w}}}\exp(-2\pi(|\xi^\tau| + |\xi^{\sigma\tau}|)y)\mathbf{e}_K(\mathrm{Tr}(\xi)x)\right\} dz \wedge d\overline{z} \end{aligned}$$

Then, noting $dz \wedge d\overline{z} = 2idy \wedge dx$

$$\begin{aligned} &\int_{\Phi_\infty\backslash\mathfrak{H}} (n!)^{-2}(\nabla^n\delta(g^{(1)})|_{\mathbb{Q}})y^s \\ &\qquad = -(-2i)^{n+1}G(\varphi)(4\pi D^{-1/2})^{-s-n-1}D^{v+w}\Gamma(s+n+1)L(s, \mu, \varphi_{\mathbb{Q}}\omega). \end{aligned}$$

By Rankin convolution method, writing $E(s) = E(s, (\psi\varphi)^2_{N'})$ for $N' = N \cap C^2 \cap \mathbb{Z}$ as in the previous section, we get

$$\begin{aligned} &\int_{\Phi\backslash\mathfrak{H}} (n!)^{-2}(\nabla^n\delta(g^{(1)})|_{\mathbb{Q}})E(s) \\ &\qquad = -\frac{(-2i)^{n+1}G(\varphi)D^{v+w}\Gamma(s+n+1)}{(4\pi D^{-1/2})^{s+n+1}}L_{C'}(s+1, ((\mu\otimes\varphi)\otimes(\mu\otimes\varphi)_\sigma)_+ \otimes \omega), \end{aligned}$$

where $C' = C \cap \mathbb{Z}$. In particular, if $\mu = \widehat{\lambda}$,

$$\begin{aligned} &\int_{\Phi\backslash\mathfrak{H}} (n!)^{-2}(\nabla^n\delta(g^{(1)})|_{\mathbb{Q}})E(s) \\ &\qquad = -\frac{(-2i)^{n+1}G(\varphi)D^{v+w}\Gamma(s+n+1)}{(4\pi D^{-1/2})^{s+n+1}}L_{C'}(s, \mathrm{Ad}(\lambda)\otimes\psi\varphi_{\mathbb{Q}}\omega)L(s, \psi\alpha\varphi_{\mathbb{Q}}\omega). \end{aligned}$$

Thus assuming that $\alpha = \psi\varphi_{\mathbb{Q}}\omega$ for $\alpha = (\frac{D}{})$ and comparing the residues at $s = 1$, we have

$$\int_{\Phi\backslash\mathfrak{H}} (n!)^{-2}(\nabla^n\delta((f|R(\varphi))^{(1)})|_{\mathbb{Q}})$$
$$= -(-i)^{n+1}\phi(N')^{-1}N'^2\sqrt{D}^{n+2v+2w+2}G(\varphi)\Gamma(1,\mathrm{Ad}(\lambda)\otimes\alpha)L_{C'}(1,\mathrm{Ad}(\lambda)\otimes\alpha).$$

In the real quadratic case, $H^2_{\mathrm{cusp}}(Y_{0,K}(N),\mathcal{L}(\widehat{\kappa},\chi;\mathbb{C}))[\widehat{\lambda}]$ is two dimensional. Thus we need to further decompose this space into 1 dimensional pieces. As explained in [**Hi88a**, Section 7] (see Proof of Theorem 7.2), the finite group $S = C_\infty/C_{\infty+}$ acts on $\mathfrak{Z} = G(\mathbb{R})_+/C_{\infty+}$, $S_{\widehat{\kappa},J}(N,\chi)$ and $H^2_{\mathrm{cusp}}(Y_{0,K}(N),\mathcal{L}(\kappa,\chi;A))$, where C_∞ (resp. $C_{\infty+}$) is the standard maximal compact subgroup of $G(\mathbb{R})$ (resp. $G(\mathbb{R})_+$). We can identify $S \cong \{\pm1\}^{I_K}$ by taking the determinant and S with the power set of I_K by $c \mapsto \{v|c_v = +1\}$. Thus we can think of $cJ \subset I_K$ for each $J \subset I_K$. Identifying F_∞ with $\mathbb{R}\times\mathbb{R}$ via $\xi \mapsto (\xi^\tau,\xi^{\sigma\tau})$, we identify $\mathfrak{Z}$ with $\mathcal{H}_\tau \times \mathcal{H}_{\sigma\tau}$ for copies of upper half planes $\mathcal{H}_{\sigma\tau}$ and $\mathcal{H}_\tau$. Then for $f \in S_{\widehat{\kappa},J}(N,\chi)$, $cf(x) = f(xc)$ for $c = (c_\tau, c_{\sigma\tau})$ with $c_v \in \{\left(\begin{smallmatrix}\pm1 & 0\\ 0 & 1\end{smallmatrix}\right)\}$. Then $cf \in S_{\widehat{\kappa},cJ}(N,\chi)$ with $\mathbf{a}(\mathfrak{n},cf) = \mathbf{a}(\mathfrak{n},f)$ for all $\mathfrak{n}$. Thus c takes $S_{\widehat{\kappa},J}(N,\chi)$ into $S_{\widehat{\kappa},cJ}(N,\chi)$, and this action of S commutes with Hecke operators $T(\mathfrak{n})$. When there is a unit $\varepsilon \in \mathfrak{r}^\times$ with $\varepsilon^\rho\det(c_\rho) > 0$ for both $\rho = \tau$ and $\sigma\tau$, $cf^{(a)}(z) = f^{(a)}(\varepsilon z)$, where $\varepsilon z_\rho = \varepsilon^\rho z^c_\rho$ with $z^c_\rho = \begin{cases} z_\rho & \text{if } \det(c_\rho) = 1,\\ \overline{z}_\rho & \text{if } \det(c_\rho) = -1.\end{cases}$ If there is not such a unit, we find $\varepsilon \in F^\times$ and another member a' of the complete set $\{a\}$ of representatives for Cl_F such that (i) $\varepsilon^\rho\det(c_\rho) > 0$ for both $\rho = \tau$ and $\sigma\tau$, (ii) $a\mathfrak{r} = \varepsilon a'\mathfrak{r}$, and (iii) $cf^{(a')}(z) = f^{(a)}(\varepsilon z)$.

We take a character $\varepsilon : S \to \{\pm1\}$ and consider the projection

$$\pi_\varepsilon : H^2_{\mathrm{cusp}}(Y_0(N),\mathcal{L}(\kappa,\chi;A)) \to H^2_{\mathrm{cusp}}\left(Y_0(N),\mathcal{L}\left(\kappa,\chi;A\left[\frac{1}{2}\right]\right)\right)[\varepsilon]$$

given by $\pi_\varepsilon(x) = \#(S)^{-1}\sum_{c\in S}\varepsilon(c)c(x)$. Through the Eichler-Shimura isomorphism introduced in Section 2.2

$$\delta : \bigoplus_{J\subset I_K} S_{\kappa,J}(N;\chi)_{/K} \;\cong\; H^2_{\mathrm{cusp}}(Y_0(N),\mathcal{L}(\kappa,\chi;\mathbb{C}))[\chi],$$

we can attach Fourier expansion to cohomology classes in the right-hand-side of the above formula, which we write as H^2_{cusp}. Thus $\phi \in H^2_{\mathrm{cusp}}$ has Fourier coefficients $\mathbf{a}(\mathfrak{n};\phi)$ for ideals $\mathfrak{n}$ given by $\mathbf{a}(\mathfrak{n};\delta(f)) = \mathbf{a}(\mathfrak{n};f)$. The action of S preserves the Fourier coefficients: $\mathbf{a}(\mathfrak{n};c(\phi)) = \mathbf{a}(\mathfrak{n};\phi)$. Note that

$$c\delta\left((f)^{(a)}\right)$$
$$= \{c\}(y^{\widehat{n}/2}|y|_{K_{\mathbb{A}}})^{-1}(cf)^{(a')}(-1)^n\{(X_\tau - z^c_\tau Y_\tau)^n(X_{\sigma\tau} - \overline{z}^c_{\sigma\tau}Y_{\sigma\tau})^n\}dz^c_\tau\wedge d\overline{z}^c_{\sigma\tau},$$

where identifying $S = \{\pm1\}^{I_K}$, we write $\{c\} = \prod_{v\in I_K} c_v$ and

$$z^c_v = \begin{cases}\overline{z}_v & \text{if } c_v = -1,\\ z_v & \text{if } c_v = +1.\end{cases}$$

Note that, identifying S with the power set of I_K, if either $c = J$ or $-J$, the restriction $\delta((cf)^{(a)})$ to $\mathfrak{H}$ just vanishes, where $J_\tau = +1$ and $J_{\sigma\tau} = -1$ because

$J = \{\tau\}$. Let $c = (-1_\tau, -1_{\sigma\tau})$. Then we have

$$\begin{aligned} &c\delta\left((f)^{(1)}\right) \\ &\quad = (y^{\widehat{n}/2}|y|_{K_{\mathbb{A}}})^{-1} f^{(1)}(-\overline{z})(-1)^n \left\{(X_\tau - z_\tau^c Y_\tau)^n (X_{\sigma\tau} - \overline{z}_{\sigma\tau}^c Y_{\sigma\tau})^n\right\} d\overline{z}_\tau \wedge dz_{\sigma\tau}. \end{aligned}$$

We execute the computation done for $\delta(g^{(1)})$ replacing it by $c\delta(g^{(1)})$ and obtain

$$\int_{Y_{0,\mathbb{Q}}(N')} (n!)^{-2} \left(\nabla^n c\delta(g^{(1)})|_{\mathbb{Q}}\right) y^s = (-1)^{n+1} \int_{Y_{0,\mathbb{Q}}(N')} (n!)^{-2} \left(\nabla^n \delta(g^{(1)})|_{\mathbb{Q}}\right) y^s.$$

Therefore, assuming $\varepsilon(-1_\tau, -1_{\sigma\tau}) = (-1)^{n+1}$, $\mu = \widehat{\lambda}$ and $\alpha = \psi\varphi_{\mathbb{Q}}\omega$, we have

$$\begin{aligned} &\int_{Y_{0,\mathbb{Q}}(N')} (n!)^{-2} \left(\nabla^n \pi_\varepsilon \delta((f|R(\varphi))^{(1)})|_{\mathbb{Q}}\right) \\ &= -(-i)^{n+1}\phi(N')^{-1}N'^2\sqrt{D}^{n+2v+2w+2} \cdot G(\varphi)\Gamma(1, \mathrm{Ad}(\lambda)\otimes\alpha)L_{C'}(1, \mathrm{Ad}(\lambda)\otimes\alpha). \end{aligned}$$

As seen in the proof of Theorem 8.1 in [**Hi94**], the twisting operator $R(\varphi)$ takes $H^2_{\mathrm{cusp}}(Y_{0,K}(N), \mathcal{L}(\widehat{\kappa}, \chi; A))[\varepsilon\varphi_\infty]$ into $H^2_{\mathrm{cusp}}(Y_{0,K}(N\cap C^2), \mathcal{L}(\widehat{\kappa}+(0,\widehat{w}), \chi\varphi^2; A))[\varepsilon]$, where we identify S with $\{\pm 1\}^{I_K} \subset K_\infty^\times$ and consider φ_∞ as a character of S. Since $\alpha = \psi\varphi_{\mathbb{Q}}\omega$, $\varphi_\infty((-1,-1)) = (-1)^n$. This shows that $\varepsilon_0 = \varepsilon\varphi_\infty$ satisfies $\varepsilon_0(-1,-1) = -1$. We record what we have proven:

PROPOSITION 4.1. *Let $F = \mathbb{Q}$ and $K = \mathbb{Q}(\sqrt{D})$ be a real quadratic field with discriminant $D > 0$ and $\kappa = (n\tau, v\tau) \in \mathbb{Z}[I]^2$ with $I = \{\tau\}$. Let φ be an arithmetic Hecke character with $\infty(\varphi) = -w(\tau+\sigma\tau) \in \mathbb{Z}[I_K]$ and of conductor C. Let $\chi = \psi \circ N_{K/\mathbb{Q}}$ for an arithmetic Hecke character ψ with $\psi_\infty(x) = x^{-n-2v}$ for all $x \in \mathbb{A}_\infty^\times = \mathbb{R}^\times$. Suppose that $\alpha = \varphi_{\mathbb{Q}}\omega\psi$ with $\omega(x) = |x|_{\mathbb{A}}^{n_\tau+2v_\tau+2w}$ and $\alpha = (\frac{D}{\ })$. If a primitive system $\mu : h_{\widehat{\kappa}}(N, \chi; \mathbb{C})_{/K} \to \mathbb{C}$ is a base change lift of $\lambda : h_\kappa(N_0, \psi; \mathbb{C})_{/\mathbb{Q}} \to \mathbb{C}$, then we have*

$$\begin{aligned} &\int_{Y_{0,\mathbb{Q}}(N')} (n!)^{-2}(\nabla^n \pi_\varepsilon \delta(g^{(1)})|_{\mathbb{Q}}) \\ &= -(-i)^{n+1}\phi(N')^{-1}N'^2\sqrt{D}^{n+2v+2w+2} G(\varphi)\Gamma(1, \mathrm{Ad}(\lambda)\otimes\alpha)L_{C'}(1, \mathrm{Ad}(\lambda)\otimes\alpha), \end{aligned}$$

where $g = f|R(\varphi)$ for the primitive form f with $f|T(\mathfrak{n}) = \mu(T(\mathfrak{n}))f$ for all $\mathfrak{n} \subset \mathcal{R}$, $C' = C\cap\mathbb{Z}$, and N' is the positive generator of $N\cap C^2\cap\mathbb{Z}$. If $\mu : h_{\widehat{\kappa}}(N, \chi; \mathbb{C})_{/K} \to \mathbb{C}$ is not a twist of any base change lift, then

$$\int_{Y_{0,\mathbb{Q}}(N')} (n!)^{-2}(\nabla^n \pi_\varepsilon \delta(g^{(1)})|_{\mathbb{Q}}) = 0.$$

PROOF. The proof of the vanishing is given in [**HaLR**, 3.12] and is basically the same as that of Proposition 3.1. □

We now define the modular periods. Let A be a Dedekind domain in $\mathbb{C}$ containing all values of $\mu(T(\mathfrak{n}))$ for ideals $\mathfrak{n} \subset \mathcal{R}$. We assume that

$$H^2_{\mathrm{cusp}}(Y_{0,K}(N), \mathcal{L}(\widehat{\kappa}, \chi; A))[\varepsilon, \mu] = A\xi_\varepsilon(\mu)$$

for a character ε of S. Then we define $\Omega_1(\varepsilon, \mu^\rho; A) \in \mathbb{C}^\times$ by

$$\pi_\varepsilon(\delta(f^\rho)) = \Omega_1(\varepsilon, \mu^\rho; A)\xi(\mu)^\rho,$$

where f^ρ is the primitive form in $S_{\widehat{\kappa},J}(N,\chi^\rho)$ with $f^\rho|T(\mathfrak{n}) = \mu(T(\mathfrak{n}))^\rho f^\rho$. This period Ω_1 is well defined up to units in A. We also define

$$\underline{\Omega}_1(\mu;\mathbb{Q}(\mu)) = (\Omega_1(\mu^\rho;\mathbb{Q}(\mu^\rho)))_\rho \in \mathbb{Q}(\mu)\otimes_{\mathbb{Q}}\mathbb{C}.$$

The period $\underline{\Omega}_1(\mu;\mathbb{Q}(\mu))$ is well defined as an element of $(\mathbb{Q}(\mu)\otimes_{\mathbb{Q}}\mathbb{C})^\times/\mathbb{Q}(\mu)^\times\otimes 1$. Again we have

$$\underline{\Omega}_1(\mu\otimes\varphi;\mathbb{Q}(\mu\otimes\varphi)) = \mathbf{G}(\varphi)^{-1}\underline{\Omega}_1(\mu;\mathbb{Q}(\mu)).$$

COROLLARY 4.2. *If $A=\mathbb{Q}(\lambda)$, we have, for all $\varepsilon : S\to\{\pm 1\}$ with $\varepsilon(-1,-1) = -1$,*

$$\frac{(1\otimes i^{n+1}\sqrt{D}^{n}\Gamma(1,\mathrm{Ad}(\lambda)\otimes\alpha))\mathbb{L}_{C'}(1,\mathrm{Ad}(\lambda)\otimes\alpha)}{\mathbf{G}(\alpha\psi)\underline{\Omega}_1(\varepsilon,\widehat{\lambda};\mathbb{Q}(\lambda))}\in\mathbb{Q}(\lambda)$$

in $\mathbb{Q}(\lambda)\otimes_{\mathbb{Q}}\mathbb{C}$, where

$$\mathbb{L}_{C'}(s,\mathrm{Ad}(\lambda)\otimes\alpha) = (L_{C'}(s,\mathrm{Ad}(\lambda^\rho)\otimes\alpha))_\rho$$

has values in $\mathbb{Q}(\lambda)\otimes_{\mathbb{Q}}\mathbb{C}$. Moreover, if A is a discrete valuation ring in $\mathbb{Q}(\lambda)$ with residual characteristic prime to $6(n!)$ and if $\varphi=\mathrm{id}$, then for the positive generator N_0 of $N\cap\mathbb{Z}$

$$i^{n+1}{N_0}^2\phi(N_0)^{-1}\frac{\sqrt{D}^{n+2v+2}\Gamma(1,\mathrm{Ad}(\lambda)\otimes\alpha)L(1,\mathrm{Ad}(\lambda)\otimes\alpha)}{\Omega_1(\varepsilon,\widehat{\lambda};A)}\in A.$$

5. Congruence and the adjoint L-values

Here we study a simple consequence of Corollaries 3.2 and 4.2 on congruence of systems of Hecke eigenvalues. To describe such congruence among cusp forms in terms of Hecke algebras and deformation rings of Galois representations, we here introduce a general notion of congruence modules and differential modules: Let R be an algebra over a Dedekind domain A. We assume that R is an A-flat module of finite type. Let $\phi : R\to A$ be an A-algebra homomorphism. We define $C_1(\phi;A) = \Omega_{R/A}\otimes_{R,\phi}\mathrm{Im}(\phi)$, which we call the differential module of ϕ. We suppose that R is reduced (that is, the nilradical of R vanishes). Then the total quotient ring $\mathrm{Frac}(R)$ can be decomposed uniquely into $\mathrm{Frac}(R)$ $=\mathrm{Frac}(\mathrm{Im}(\phi))\times X$ as an algebra direct product. Let $\mathfrak{a}=\mathrm{Ker}(R\to X)$. Then we put $C_0(\phi;A) = (R/\mathfrak{a})\otimes_{R,\phi}\mathrm{Im}(\phi)\cong\mathrm{Im}(\phi)/(\mathrm{Im}(\phi)\cap R)$ (cf. [**Hi88b**, Section 6]), which is called the congruence module of ϕ but is actually a ring. Here the intersection $\mathrm{Im}(\phi)\cap R$ is taken in $\mathrm{Frac}(R)$. Suppose now that A is a subring of a number field in $\overline{\mathbb{Q}}$. Since $\mathrm{Spec}(C_0(\phi;A))$ is the scheme theoretic intersection of $\mathrm{Spec}(\mathrm{Im}(\phi))$ and $\mathrm{Spec}(R/\mathfrak{a})$ in $\mathrm{Spec}(R)$, a prime $\mathfrak{p}$ is in the support of $C_0(\phi;A)$ if and only if there exists an A-algebra homomorphism $\phi' : R\to\overline{\mathbb{Q}}$ factoring through $R/\mathfrak{a}$ such that $\phi(a)\equiv\phi'(a)\bmod\mathfrak{p}$ for all $a\in R$. In other words, $\phi\bmod\mathfrak{p}$ factors through $R/\mathfrak{a}$ and can be lifted to ϕ'.

Let $K/\mathbb{Q}$ be a quadratic extension with discriminant D. Let $\alpha = (\frac{D}{})$. We consider a system of Hecke eigenvalues $\lambda : h_\kappa(D,\psi;A)\to A$ for a discrete valuation ring A in $\mathbb{Q}(\lambda)$ with residue field $\mathbb{F}$ of characteristic $p>2$, where $\psi(x)=\alpha(x)|x|_{\mathbb{A}}^{-n}$. From our assumption that $\psi_\infty(x) = x^{-n-2v}$ for all $x\in\mathbb{R}^\times$, we conclude that $v=0$ and

$$n\ \text{is}\ \begin{cases}\text{odd} & \text{if } K \text{ is imaginary},\\ \text{even} & \text{if } K \text{ is real}.\end{cases}$$

The space we are looking into has the Neben character α and level D, under the classical notation, $S_{n+2}(\Gamma_0(D),(\frac{D}{}))$. Let $\mathcal{O}$ be the completion under the $\mathfrak{m}_A$-adic topology for the maximal ideal $\mathfrak{m}_A$ of A. Let $\mathfrak{m}$ be the maximal ideal of $\mathcal{O}$. We consider the Galois representation $\rho_\lambda : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathcal{O})$ which is the $\mathfrak{m}_A$-adic member of the compatible system $\rho(\lambda)$ associated to λ. We define the residual representation $\overline{\rho} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{F})$ by ρ_λ mod $\mathfrak{m}$. Let $\mathcal{G} = \mathrm{Gal}(K^{(p)}/\mathbb{Q})$ for the maximal extension $K^{(p)}/K$ unramified outside $\{p, \infty\}$. Then ρ_λ factors through $\mathcal{G}$. Let E be a number field in $K^{(p)}$ with integer ring $\mathfrak{r}$, and put $\mathcal{H} = \mathrm{Gal}(K^{(p)}/E)$. For any representation ρ of $\mathcal{G}$, we write ρ_E for the restriction of ρ to $\mathcal{H}$. We consider the following condition:

(AI_E) $\quad\overline{\rho}_E$ *is absolutely irreducible.*

We now consider deformations of $\overline{\rho}$, introduced by Mazur [**Ma**], over the category $CNL_{\mathcal{O}}$ of complete local noetherian $\mathcal{O}$-algebras with residue field $\mathbb{F}$. A Galois representation $\rho : \mathcal{H} \to \mathrm{GL}_2(B)$ for $B \in CNL_{\mathcal{O}}$ is a deformation of $\overline{\rho}_E$ if ρ_E mod $\mathfrak{m}_B$ coincides with $\overline{\rho}$ as matrix representations. We look into the deformation functor $\mathcal{F} : CNL_{\mathcal{O}} \to SETS$ given by

$$\mathcal{F}_E(B) = \{\rho : \mathcal{H} \to \mathrm{GL}_2(B) | \rho \bmod \mathfrak{m}_B = \overline{\rho}\} / \approx,$$

where "$\approx$" is the conjugation by elements in $1 + M_2(\mathfrak{m}_B)$. We impose more conditions on deformations. A deformation ρ is called p-ordinary over a number field E if $\rho|_{\mathcal{D}} \cong \left(\begin{smallmatrix} \delta_{\mathcal{D}} & * \\ 0 & \varepsilon_{\mathcal{D}} \end{smallmatrix}\right)$ with an unramified character $\delta_{\mathcal{D}}$ on every decomposition subgroup $\mathcal{D}$ of $\mathcal{H}$ at $\mathfrak{p}|p$. We like to impose some of the following two conditions depending on the situation:

(ord_E) ρ *is p-ordinary over E with two distinct characters $\delta_{\mathcal{D}}$ and $\varepsilon_{\mathcal{D}}$ for all $\mathcal{D}$;*
(fl_E) $\rho|_{\mathcal{D}}$ *is realized on the generic fibre of a finite flat group scheme over $\mathfrak{r}_{\mathfrak{p}}$ for all $\mathfrak{p}|p$.*

For primes $\mathfrak{l}|D$ outside p, we impose the following condition:

$$(\alpha_E)\rho \cong \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix} \text{ on the inertia subgroup } I_{\mathfrak{l}} \subset \mathcal{H} \text{ for each prime } \mathfrak{l}|D \text{ and } \mathfrak{l} \nmid p.$$

Let ν be the p-adic cyclotomic character. By class field theory, we regard ψ as a character of $\mathcal{G}$. Then $\det(\rho_\lambda) = \nu\psi$. Hereafter we suppose that $p \nmid D$. We consider the following subfunctor of $\mathcal{F}_E$:

$$\mathcal{F}'_E(B) = \begin{cases} \{\rho \in \mathcal{F}_E(B) |\ \rho \text{ satisfies } (\mathrm{ord}_E), (\alpha_E) \text{ and } \det(\rho) = (\nu\psi)_E\} & \text{if } n > 0, \\ \{\rho \in \mathcal{F}_E(B) |\ \rho \text{ satisfies } (\mathrm{ord}_E), (fl_E), (\alpha_E) \text{ and } \det(\rho) = (\nu\psi)_E\} & \text{if } n = 0. \end{cases}$$

Under (AI_E) and (ord_E) for $\overline{\rho}$, this functor is representable by a universal couple (R_E, ϱ_E) (see [**Ma**] and [**Hi96**, Appendix]).

We put $h_\kappa(D, \psi; \mathcal{O}) = h_\kappa(D, \psi; A) \otimes_A \mathcal{O}$ and define $h_\kappa(Dp, \psi; \mathcal{O})$ similarly. Let

$$e_0 = \lim_{n\to\infty} T(p)^{n!} \in h_\kappa(D, \psi; \mathcal{O}) \quad \text{and} \quad e = \lim_{n\to\infty} T(p)^{n!} \in h_\kappa(Dp, \psi; \mathcal{O}).$$

Since $T(p)$ of level D and that of level Dp are different, we know $e_0 \neq e$ and we have a surjective $\mathcal{O}$-algebra homomorphism of $eh_\kappa(Dp, \psi; \mathcal{O})$ onto $e_0h_\kappa(D, \psi; \mathcal{O})$ taking $T(n)$ to $T(n)$ for all n prime to p [**MaT**]. If $n > 0$, $e_0h_\kappa(D, \psi; \mathcal{O}) \cong eh_\kappa(Dp, \psi; \mathcal{O})$, which is a consequence of [**Hi86**, Proposition 4.7]. We assume that λ factors through $e_0h_\kappa(D, \psi; \mathcal{O})$, which is equivalent to $\lambda(T(p)) \in A^\times$ and implies that ρ_λ is p-ordinary. Let $h = h_{\mathbb{Q}}$ (resp. $h'_{\mathbb{Q}}$) be the unique local ring of $e_0h_\kappa(D, \psi; \mathcal{O})$ (resp. $eh_\kappa(Dp, \psi; \mathcal{O})$) through which λ factors. Thus if $n > 0$, $h_{\mathbb{Q}} \cong h'_{\mathbb{Q}}$. As is well known, under $(AI_{\mathbb{Q}})$, there is a unique Galois representation $\rho_{\mathbb{Q}}$, up to isomorphisms, such that $\mathrm{Tr}(\rho_{\mathbb{Q}}(Frob_\ell)) = T_h(\ell)$ for all primes ℓ outside Dp (for uniqueness, see [**C**]), where $Frob_\ell$ is the Frobenius element in $\mathcal{G}$ at the prime ℓ, and $T_h(\ell)$ is the projection of $T(\ell)$ to $h_{\mathbb{Q}}$. Let $k(E) = E(\sqrt{(-1)^{(p-1)/2}p})$. It has been proven by Taylor and Wiles [**W**] and [**TW**] (see also [**Fu**] when $n = 0$) that, under $(AI_{k(\mathbb{Q})})$ and $(\mathrm{ord}_{\mathbb{Q}})$ for $\overline{\rho}$, $\mathcal{F}'_{\mathbb{Q}}$ is representable by the pair $(h_{\mathbb{Q}}, \rho_{\mathbb{Q}})$. Thus in this case, the natural morphism $\pi_{\mathbb{Q}} : R_{\mathbb{Q}} \to h_{\mathbb{Q}}$ with $\pi_{\mathbb{Q}}\varrho_{\mathbb{Q}} \approx \rho_{\mathbb{Q}}$ is a surjective isomorphism in $CNL_{\mathcal{O}}$. When E is totally real, this result is generalized by Fujiwara [**Fu**] under certain assumptions. We describe it here for real quadratic K. Since $K^{(p)}$ is unramified outside $\{p, \infty\}$, $\mathcal{F}'_K$ is (basically) defined by ramification condition (ord_K) and the determinant condition. We define in the same way the idempotent $e = \lim_{n\to\infty} T(p)^{n!} \in h_{\widehat{\kappa}}(p, \mathrm{id}; \mathcal{O})$ and $e_0 \in h_{\widehat{\kappa}}(1, \mathrm{id}; \mathcal{O})$. Let h_K (resp. h'_K) be the local ring of $e_0 h_{\widehat{\kappa}}(1, \mathrm{id}; \mathcal{O})$ (resp. $e h_{\widehat{\kappa}}(p, \mathrm{id}; \mathcal{O})$) through which the base change lift $\widehat{\lambda}$ factors through. We again have $h_K \cong h'_K$ if $n > 0$. We have a modular deformation $\rho_K : \mathcal{H} \to \mathrm{GL}_2(h_K)$. Then Fujiwara has proven that $\mathcal{F}'_K$ is representable by (h_K, ρ_K) under the following condition in addition to $(AI_{k(K)})$ and (ord_K) for $\overline{\rho}_K$:

(unr) *p is unramified in K.*

We write $\pi_K : R_K \cong h_K$ for the isomorphism inducing $\pi_K\varrho_K \approx \rho_K$.

When it is necessary to indicate the dependence on E, we write $\lambda_{\mathbb{Q}}$ for λ and λ_K for $\widehat{\lambda}$. Since the conductor of the Neben character coincides with the level, h_E and the Hecke algebras are reduced. In the course of the proof of the above result: $R_E \cong h_E$ for $E = K$ or $\mathbb{Q}$, it is shown that R_E is a local complete intersection. This fact is basically equivalent to

$$\text{(C1)} \qquad |C_1(\lambda_E; \mathcal{O})| = |C_0(\lambda_E; \mathcal{O})| \quad \text{for } E = \mathbb{Q} \text{ and real } K,$$

where $\lambda_E : h_E \to \mathcal{O}$. When $E = \mathbb{Q}$, the above fact implies that, for any given character $\varepsilon : S = \{\pm 1\} \to \{\pm 1\}$,

$$(\mathrm{mlt}_{\mathbb{Q}}) \qquad H^1_{\mathrm{cusp}}(Y_{0,\mathbb{Q}}(D), \mathcal{L}(\kappa, \psi; \mathcal{O}))[h_{\mathbb{Q}}, \varepsilon] \cong h_{\mathbb{Q}} \cong R_{\mathbb{Q}} \quad \text{as } h_{\mathbb{Q}}\text{-modules},$$

where the left-hand-side is the eigenspace for $h_{\mathbb{Q}}$, in other words, writing 1_h for the idempotent of $h_{\mathbb{Q}}$ in $h_\kappa(D, \psi; \mathcal{O})$,

$$H^1_{\mathrm{cusp}}(Y_{0,\mathbb{Q}}(D), \mathcal{L}(\kappa, \psi; \mathcal{O}))[h_{\mathbb{Q}}, \varepsilon] = 1_h(H^1_{\mathrm{cusp}}(Y_{0,\mathbb{Q}}(D), \mathcal{L}(\kappa, \psi; \mathcal{O}))[\varepsilon]).$$

Since $E = \mathbb{Q}$, there are only two ε; one is trivial, which we write as "+", and we write the other as "−". By virtue of $(\mathrm{mlt}_{\mathbb{Q}})$, we can compute $C_0(\lambda; \mathcal{O})$ using cohomology groups. To explain this, we write $L(B) = H^1_{\mathrm{cusp}}(Y_{0,\mathbb{Q}}(D), \mathcal{L}(\kappa, \psi; B))[\varepsilon]$. Then $L(\mathcal{O}) = L(A) \otimes_A \mathcal{O}$. Decomposing $\mathrm{Frac}(h_\kappa(D, \psi; A)) = \mathrm{Frac}(\mathrm{Im}(\lambda)) \times X$, we define $L^\lambda(A)$ to be the image of $L(A)$ in $L(A) \otimes_{h_\kappa(D,\psi;A)} \mathrm{Frac}(\mathrm{Im}(\lambda))$ and a cohomological congruence module by

$$C_0^H(\lambda; A) = L^\lambda(A)/(L(A) \cap L^\lambda(A)).$$

Then $(\mathrm{mlt}_{\mathbb{Q}})$ shows that $C_0(\lambda_E; \mathcal{O}) \cong C_0^H(\lambda; A)$. It is shown in [**Hi81**] and [**Hi88b**] that the p-adic absolute value of $\Gamma(1, \mathrm{Ad}(\lambda))L(1, \mathrm{Ad}(\lambda))/\Omega_1(+, \lambda; A)\Omega_1(-, \lambda; A)$ is the inverse of the order of the right-hand-side module of the above equation. That is, under $(AI_{\mathbb{Q}})$ and $(\mathrm{ord}_{\mathbb{Q}})$ for $\overline{\rho}$, if $p \nmid 6D$,

$$\text{(C2)} \quad \left|\frac{\Gamma(1, \mathrm{Ad}(\lambda))L(1, \mathrm{Ad}(\lambda))}{\Omega_1(+, \lambda; A)\Omega_1(-, \lambda; A)}\right|_p^{-r} = \#(C_0^H(\lambda; A)) = \#(C_0(\lambda; A)) = \#(C_1(\lambda; A)),$$

where $r = r(\mathcal{O}) = \mathrm{rank}_{\mathbb{Z}_p} \mathcal{O}$, and $|\ |_p$ is the p-adic absolute value of A normalized so that $|p|_p = p^{-1}$. It is easy to see that for a non-zero element $\eta(\lambda) \in A$, $C_0(\lambda; A) \cong$

$A/\eta(\lambda)A$, and (C1) is equivalent to saying that the L-value is equal to $\eta(\lambda)$ up to A-units. Even if $p|D$, there is a similar formula (see [**Hi88b**] for details).

The Selmer group over E of $\mathrm{Ad}(\rho_\lambda)$ is defined as follows. Let $V(\lambda)$ be the $\mathcal{O}$-free module of rank 2 on which $\mathcal{G}$ acts via ρ_λ. For each decomposition subgroup $\mathcal{D}_\mathfrak{p} \subset \mathcal{H}$ at $\mathfrak{p}|p$ for primes $\mathfrak{p}$ of E, we write $V_\mathfrak{p}$ for the $\delta_\mathcal{D}$ eigenspace in $V(\lambda)$. For primes $\mathfrak{l}|D$ prime to p, we write $V_\mathfrak{l}$ for the subspace fixed by the inertia at $\mathfrak{l}$. Let $\mathcal{K}=\mathrm{Frac}(\mathcal{O})$. We identify $\mathrm{Ad}(\rho_\lambda)$ with trace 0 subspace W of $\mathrm{Hom}_\mathcal{O}(V(\lambda),V(\lambda))$. We put $W_\mathfrak{l} = \{\phi \in W | \phi(V_\mathfrak{l}) = 0\}$ for $\mathfrak{l}|Dp$. Define, for $\mathfrak{l}$ ramifying in $K^{(p)}/E$, writing $\mathcal{I}_\mathfrak{l}$ for the inertia subgroup of $\mathfrak{l}$ in $\mathcal{H}$,

$$L_\mathfrak{l} = \mathrm{Ker}(H^1(\mathcal{D}_\mathfrak{l}, W^*) \to H^1(\mathcal{I}_\mathfrak{l}, (W/W_\mathfrak{l})^*)),$$

where $X^* = X \otimes_\mathcal{O} \mathcal{K}/\mathcal{O}$. Then we put

$$\mathrm{Sel}(\mathrm{Ad}(\rho_\lambda))_{/E} = \bigcap \mathrm{Ker}(H^1(\mathcal{H}, W^*) \to H^1(\mathcal{D}_\mathfrak{l}, W^*)/L_\mathfrak{l}),$$

where $\mathfrak{l}$ runs over all primes ramifying in $K^{(p)}/E$. It is a general fact [**MaT**] (see also [**Hi96**, 3.2]) that

(C3) $\quad \mathrm{Sel}(\mathrm{Ad}(\rho_\lambda))_{/E} \cong \mathrm{Hom}_{\mathbb{Z}_p}(C_1(\lambda_E \circ \pi_E : R_E \to \mathcal{O}; \mathcal{O}), \mathbb{Q}_p/\mathbb{Z}_p)$ if $n > 0$.

Thus combining all we said, we get the following order formula of the Selmer group under $(AI_{k(\mathbb{Q})})$, $(\mathrm{ord}_\mathbb{Q})$ for $\overline{\rho}$ and $p \nmid 6D$,

(CN1) $$\frac{\Gamma(1,\mathrm{Ad}(\lambda))L(1,\mathrm{Ad}(\lambda))}{\Omega_1(+,\lambda;A)\Omega_1(-,\lambda;A)} = \eta(\lambda) \quad \text{up to } A\text{-units, and}$$

(CN2) $$\left|\frac{\Gamma(1,\mathrm{Ad}(\lambda))L(1,\mathrm{Ad}(\lambda))}{\Omega_1(+,\lambda;A)\Omega_1(-,\lambda;A)}\right|_p^{-r(\mathcal{O})} = \#(\mathrm{Sel}(\mathrm{Ad}(\rho_\lambda))_{/\mathbb{Q}}) \quad \text{if } n > 0.$$

This is a non-abelian generalization of a classical analytic class number formula (see [**W**, Chapter 4] and [**HiTU**]). The definition of the Selmer group can be interpreted by Fontaine's theory, and the above formula can be viewed as an example of the Tamagawa number formula of Bloch and Kato for the motive $M(\mathrm{Ad}(\lambda))$ (see [**W**, p. 466] and [**BK**, Section 5]). By using this interpretation, we can modify the definition of the Selmer group, and the formula (CN2) is valid even for $n = 0$ for the modified Selmer group.

There is a partial generalization of the above facts for quadratic fields K. We define $L_K(B) = H^{r_1(K)+r_2(K)}_{\mathrm{cusp}}(Y_{0,K}(1), \mathcal{L}(\widehat{\kappa}, id; B))[\varepsilon]$, where we fix for the moment a character $\varepsilon : S \to \{\pm 1\}$ when K is real. If K is imaginary, we just forget about ε. Then we define the cohomological congruence module $C_0^H(\widehat{\lambda}; A)$ by $L_K^{\widehat{\lambda}}(A)/(L_K^{\widehat{\lambda}}(A) \cap L_K(A))$, where decomposing $\mathrm{Frac}(h_{\widehat{\kappa}}(1,\mathrm{id};A)) = \mathrm{Frac}(\mathrm{Im}(\widehat{\lambda})) \times X$, $L_K^{\widehat{\lambda}}(A)$ is the image of $L_K(A)$ in $L_K(A) \otimes_{h_{\widehat{\kappa}}(1,\mathrm{id};A)} \mathrm{Frac}(\mathrm{Im}(\widehat{\lambda}))$. It has been proven by Urban [**U**], under $(\mathrm{ord}_\mathbb{Q})$ for $\overline{\rho}$, when K is imaginary and $p \nmid 6D$,

(C4) $$\left|\frac{\Gamma(1,\mathrm{Ad}(\widehat{\lambda}))L(1,\mathrm{Ad}(\widehat{\lambda}))}{\Omega_1(\widehat{\lambda};A)\Omega_2(\widehat{\lambda};A)}\right|_p^{-r(\mathcal{O})} = \#(C_0^H(\widehat{\lambda};A)).$$

Here are several remarks to be made. In [**U**], (0) the result is more general covering non-base change lift, (i) the normalization of the period is different from the one we made here by a power of 2π, (ii) there is a rational constant α_n showing up in [**U**, Theorem A], which is not explicitly computed (see [**U**, 5.5a]) as a product of factorials. The computation of α_n can be done (see $c_1(s)$ below (Δ) in Section 8 in the text), and α_n is included in the Gamma factor. Finally (iii) it is assumed in [**U**] that the level N of $\widehat{\lambda}$ is sufficiently large to ensure the smoothness of $Y_{0,K}(N)$, which guarantees the duality between (the p-ordinary parts of) $H^1_{\text{cusp}}(Y_{0,K}(N), \mathcal{L}(\widehat{\lambda}, \chi; A))$ and $H^2_{\text{cusp}}(Y_{0,K}(N), \mathcal{L}(\widehat{\lambda}, \chi; A))$. The duality is a key to the proof of the above formula. However, for any N, we can take a multiple N' so that $Y_{0,K}(N')$ is smooth. Then if $p \nmid [Y_{0,K}(N') : Y_{0,K}(N)]$, by using the restriction-corestriction technique, we can recover the duality. By varying N', the common divisor of the covering degree $[Y_{0,K}(N') : Y_{0,K}(N)]$ is made of primes appearing in the order of elliptic elements in $\mathrm{SL}_2(K)$, which is in turn a product of primes dividing the order of roots of unity ζ such that $[K(\zeta) : K] = 2$ (see [**Hi88a**, Lemma 7.1]). Since K is imaginary quadratic, the possibility of such primes are only 2 and 3. In this way, we get the result for $p > 3$ and for level 1.

Although only imaginary quadratic fields K are treated in [**U**], all argument can be generalized to arbitrary E at least for p-ordinary λ. In particular, we get for real K, supposing ($\mathrm{ord}_{\mathbb{Q}}$) for $\overline{\rho}$ and $p \nmid 6D$ (see Section 8 in the case of $K = F \times F$),

$$\text{(C5)} \qquad \left|\frac{\Gamma(1, \mathrm{Ad}(\widehat{\lambda}))L(1, \mathrm{Ad}(\widehat{\lambda}))}{\Omega_1(\varepsilon, \widehat{\lambda}; A)\Omega_1(-\varepsilon, \widehat{\lambda}; A)}\right|_p^{-r(\mathcal{O})} = \#(C_0^H(\widehat{\lambda}; A)),$$

where $(-\varepsilon)(-1_\tau) = -(\varepsilon(-1_\tau))$ for embeddings $\tau \in I_K$.

We consider the following condition:

$$(\mathrm{mlt}_K) \qquad H^{r_1(K)+r_2(K)}_{\text{cusp}}(Y_{0,K}(1), \mathcal{L}(\widehat{\kappa}, \mathrm{id}; \mathcal{O}))[h_K, \varepsilon] \cong h_K \cong R_K \text{ as } h_K\text{-modules.}$$

When K is real, it is plausible to get (mlt_K), under (cl) and (unr), as an application of the method of Taylor, Wiles and Fujiwara [**Fu**], where a similar statement is proven for the modular cohomology group obtained from everywhere unramified definite quaternion algebra over K. When K is imaginary, the assertion (mlt_K) might follow similarly, but it is certainly more difficult. Anyway in this paper, we do not touch this point, but we would just like to remark that the generalization of (CN1-2) for quadratic K follows from (mlt_K).

As we have shown in [**DHI**], there is a natural action of $\mathrm{Gal}(K/\mathbb{Q}) = \langle\sigma\rangle$ on h_K. This action brings $T(\mathfrak{n})$ to $T(\mathfrak{n}^\sigma)$ and induces an action on $C_1(\widehat{\lambda}; \mathcal{O})$. When p is odd, we get a decomposition:

$$C_1(\widehat{\lambda}; \mathcal{O}) = C_1(\widehat{\lambda}; \mathcal{O})[\mathrm{id}] \oplus C_1(\widehat{\lambda}; \mathcal{O})[\alpha],$$

where "$[\alpha]$" indicates α-eigenspace regarding α as the unique non-trivial character of $\mathrm{Gal}(K/\mathbb{Q})$.

CONJECTURE 5.1. *Suppose* (AI_K), $\lambda(T(p)) \in A^\times$, ($\mathrm{ord}_{\mathbb{Q}}$) *for* $\overline{\rho}$ *and that* $p \nmid 6D$. *Then*

$$\text{(CN3)} \qquad \left|\frac{\Gamma(1, \mathrm{Ad}(\lambda) \otimes \alpha)L(1, \mathrm{Ad}(\lambda) \otimes \alpha)}{\Omega_d(\varepsilon, \widehat{\lambda}; A)}\right|_p^{-r(\mathcal{O})} = \#(C_1(\widehat{\lambda}; \mathcal{O})[\alpha]),$$

where $\varepsilon(-1, -1) = -1$ *when* K *is real, we disregard* ε *if* K *is imaginary, and*

$$d = \begin{cases} 2 & \text{if } K \text{ is imaginary,} \\ 1 & \text{if } K \text{ is real.} \end{cases}$$

When K is real, we have

$$\text{(P)} \qquad \frac{\Omega_1(\varepsilon,\widehat{\lambda};A)}{\Omega_1(+,\lambda;A)\Omega_1(-,\lambda;A)} \in A^\times.$$

This conjecture is made in [**DHI**] for even Dirichlet characters in place of quadratic α. Some numerical example supporting the conjecture is given there for real K. The point here is the inclusion of imaginary quadratic characters α into the scope. We prove here a consequence of the conjecture on congruence among systems of Hecke eigenvalues without referring to the conjecture.

Let M be a sufficiently large number field containing all Hecke eigenvalues on $S_{\widehat{\kappa}}(1,\mathrm{id})$ and such that $H^q_{\mathrm{cusp}}(Y_{0,K}(1),\mathcal{L}(\widehat{\kappa},\mathrm{id};O))[\varepsilon,\mu]$ is O-free of rank 1 for all Hecke eigensystems $\mu : h_{\widehat{\kappa}}(1,\mathrm{id};O) \to O$, where O is the integer ring of M. Such M always exists because the above module is an O-module projective of rank 1. For two systems $\widehat{\lambda} \neq \mu$ of Hecke eigenvalues, we write, for a prime $\mathfrak{p}$ in O, $\widehat{\lambda} \equiv \mu \bmod \mathfrak{p}$ if $\widehat{\lambda}(T(\mathfrak{n})) \equiv \mu(T(\mathfrak{n})) \bmod \mathfrak{p}$ for all integral ideals $\mathfrak{n} \subset \mathcal{R}$. If this happens, $\widehat{\lambda} \bmod \mathfrak{p}$ factors through $C_0(\widehat{\lambda};O)$. Thus we call primes in the support of $C_0(\widehat{\lambda};O)$ congruence primes of $\widehat{\lambda}$. We write $\widehat{\lambda} \equiv^H \mu \bmod \mathfrak{p}$ if for a generator $\xi(\widehat{\lambda})$ of $H^q_{\mathrm{cusp}}(Y_{0,K}(1),\mathcal{L}(\widehat{\kappa},\mathrm{id};O))[\varepsilon,\widehat{\lambda}]$, there is an element $\xi(\mu) \in H^q_{\mathrm{cusp}}(Y_{0,K}(1),\mathcal{L}(\widehat{\kappa},\mathrm{id};O))[\varepsilon,\mu]$ such that

$$\xi(\widehat{\lambda}) - \xi(\mu) \in \mathfrak{p}H^q_{\mathrm{cusp}}(Y_{0,K}(1),\mathcal{L}(\widehat{\kappa},\mathrm{id};O)),$$

where $q = r_1(K) + r_2(K)$. Of course, we have

$$\widehat{\lambda} \equiv^H \mu \bmod \mathfrak{p} \Rightarrow \widehat{\lambda} \equiv \mu \bmod \mathfrak{p}.$$

The converse follows if (mlt_K) holds. The cohomological congruence $\widehat{\lambda} \equiv^H \mu \bmod \mathfrak{p}$ is equivalent to $\mathfrak{p} \in \mathrm{Supp}(C^H_0(\widehat{\lambda};O))$. Conjecture 5.1 implies that for non-base change $\mu : h_{\widehat{\kappa}}(1,\mathrm{id};O) \to O$,

$$\widehat{\lambda} \equiv \mu \bmod \mathfrak{p} \text{ for a non-base change} \mu \iff \mathfrak{p} \Big| \frac{\Gamma(1,\mathrm{Ad}(\lambda)\otimes\alpha)L(1,\mathrm{Ad}(\lambda)\otimes\alpha)}{\Omega_d(\varepsilon,\widehat{\lambda};O)}$$

under the assumption and the notation of the conjecture.

THEOREM 5.2. *Let the notation be as in the conjecture.*

1. *Suppose that $p \nmid 6D(n!)$. If $\widehat{\lambda} \equiv^H \mu \bmod \mathfrak{p}$ for a prime $\mathfrak{p}|p$ of O and a non-base-change μ, then*
$$\mathfrak{p} \Big| \frac{\Gamma(1,\mathrm{Ad}(\lambda)\otimes\alpha)L(1,\mathrm{Ad}(\lambda)\otimes\alpha)}{\Omega_d(\varepsilon,\widehat{\lambda};O)} \in O\left[\frac{1}{6D(n!)}\right].$$
2. *Let K be a real quadratic field. Suppose the assertion (P) of the conjecture for a choice of ε in addition to $p \nmid 6D$, $\lambda(T(p)) \in A^\times$, $(AI_{k(\mathbb{Q})})$ and $(\mathrm{ord}_{\mathbb{Q}})$. Then if a prime $\mathfrak{p}$ with $\mathfrak{p}|p$ of O divides*
$$\frac{\Gamma(1,\mathrm{Ad}(\lambda)\otimes\alpha)L(1,\mathrm{Ad}(\lambda)\otimes\alpha)}{\Omega_1(\varepsilon,\widehat{\lambda};O)}$$
but is prime to $\Gamma(1,\mathrm{Ad}(\lambda))L(1,\mathrm{Ad}(\lambda))/\Omega_1(+,\lambda;O)\Omega_1(-,\lambda;O)$, then there exists a non-base-change lift μ such that $\mu \equiv \widehat{\lambda} \bmod \mathfrak{p}$.

PROOF. Let A be the valuation ring $O_{\mathfrak{p}}$. We consider the following sequence of maps:

$$i^* : H^q_{\text{cusp}}(Y_{0,K}(1), \mathcal{L}(\widehat{\kappa}, \text{id}; A)) \longrightarrow H^q_{\text{cusp}}(Y_{0,\mathbb{Q}}(1), \mathcal{L}(\widehat{\kappa}, \text{id}; A)|_{Y_{0,\mathbb{Q}}(1)});$$
$$\pi_* : H^q_{\text{cusp}}(Y_{0,\mathbb{Q}}(1), \mathcal{L}(\widehat{\kappa}, \text{id}; A)|_{Y_{0,\mathbb{Q}}(1)}) \longrightarrow H^q_c(Y_{0,\mathbb{Q}}(1), A) \cong A.$$

The map i^* is induced by the inclusion $i : Y_{0,\mathbb{Q}}(1) \hookrightarrow Y_{0,K}(1)$ and π_* is induced by the morphism of sheaves $\pi : \mathcal{L}(\widehat{\kappa}, \text{id}; A)|_{Y_{0,\mathbb{Q}}(1)} \to A$, which is induced by $P(X,Y) \mapsto (n!)^{-2}\nabla^n P(X,Y)$. Thus the composite $Ev = \pi_* \circ i^*$ is well defined integrally over A if $\mathfrak{p}$ is prime to $n!$. Suppose that $\xi(\widehat{\lambda}) - \xi(\mu) \in \mathfrak{p}H^q_{\text{cusp}}(Y_{0,K}(1), \mathcal{L}(\widehat{\kappa}, \text{id}; A))$. By Propositions 3.1 and 4.1 and their proof, $Ev(\xi(\widehat{\lambda}))$ is equal to the L-value

$$\frac{\Gamma(1, \text{Ad}(\lambda) \otimes \alpha)L(1, \text{Ad}(\lambda) \otimes \alpha)}{\Omega_d(\varepsilon, \widehat{\lambda}; O)}$$

up to A-units because $p \nmid 6D$. On the other hand, if μ is not a base change lift, twists of μ by any Hecke characters cannot be base change lift, because μ is of level 1. Thus again by Propositions 3.1 and 4.1, $Ev(\xi(\mu)) = 0$. Thus $Ev(\xi(\widehat{\lambda})) = Ev(\xi(\widehat{\lambda}) - \xi(\mu))$, which is divisible by $\mathfrak{p}$ by definition of the cohomological congruence. This proves the assertion (1).

Under the assumption (P), we see that

$$\mathfrak{p}\Big|\frac{\Gamma(1, \text{Ad}(\lambda) \otimes \alpha)L(1, \text{Ad}(\lambda) \otimes \alpha)}{\Omega_1(\varepsilon, \widehat{\lambda}; O)} \Rightarrow \mathfrak{p}\Big|\frac{\Gamma(1, \text{Ad}(\widehat{\lambda}))L(1, \text{Ad}(\widehat{\lambda}))}{\Omega_1(\varepsilon, \widehat{\lambda}; O)\Omega_1(-\varepsilon, \widehat{\lambda}; O)},$$

because

$$\frac{\Gamma(1, \text{Ad}(\widehat{\lambda}))L(1, \text{Ad}(\widehat{\lambda}))}{\Omega_1(\varepsilon, \widehat{\lambda}; A)\Omega_1(-\varepsilon, \widehat{\lambda}; A)} = \frac{\Gamma(1, \text{Ad}(\lambda))L(1, \text{Ad}(\lambda))\Gamma(1, \text{Ad}(\lambda) \otimes \alpha)L(1, \text{Ad}(\lambda) \otimes \alpha)}{\Omega_1(+, \lambda; A)^2\Omega_1(-, \lambda; A)^2}$$

up to A-units. Thus by $(C5)$, we have a congruence $\widehat{\lambda} \equiv \mu \bmod \mathfrak{p}$. As shown in [**DHI**], $C_1(\widehat{\lambda}; A)[\text{id}] \cong C_1(\lambda; A)$. This combined with $(CN1)$ shows that the factor $\mathfrak{p}$ has to be an associated prime of $C_1(\widehat{\lambda}; A)[\alpha]$, which shows the result. □

6. The case of quadratic extensions of totally real fields

Let F be a totally real field and K be a quadratic extension of F. We use the notation introduced in Section 2. Thus $H = \text{Res}_{\mathfrak{r}/\mathbb{Z}}\text{GL}(2)$ and $G = \text{Res}_{\mathcal{R}/\mathbb{Z}}\text{GL}(2)$. We consider a weight $\kappa = (n, v) \in \mathbb{Z}[I]^2$ with $n \geq 0$. Then we write $\widehat{\kappa} = (\widehat{n}, \widehat{v}) = \text{Inf}(\kappa)$ in $\mathbb{Z}[I_K]^2$. We then consider a cohomological primitive form $f_0 \in S_{\kappa,I}(N_0, \psi)_{/F}$ with $f_0|T(\mathfrak{n}) = \lambda(T(\mathfrak{n}))f_0$. We choose a subset Ψ of I_K such that $I_K = \Psi \sqcup \sigma\Psi$ for the generator σ of $\text{Gal}(K/F)$. Then we put $J = \Sigma_K(\mathbb{R}) \cap \Psi$ and choose $\Sigma_K(\mathbb{C})$ in I_K so that $\Psi = \Sigma_K(\mathbb{C}) \cup J$. Let $f \in S_{\kappa,J}(N, \chi)_{/K}$ be the base change lift of f_0 with $f|T(\mathfrak{n}) = \widehat{\lambda}(T(\mathfrak{n}))f$. Thus $\chi = \psi \circ N_{K/F}$. For each $t = \left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right) \in H(\mathbb{A}^{(\infty)})$, we consider

$$U_a = tU_0(N)t^{-1}, \quad \Gamma^{(a)} = U_aG(\mathbb{R})_+ \cap G(\mathbb{Q}) \quad \text{and} \quad \Phi^{(a)} = H(\mathbb{Q})_+ \cap \Gamma^{(a)},$$

where $H(\mathbb{Q})_+ = H(\mathbb{Q}) \cap H(\mathbb{R})_+$. Here note that $Y_{0,F}(N') = \sqcup_a \Phi^{(a)}\backslash\mathfrak{H}$ for $\mathfrak{H} = H(\mathbb{R})_+/C_{\infty+}$, where a runs over a complete set of representatives for $F^\times_{\mathbb{A}}/F^\times\widehat{\mathfrak{r}}^\times F^\times_{\infty+}$

with the identity component $F^{\times}_{\infty+}$ of $F^{\times}_{\infty}$. Let $\mathfrak{d} = \mathfrak{d}_K$ be the absolute different of $K/\mathbb{Q}$. Let Cl_X be the strict class group of X, that is

$$\{fractional\ ideals\ of\ X\}/\{(\xi)|\xi \in X, \xi \gg 0\},$$

where $\xi \gg 0$ means that image of ξ is positive for every real embedding of X.

For $z = y^{-1/2}\left(\begin{smallmatrix} y & x \\ 0 & 1\end{smallmatrix}\right) \in \mathrm{SL}_2(K_\infty) \subset G(\mathbb{R})_+$ for real positive y

$$f^{(a)}(z) = f(tz) = y^{\widehat{n}/2}|ay|_{K_{\mathbb{A}}}\left\{\sum_{\xi\in K^\times} \widehat{\lambda}(T(\xi a\mathfrak{d}))\xi^{-\widehat{v}}W(\xi y)\mathbf{e}(\xi x)\right\},$$

where ξ runs over all elements in $a^{-1}\mathfrak{d}^{-1}$ such that $\xi^\tau > 0$ for all $\tau \in J$ and $\xi^\tau < 0$ for all $\tau \in \Sigma_K(\mathbb{R}) - J$. Here, writing $n^*_\tau = 2n_\tau + 2$ for $\tau \in \Sigma_K(\mathbb{C})$

$$W(y) = \prod_{\tau\in\Sigma_K} W_\tau(y_\tau)$$

with

$$W_\tau(y) = \begin{cases} \sum_\alpha \binom{n^*_\tau}{\alpha}\left(\frac{y}{\sqrt{-1}|y|}\right)^{n_\tau+1-\alpha} K_{\alpha-n_\tau-1}(4\pi|y|)S_\tau^{n^*_\tau-\alpha}T_\tau^\alpha & \text{if } \tau\in\Sigma_K(\mathbb{C}),\\ \exp(-2\pi|y|) & \text{if } \tau\in\Sigma_K(\mathbb{R}),\end{cases}$$

where α runs over integers with $0 \le \alpha \le n^*_\tau$.

For each subset Y of I_X and $m \in \mathbb{Z}[I_X]$, we write $m! = \prod_\tau m_\tau!$ and $m(Y) = \sum_{\tau\in Y} m_\tau\tau$; in particular, we write $n(\mathbb{C}) = \mathrm{Res}^K_F\widehat{n}(\Sigma_K(\mathbb{C}))$ and $n(\mathbb{R}) = \mathrm{Res}^K_F\widehat{n}(J)$. For $x \in X\otimes_{\mathbb{Q}}\mathbb{C}$, we write $x^m = \prod_{\tau\in I_X} x^{\tau m_\tau}$ and $x^Y = \prod_{\tau\in Y} x^\tau$. For some specific number, $x = \pi$ or $x = 2i$, we write x^m and x^Y identifying x with a tuple $(x, x, \dots, x) \in \prod_{\tau\in I_X}\mathbb{C} = X\otimes_{\mathbb{Q}}\mathbb{C}$; thus, for example, $(2\pi i)^m = (2\pi i)^{\sum_\tau m_\tau}$. Note that $\Sigma_K(\mathbb{C})c = \sigma\Sigma_K(\mathbb{C})$. Then we consider $\nabla_\tau = \partial^2/\partial X_{\sigma\tau}\partial Y_\tau - \partial^2/\partial X_\tau\partial Y_{\sigma\tau}$ and write $\nabla^n = \prod_{\tau\in\Psi}\nabla_\tau^{n_\tau}$. We then compute the pull back $\nabla^n\delta(g^{(a)})|_F = i^*\nabla^n\delta(g^{(a)})$ for $g = f|R(\varphi)$ and $i : \mathfrak{H} \hookrightarrow \mathfrak{Z}$:

$$\begin{aligned}(n!)^{-2}\nabla^n\delta(g^{(a)})|_F &= (n!)^{-2}\nabla^n\delta_J(g^{(a)})|_F\\ &= (-1)^J(-2i)^{n(\mathbb{R})+J}\sum_{0\le j\le n(\mathbb{C})}\binom{n(\mathbb{C})}{j}\left((g^{(a)})_{2j} + (g^{(a)})_{2j+2}\right)y^{n(\mathbb{R})}d\mu(z),\end{aligned}$$

where we write

$$g^{(a)} = \sum_{0\le\alpha\le n^*} g^{(a)}_\alpha\binom{n^*}{\alpha}S^{n^*-\alpha}T^\alpha \text{ for } n^* = \textstyle\sum_{\tau\in\Sigma_K(\mathbb{C})}(n_\tau + n_{\tau\sigma} + 2)\tau,$$

and

$$d\mu(z) = \bigwedge_{\tau\in I_F} y_\tau^{-2}dy_\tau\wedge dx_\tau.$$

We now choose $\Delta \in K^\times$ such that $\Delta^\sigma = -\Delta$, $\Sigma_K(\mathbb{C}) = \{\tau \in I_K(\mathbb{C})|\,\mathrm{Im}(\Delta^\tau) > 0\}$ and $J = \{t \in \Sigma_K(\mathbb{R})|\Delta^\tau > 0\}$, where $I_K(\mathbb{C}) = \Sigma_K(\mathbb{C}) \sqcup \Sigma_K(\mathbb{C})c$ is the set of all complex embeddings of K into $\mathbb{C}$. Then we see

$$\xi\Delta^{-1} \in \left\{x \in a^{-1}\mathfrak{d}^{-1}|x^\sigma = -x\right\} \iff \xi \in a^{-1}\mathfrak{d}^{-1}\Delta\cap F.$$

Note that $a^{-1}\mathfrak{d}^{-1}\Delta$ is an ideal of F and $I_K = \Psi\sqcup\sigma\Psi$. For $\xi \in F$,

$$N_{F/\mathbb{Q}}(\xi\Delta^{-1}\mathfrak{d}) = |N_{F/\mathbb{Q}}(\xi)|D_F|\Delta^{-\Psi}|N_{F/\mathbb{Q}}(D_{K/F})^{1/2},$$

where $\Delta^{-\Psi} = \prod_{\tau\in\Psi}(\Delta^\tau)^{-1}$.

The stabilizer $\Phi_\infty^{(a)}$ of the cusp ∞ of $\Phi^{(a)}$ is an extension of $a\mathfrak{r}$ by $\mathfrak{r}_+^\times = \{\varepsilon \in \mathfrak{r}^\times | \varepsilon \gg 0\}$, that is, the following sequence is exact:

$$0 \to a\mathfrak{r} \to \Phi^{(a)} \xrightarrow{\det} \mathfrak{r}_+^\times \to 1.$$

Let φ be an arithmetic Hecke character of conductor C and with infinity type $-w$. Note that $(n!)^{-2}(\nabla^n \delta(g^{(a)})|_F)$ is a differential form on $Y_{0,F}(N')$ with values in $L((0, n+2v+\mathrm{Res}(w)), (\psi\varphi_F)^2; \mathbb{C})$ because we have (cf. [**Hi94**, 11.2a])

$$L((\widehat{n}, \widehat{v}+w), \chi\varphi^2; \mathbb{C})|_F \cong \bigoplus_{0\le j\le 2n} L((2n-2j, 2v+\mathrm{Res}(w)+j), (\psi\varphi_F)^2; \mathbb{C}).$$

Let $\omega = |\ |_{F_\mathbb{A}}^m$ for $m\mathbf{1}_F = n+2v+\mathrm{Res}(w)$. Thus we have

$$\int_{\Phi\backslash\mathfrak{H}} (n!)^{-2}(\nabla^n\delta(g^{(a)})|_F)|a|_{K_\mathbb{A}}^s y^{s\mathbf{1}} = (-1)^J(-2i)^{(n(J))+J}|a|_{K_\mathbb{A}}^s$$
$$\times \int_{F_\mathbb{R}/\mathfrak{r}_+^\times}\int_{F_\infty/a\mathfrak{r}} \sum_{0\le j\le n(\mathbb{C})} \binom{n(\mathbb{C})}{j} ((g^{(a)})_{2j} + (g^{(a)})_{2j+2}) dx y^{(s-2)\mathbf{1}+n(\mathbb{R})} dy$$
$$= \omega(a)^{-1} c_1(s) G(\varphi) L_a(s, \widehat{\lambda}, \varphi\omega),$$

where $r_2 = r_2(K)$ is the number of complex places of K and

$$L_a(s,\widehat{\lambda},\varphi\omega) = \sum_{0\ll\xi\in a^{-1}\mathfrak{d}^{-1}\Delta\cap F} \widehat{\lambda}(T(\xi a\Delta^{-1}\mathfrak{d}))\varphi_F\omega(\xi a\mathfrak{d}\Delta^{-1})N_{F/\mathbb{Q}}(\xi a\Delta^{-1}\mathfrak{d})^{-s-1}$$
$$c_1(s) = (-1)^{n(J)}\sqrt{-1}^{n(J)+J}2^{J(1-s)-2\mathbf{1}}|D_F|^{(3/2)+m+s}N(D_{K/F})^{-(m+s+1)/2}$$
$$\times \left\{\prod_{\tau\in\Psi(\mathbb{C})}(1+(-1)^{m+1})\right\}\left(\frac{\Gamma(\frac{s}{2})\Gamma(\frac{s}{2})}{\Gamma(s)}\right)^{r_2(K)}\left\{\prod_{\tau\in I}\Gamma_\mathbb{C}(s+n_\tau+1)\right\}.$$

Here $\xi \gg 0$ implies $\xi^\tau > 0$ for all $\tau \in I$, and we have used the fact that

$$\int_{F_\mathbb{R}/a\mathfrak{r}} dx = |D_F|^{1/2}|a|_{F_\mathbb{A}}^{-1}$$

for the discriminant D_F of $F/\mathbb{Q}$;

$$f^{(a)}(z) = y^{\widehat{n}/2}|ay|_{K_\mathbb{A}} \sum_\xi \lambda(T(\xi a\mathfrak{d}))\xi^{-\widehat{v}}W(\xi y)\mathbf{e}_K(\xi x),$$

where ξ runs over all elements in $a^{-1}\mathfrak{d}^{-1}$ such that $\xi^\tau > 0$ for all $\tau \in J$ and $\xi^\tau < 0$ for all $\tau \in \Sigma_K(\mathbb{R}) - J$, and

$$y^{\widehat{n}/2} = y^n \quad \text{and} \quad |a|_{K_\mathbb{A}} = |a|_{F_\mathbb{A}}^2.$$

We now look into

$$\sum_{a\in Cl_F} \omega(a) \int_{\Phi_\infty^{(a)}\backslash\mathfrak{H}} (n!)^{-2}(\nabla^n\delta(g^{(a)})|_F)|a|_{F_\mathbb{A}}^s y^{s\mathbf{1}}.$$

As a runs over a complete set of representatives in $F_\mathbb{A}^\times$ for Cl_F, the set of ideals $\{a\Delta\mathfrak{d}\}$ gives again a complete set of representatives for the ideal class group Cl_F

because $\Delta\mathfrak{d}$ is an ideal of F. Thus we get

$$\sum_{a\in Cl_F}\omega(a)\int_{\Phi_\infty^{(a)}\backslash\mathfrak{H}}(n!)^{-2}(\nabla^n\delta(g^{(a)})|_F)|a|^s_{F_{\mathbb{A}}}y^{s\mathbf{1}}=c_1(s)G(\varphi)L(s,\widehat{\lambda},\varphi_F\omega).$$

By the Rankin convolution method, we get

$$\int_{\Phi_\infty^{(a)}\backslash\mathfrak{H}}(n!)^{-2}(\nabla^n\delta(g^{(a)})|_F)y^{s\mathbf{1}}=\int_{\Phi^{(a)}\backslash\mathfrak{H}}(n!)^{-2}(\nabla^n\delta(g^{(a)})|_F)y^{s\mathbf{1}}\mathcal{E}^{(a)}(s),$$

where

$$\mathcal{E}^{(a)}(s)=\sum_{\delta\in\Phi^{(a)}/\Phi_\infty^{(a)}}\varphi^2_{N'}\psi^2_{N'}(\gamma)y^{s\mathbf{1}}\circ\gamma.$$

We now put

$$E(s)=E(x,s)=\sum_{a\in Cl_F}\omega(a)|a|^s_{F_{\mathbb{A}}}\mathcal{E}^{(a)}(s)$$

as a function of $x\in H(\mathbb{A})$. Writing $Y=Y_{0,F}(N')=\sqcup_{a\in Cl_F}Y^{(a)}$, if $(\omega\psi\varphi_F)^2=\mathrm{id}$, by (RES3) in Appendix, we get

$$\begin{aligned}&\mathrm{Res}_{s=1}\zeta_{F,N'}(2s)\int_Y E(s)(n!)^{-2}(\nabla^n\delta(g)|_F)\\&=N_{F/\mathbb{Q}}(N')^{-2}\phi(N')\frac{2^{[F:\mathbb{Q}]-1}\pi^{[F:\mathbb{Q}]}R_\infty}{w(F)|D_F|}\sum_a\omega(a)\int_{Y^{(a)}}(n!)^{-2}(\nabla^n\delta(g)|_F),\end{aligned}$$

where R_∞ is the regulator of F. Let $\varepsilon:S=C_\infty/C_{\infty+}=\{\pm1\}^{\Sigma_K(\mathbb{R})}\to\{\pm1\}$ be a character such that

$$\varepsilon((-1,-1)_\tau)=(-1)^{n_\tau+1}\quad\text{for all}\quad\tau\in J,$$

where $(-1,-1)_\tau$ is an element of S whose component is equal to 1 outside $\{\tau,\sigma\tau\}$ and is equal to -1 at τ and $\sigma\tau$. We consider the projection π_ε to the ε-eigenspace. In the same manner as in Section 4, under the condition of the parity of ε as above, the application of π_ε on the integrand does not have any effect on the outcome of the computation. Thus if $\omega\psi\varphi_F=\alpha$,

$$\begin{aligned}&\mathrm{Res}_{s=1}\zeta_{F,N'_F}(2s)\int_Y E(s)(n!)^{-2}(\nabla^n\pi_\varepsilon\delta(g)|_F)\\&\qquad=c_1(1)\{\mathrm{Res}_{s=1}\zeta_{F,N'_F}(s)\}L(1,\mathrm{Ad}(\lambda)\otimes\alpha).\end{aligned}$$

Thus we get

$$\begin{aligned}\Gamma(1,\mathrm{Ad}(\lambda)\otimes\alpha)L_{C'}(1,\mathrm{Ad}(\lambda)\otimes\alpha)&=c_0N(D_{K/F})^{1/2}G(\varphi)^{-1}i^{n(J)+J}\\&\times h(F)^{-1}\sum_a\omega(a)\int_{Y^{(a)}}(n!)^{-2}(\nabla^n\pi_\varepsilon\delta(g)|_F),\end{aligned}$$

where

$$c_0=2^{r_2(K)-(r_1(K)/2)-1}(N_{F/\mathbb{Q}}N')^{-2}\phi(N')|D_F|^{-m-3}N_{F/\mathbb{Q}}(D_{K/F})^{(m+1)/2}\in\mathbb{Q}.$$

Let μ be a system of Hecke eigenvalues of level N and with character χ. Let $d=[F:\mathbb{Q}]$ and A be a Dedekind domain in $\overline{\mathbb{Q}}$ containing $\mu(\mathfrak{n})$ for all ideals $\mathfrak{n}\subset\mathfrak{r}$. For each character $\varepsilon:S\to\{\pm1\}$, we consider $H^d(Y_{0,K}(N),\mathcal{L}(\widehat{\kappa},\chi;A))[\varepsilon,\mu]$ which is projective of rank 1 over A. Extending A a bit, we may assume that it is free of rank 1 over A. Let ξ be the generator of this free A-module of rank 1. Let f be the normalized Hecke eigenform with eigenvalues μ. We define

$$[\pi_\varepsilon \delta_{J,\emptyset}(f)] = \Omega_{\Sigma_K(\mathbb{R}),2\Sigma_K(\mathbb{C})}(\varepsilon, \mu; A)\xi,$$

where $[\phi]$ indicates the cohomology class of a closed form ϕ. Here the subscript "$\Sigma_K(\mathbb{R}), 2\Sigma_K(\mathbb{C})$" indicates that the cohomology class is of degree 1 at archimedean places in $\Sigma_K(\mathbb{R})$ and of degree 2 at archimedean places in $\Sigma_K(\mathbb{C})$. When $A = \mathbb{Q}(\mu)$, we have the cohomological conjugate

$$\xi^\rho \in H^d(Y_{0,K}(N), \mathcal{L}(\widehat{\kappa}\rho, \chi^\rho; A))[\varepsilon, \mu^\rho]$$

and define automorphic conjugate f^ρ by $\mathbf{a}(\mathfrak{n}, f^\rho) = \mathbf{a}(\mathfrak{n}, f)^\rho$. Then we define

$$[\pi_\varepsilon \delta_{J,\emptyset}(f^\rho)] = \Omega_{\Sigma_K(\mathbb{R}),2\Sigma_K(\mathbb{C})}(\varepsilon, \mu^\rho; A)\xi^\rho,$$

and put

$$\underline{\Omega}_{\Sigma_K(\mathbb{R}),2\Sigma_K(\mathbb{C})}(\varepsilon, \mu; A) = (\Omega_{\Sigma_K(\mathbb{R}),2\Sigma_K(\mathbb{C})}(\varepsilon, \mu^\rho; A))_{\rho \in I_{\mathbb{Q}(\mu)}}$$

as an element of $(\mathbb{Q}(\mu) \otimes_{\mathbb{Q}} \mathbb{C})^\times$. Then we get similarly to Corollaries 3.2 and 4.2 the following result.

THEOREM 6.1. *Suppose that we have an arithmetic Hecke character φ of K and ω of F such that (i) the conductor of ω is* 1 *and (ii) $\psi\varphi_F\omega = \alpha$. Then we have*

$$\frac{(1 \otimes \Gamma(1, \mathrm{Ad}(\lambda) \otimes \alpha))\mathbb{L}_{C'}(1, \mathrm{Ad}(\lambda) \otimes \alpha)}{(1 \otimes N_{F/\mathbb{Q}}(D_{K/F})^{1/2} i^{n(J)+J})\mathbf{G}(\psi\alpha)\underline{\Omega}_{\Sigma_K(\mathbb{R}),2\Sigma_K(\mathbb{C})}(\varepsilon, \widehat{\lambda}; \mathbb{Q}(\lambda))} \in \mathbb{Q}(\lambda) \subset \mathbb{Q}(\lambda) \otimes_{\mathbb{Q}} \mathbb{C}$$

for any ε with $\varepsilon((-1,-1)_\tau) = -1$ for all $\tau \in J$. Moreover if φ can be chosen to be the identity character, then for all valuation ring A of $\mathbb{Q}(\lambda)$ with residual characteristic prime to $2e(n!)D_K$, we have

$$\frac{(N_{F/\mathbb{Q}}N_0)^2 \phi(N_0)^{-1} N_{F/\mathbb{Q}}(D_{K/F})^{1/2} i^{n(J)+J} \Gamma(1, \mathrm{Ad}(\lambda) \otimes \alpha) L(1, \mathrm{Ad}(\lambda) \otimes \alpha)}{\Omega_{\Sigma_K(\mathbb{R}),2\Sigma_K(\mathbb{C})}(\varepsilon, \widehat{\lambda}; A)} \in A,$$

where $N_0 = N \cap \mathfrak{r}$ and e is the least common multiple of the order of maximal torsion subgroups of $\Gamma^{(a)}/\mathcal{R}^\times$ for all a.

If either K is totally real or a CM field, we can always find ω and φ as in the theorem (see Lemma 2.2). If K has both complex and real places, this condition really imposes a restriction. Probably one could remove this condition taking the integral over $Y(S)$ for a smaller subgroup S in $U_0(N')$ allowing ω with non-trivial conductor, but we might lose more Euler factors of the adjoint L in the process. Since this would further complicate our computation, we do not look into this point in the present paper.

We can formulate the divisibility of the L-value by congruence primes of $\widehat{\lambda}$ as is done in Section 5 for quadratic fields, which we leave to the reader.

The parity restriction on ε is explained by the following fact:

$$\varphi((-1,-1)_\tau) = \varphi_F((-1)_\tau) = \psi((-1)_\tau)\alpha((-1)_\tau) = (-1)^{n_\tau}.$$

Thus the condition that $\varepsilon((-1,-1)_\tau) = -1$ is equivalent to

$$\varepsilon\varphi_\infty((-1,-1)_\tau) = (-1)^{n_\tau+1},$$

and we have

$$\Omega_{\Sigma_K(\mathbb{R}),2\Sigma_K(\mathbb{C})}(\varepsilon\varphi_\infty, \widehat{\lambda} \otimes \varphi; \mathbb{Q}(\lambda)) = G(\varphi)^{-1}\Omega_{\Sigma_K(\mathbb{R}),2\Sigma_K(\mathbb{C})}(\varepsilon, \widehat{\lambda}; \mathbb{Q}(\lambda)).$$

7. Quadratic extensions K of an imaginary quadratic F

Now we assume F to be an imaginary quadratic field and keep notation introduced in the previous sections. We write σ for the generator of $\mathrm{Gal}(K/F)$, and fix an embedding $\tau : K \to \mathbb{C}$. We then write $J' = \{\sigma\tau\}$. For each $\iota \in I_K$, we write $[\iota]$ for the archimedean place arising from ι. Thus this correspondence induces a linear map $[\,] : \mathbb{Z}[I_K] \to \mathbb{Z}[\Sigma]$ for the set $\Sigma = \Sigma_K = I_K/\langle c\rangle$ of archimedean places. If no confusion is likely, we write Σ for Σ_K to simplify the notation. We decompose $I_K = \Psi \sqcup \sigma\Psi$ with $\Psi = \{\tau, \tau c\}$. We identify Σ with $\{\tau, \sigma\tau\}$. Let f be a cohomological modular form on G of weight $(\widehat{n}, \widehat{v})$. Then we can write for $z \in \mathfrak{Z} = G(\mathbb{R})/Z(\mathbb{R})C_\infty$ given by $y^{-1/2}\left(\begin{smallmatrix} y & x \\ 0 & 1\end{smallmatrix}\right) \in \mathrm{SL}_2(K_\infty)$

$$f^{(a)}(z) = f\left(ty^{-1/2}\begin{pmatrix} y & x \\ 0 & 1\end{pmatrix}\right) = y^{\widehat{n}/2}|ay|_{K_A}\left\{\sum_{\xi\in K^\times}\widehat{\lambda}(T(\xi a\mathfrak{d}))\xi^{-\widehat{v}}W(\xi y)\mathbf{e}(\xi x)\right\}$$

where

$$W(y) = \sum_{0\le\alpha\le\widehat{n}^*}\binom{\widehat{n}^*}{\alpha}\left(\frac{y}{\sqrt{-1}|y|}\right)^{\widehat{n}+\mathbf{1}-\alpha}K_{\alpha-\widehat{n}-\mathbf{1}}(4\pi|y|)S^{\widehat{n}^*-\alpha}T^\alpha \quad (\widehat{n}^* = [\widehat{n}] + 2\Sigma).$$

We write $f^{(a)}(z) = \sum_{0\le\alpha\le\widehat{n}^*} f^{(a)}_\alpha(z)\binom{\widehat{n}^*}{\alpha}S^{\widehat{n}^*-\alpha}T^\alpha$. Then by a computation similar to (δ) in Section 3 and [**Hi94**, Section 2], $z^{-1}\delta_{J'}(f^{(a)})$ for $z = y^{-1/2}\left(\begin{smallmatrix} y & x \\ 0 & 1\end{smallmatrix}\right)$ is given by the following formula through replacing $U_{\alpha_\rho}U_{\beta_\theta}$ by $f_{[\alpha_\rho\rho+\beta_\theta\theta]}$:

$$\begin{aligned}
&y_{\sigma\tau}^{-1}y_\tau^{-2}\sum_{0\le j\le\widehat{n}}(-1)^{n_\tau-j_{\tau c}+n_{\sigma\tau}-j_{\sigma\tau c}}\binom{\widehat{n}}{j}X^{\widehat{n}-j}Y^j\\
&\times\left\{U_{n_{\tau c}+j_\tau-j_{\tau c}}dy_\tau\wedge dx_\tau - 4U_{n_{\tau c}+j_\tau-j_{\tau c}+1}dx_\tau\wedge d\overline{x}_\tau + U_{n_{\tau c}+j_\tau-j_{\tau c}+2}dy_\tau\wedge d\overline{x}_\tau\right\}\\
&\bigwedge\left\{U_{n_{\sigma\tau c}+j_{\sigma\tau}-j_{\sigma\tau c}}dx_{\sigma\tau} - 2U_{n_{\sigma\tau c}+j_{\sigma\tau}-j_{\sigma\tau c}+1}dy_{\sigma\tau} - U_{n_{\sigma\tau c}+j_{\sigma\tau}-j_{\sigma\tau c}+2}d\overline{x}_{\sigma\tau}\right\},
\end{aligned}$$

where we have written $X^{\widehat{n}-j}Y^j$ for $\prod_{\rho\in I_K}X_\rho^{\widehat{n}_\rho-j_\rho}Y_\rho^{j_\rho}$, $\binom{\widehat{n}}{j}$ for $\prod_{\rho\in I_K}\binom{\widehat{n}_\rho}{j_\rho}$, and j runs over $j \in \mathbb{Z}[I_K]$ with $0 \le j_\rho \le \widehat{n}_\rho$ for all $\rho \in I_K$. We get a non-trivial result only for base change lift; so, we may assume that $n_\tau = n_{\tau c} = n_{\sigma\tau} = n_{\sigma\tau c}$. The action of z does not affect the outcome of the differential operator $\nabla^n = (\nabla_\tau\nabla_{\tau c})^{n_\tau}$ applied to $\delta_{J'}(f)|_F$ ($\nabla_\rho = \partial^2/\partial X_{\sigma\rho}\partial Y_\rho - \partial^2/\partial X_\rho\partial Y_{\sigma\rho}$); so, we forget about it. We then restrict the differential form to $\mathfrak{H} = H(\mathbb{R})/Z_H(\mathbb{R})C_\infty$ and hence, we may assume that $y = y_\tau = y_{\sigma\tau}$ and $x = x_\tau = x_{\sigma\tau}$. Thus we get

$$\begin{aligned}
&(n!)^{-2}\nabla^n\delta_{J'}(f)|_F\\
&\quad = (-1)^{n_\tau}\sum_{0\le j\le n} -(-1)^j\binom{n}{j}\{f_{[n'+2J']} - 8f_{[n'+\Sigma]} + f_{[n'+2(\Sigma-J')]}\}d\mu(z),
\end{aligned}$$

where $d\mu = y^{-3}dy\wedge dx\wedge d\overline{x}$, $\binom{n}{j} = \binom{n_\tau}{j_\tau}\binom{n_{\tau c}}{j_{\tau c}}$, and $n' = \widehat{n}(\Sigma) + j^\#$ for $j^\# = (j_\tau - j_{\tau c})\tau + (j_{\tau c} - j_\tau)\sigma\tau$. We now compute for $X = J', \Sigma - J', \Sigma$, writing x for the number of elements in X and g for $f|R(\varphi)$,

$$\int_{\Phi_\infty^{(a)}\backslash\mathfrak{Z}} g^{(a)}_{[n'+2X]}|ay|^s_{F_\mathbb{A}}d\mu(z)$$
$$= -(-1)^{j+x}2^{2n_\tau+2s}\sqrt{-1}\mathrm{vol}(F_\mathbb{R}/a\mathfrak{r})G(\varphi)|a|^{s+2}_{F_\mathbb{A}}(|D_F|N_{F/\mathbb{Q}}(D_{K/F})^{1/2})^{1+s}$$
$$\cdot\frac{\Gamma(n_\tau+1+s)^2\Gamma(n_\tau+s+j_\tau-j_{\tau c}+x)\Gamma(n_\tau+s+j_\tau-j_{\tau c}+2-x)}{\Gamma(2n_\tau+2+2s)}$$
$$\cdot\sum_{\xi\in F^\times}\widehat{\lambda}(T(\xi a\Delta^{-1}\mathfrak{d}))\varphi(\xi a\Delta^{-1}\mathfrak{d})\xi^{-n-2v-\mathrm{Res}(w)}\Delta^{\widehat{v}+w+\widehat{n}/2}N_{F/\mathbb{Q}}(\xi\Delta^{-1}\mathfrak{d})^{-1-s},$$

where $\Delta\in K^\times$ satisfies $\Delta^\sigma=-\Delta$ and we have used the identity:

$$N_{F/\mathbb{Q}}(\Delta^{-1}\mathfrak{d})=|D_F||\Delta^{-\tau}|^2N_{F/\mathbb{Q}}(D_{K/F})^{1/2}$$

along with the convention that $\widehat{\lambda}(T(\mathfrak{n}))=0$ for non-integral ideals $\mathfrak{n}$.

Now we suppose to have a Hecke character ω of conductor 1 of $F_\mathbb{A}^\times/F^\times$ with $\infty(\omega)=n+2v+\mathrm{Res}(w)$. For that, we may need to allow algebraic character φ not necessarily arithmetic to have such ω (see Lemma 2.2). Note that

$$\omega(\xi a\Delta^{-1}\mathfrak{d})=\omega(a\Delta^{-1}\mathfrak{d})\xi^{n+2v+\mathrm{Res}(w)}.$$

Then we have, for $g=f|R(\varphi)$,

$$\int_{\Phi_\infty^{(a)}\backslash\mathfrak{H}}(n!)^{-2}(\nabla^n\delta_{J'}(g^{(a)})|_F)|ay|^s_{F_\mathbb{A}}(-1)^{n_\tau+1}2^{2n_\tau+2s}(4\pi)^{-2n_\tau-2-2s}\sqrt{-1}\mathrm{vol}(F_\mathbb{R}/a\mathfrak{r})$$
$$\cdot G(\varphi)(|D_F|N_{F/\mathbb{Q}}(D_{K/F})^{1/2})^{1+s}\times\omega(a\Delta^{-1}\mathfrak{d})^{-1}\Delta^{\widehat{v}+w+\widehat{n}/2}\Gamma(n_\tau+1+s)^2$$
$$\cdot\{G_0(s)+8G_1(s)+G_2(s)\}L_a(s,\widehat{\lambda},\varphi_F\omega),$$

where

$$L_a(s,\widehat{\lambda},\varphi_F\omega)=\sum_{\xi\in F^\times/\mathfrak{r}^\times}\widehat{\lambda}(T(\xi a\Delta^{-1}\mathfrak{d}))\varphi\omega(\xi a\Delta^{-1}\mathfrak{d})N_{F/\mathbb{Q}}(\xi a\Delta^{-1}\mathfrak{d})^{-1-s},$$

$$G_x(s)=\sum_{0\le j\le n}\binom{n}{j}\frac{\Gamma(n_\tau+s+j_\tau-j_{\tau c}+x)\Gamma(n_\tau+s+j_\tau-j_{\tau c}+2-x)}{\Gamma(2n_\tau+2+2s)}$$

for $x=0,1,2$. Using the formula in [**Hi94**, p. 505] twice, we get

$$G_x(s)=\frac{\Gamma(s+x)\Gamma(s+2-x)}{\Gamma(2+2s)},$$
$$G_0(s)+8G_1(s)+G_2(s)=2(5s+1)\frac{\Gamma(s)\Gamma(s+1)}{\Gamma(2+2s)}.$$

Then using the fact that $\mathrm{vol}(F_\infty/\mathfrak{r})=2^{-1}|D_F|^{1/2}$, we have

$$\sum_{a\in Cl_F}\omega(a)\int_{\Phi_\infty^{(a)}\backslash\mathfrak{H}}(n!)^{-2}(\nabla^n\delta_{J'}(g^{(a)})|_F)|a|^s_{F_\mathbb{A}}y^{s\mathbf{1}}=c_1(s)L(s,\widehat{\lambda},\varphi_F\omega),$$

where

$$c_1(s)=\sqrt{-1}|D_F|^{1/2}(|D_F|N_{F/\mathbb{Q}}(D_{K/F})^{1/2})^{1+s}G(\varphi)\omega(\Delta^{-1}\mathfrak{d})^{-1}\Delta^{\widehat{v}+w+\widehat{n}/2}$$
$$\cdot(-1)^{n_\tau+1}2^{2n_\tau+2s}(4\pi)^{-2n_\tau-2-2s}(5s+1)\frac{\Gamma(s)\Gamma(s+1)}{\Gamma(2+2s)}\Gamma(n_\tau+1+s)^2.$$

By Rankin's convolution method, we get

$$\int_{\Phi_\infty^{(a)}\backslash\mathfrak{H}}(n!)^{-2}(\nabla^n\delta_{J'}(g^{(a)})|_F)y^{s\mathbf{1}}=\int_{\Phi^{(a)}\backslash\mathfrak{H}}(n!)^{-2}(\nabla^n\delta_{J'}(g^{(a)})|_F)\mathcal{E}^{(a)}(s),$$

where

$$\mathcal{E}^{(a)}(s)=\sum_{\gamma\in\Phi^{(a)}/\Phi_\infty^{(a)}}\varphi_{N'}^2\psi_{N'}^2(\gamma)y^{s\mathbf{1}}\circ\gamma.$$

Put $E(s)=\sum_{a\in Cl_F}\omega(a)|a|_{F_\mathbb{A}}^s\mathcal{E}^{(a)}(s)$. If $(\omega\psi\varphi_F)^2=\mathrm{id}$, by (RES3) in Appendix, we get for $Y=Y_{0,F}(N')=\sqcup_{a\in Cl_F}Y^{(a)}$

$$\begin{aligned}\mathrm{Res}_{s=1}\zeta_{F,N'}(2s)&\int_Y E(s)(n!)^{-2}(\nabla^n\delta(g)|_F)\\&=N_{F/\mathbb{Q}}(N')^{-2}\phi(N')\frac{2\pi^2R_\infty}{|D_F|}\sum_{a\in Cl_F}\omega(a)\int_{Y^{(a)}}(\nabla^n\delta(g^{(a)})|_F).\end{aligned}$$

On the other hand, if $\alpha=\psi\omega\varphi_F$, we have

$$\begin{aligned}\mathrm{Res}_{s=1}\zeta_{F,N'}(2s)&\int_Y E(s)(n!)^{-2}(\nabla^n\delta(g)|_F)\\&=c_1(1)\{\mathrm{Res}_{s=1}\zeta_{F,N'}(s)\}L_{C'}(1,\mathrm{Ad}(\lambda)\otimes\alpha).\end{aligned}$$

Comparing the two expressions, we get

$$\begin{aligned}\Gamma(1,\mathrm{Ad}(\lambda)\otimes\alpha)L_{C'}(1,\mathrm{Ad}(\lambda)\otimes\alpha)&=c_0G(\varphi)^{-1}\omega(\Delta^{-1}\mathfrak{d})\Delta^{-\widehat{v}-w-\widehat{n}/2}\\&\cdot\sqrt{-1}h(F)^{-1}\sum_{a\in Cl_F}\omega(a)\int_{Y^{(a)}}(n!)^{-2}(\nabla^n\delta(g^{(a)})|_F),\end{aligned}$$

where $c_0=2(-1)^{n_\tau+1}|D_F|^{-3}N_{F/\mathbb{Q}}(D_{K/F})^{-1}\in\mathbb{Q}$.

At the beginning, we fixed an embedding τ of K into $\mathbb{C}$. We study what happens if we start with $\sigma\tau$ instead of τ. The result is the same, but J' will be replaced by $\{\tau\}$. Note that $\{\delta_{J'}(f)\}_{J'\subset\Sigma_K(\mathbb{C})}$ forms a base of $H^3_{\mathrm{cusp}}(Y_0(N),\mathcal{L}(\widehat{\kappa},\chi;\mathbb{C}))[\widehat{\lambda}]$. Thus

$$H^3(\mathbb{Q}(\lambda))[\widehat{\lambda}]=H^3_{\mathrm{cusp}}(Y_0(N),\mathcal{L}(\widehat{\kappa},\chi;\mathbb{Q}(\lambda)))[\widehat{\lambda}]\cong\mathbb{Q}(\lambda)^2.$$

Then σ acts on $Y_0(N)$ via the Galois action: $x\mapsto x^\sigma$ on $G(\mathbb{A})$, because $N^\sigma=N$. We let σ act on $L(n;\mathbb{Q}(\lambda))$ by $\sigma(X_\tau,Y_\tau)=(X_{\sigma\tau},Y_{\sigma\tau})$. Thus for a differential form ϕ on $\mathfrak{Z}$ with values in $L(\widehat{n};\mathbb{C})$ such that $\gamma^*\phi=\gamma\phi$, we see

$$\gamma^*(\sigma(\sigma^*\phi))=\sigma\sigma^*(\gamma^\sigma)^*\phi=\sigma\gamma^\sigma(\sigma^*\phi)=\gamma\sigma(\sigma^*\phi).$$

Thus via $\phi\mapsto\sigma\sigma^*\phi$, $\mathrm{Gal}(K/F)$ acts on $H^3_{\mathrm{cusp}}(Y_0(N),\mathcal{L}(\widehat{\kappa},\chi;\mathbb{C}))$. We see that $\sigma^*\delta_{\{\tau\}}(f)=\sigma\delta_{\{\sigma\tau\}}(f)$. Since $\widehat{\lambda}$ is stable under σ, $H^3[\widehat{\lambda}]$ is stable under the action of σ. Thus $H^3(\mathbb{Q}(\lambda))[\widehat{\lambda}][\sigma-1]$ is one dimensional, and we take a generator ξ such that

$$H^3(\mathbb{Q}(\lambda))[\widehat{\lambda}][\sigma-1]=\mathbb{Q}(\lambda)\xi.$$

The action of σ defined above commutes with the Galois action induced covariantly by the Galois action on $L(\widehat{\kappa},\chi;\mathbb{Q}(\lambda))$. We write ξ^ρ for the Galois conjugate of ξ under the latter action. We note that $H^3(\mathbb{C})[\widehat{\lambda}^\rho][\sigma-1]=\mathbb{C}(\delta_{\{\tau\}}(f^\rho)+\delta_{\{\sigma\tau\}}(f^\rho))$. We then define a complex number $\Omega_{(1,2)}(\widehat{\lambda}^\rho;\mathbb{Q}(\lambda))\in\mathbb{C}^\times$ by

$$(\delta_{\{\tau\}}(f^\rho)+\delta_{\{\sigma\tau\}}(f^\rho))=\Omega_{(1,2)}(\widehat{\lambda}^\rho;\mathbb{Q}(\lambda))\xi^\rho.$$

We put

$$\underline{\Omega}_{(1,2)}(\widehat{\lambda}, \mathbb{Q}(\lambda)) = (\Omega_{(1,2)}(\widehat{\lambda}^{\rho}; \mathbb{Q}(\lambda)))_{\rho \in I_{\mathbb{Q}(\lambda)}} \in (\mathbb{Q}(\lambda) \otimes_{\mathbb{Q}} \mathbb{C})^{\times}.$$

Note that $\Delta^{-1}\mathfrak{d} = \mathfrak{d}_F \xi$. Thus for a finite idele d of F generating $\mathfrak{d}_F$, we have $\omega(\Delta^{-1}\mathfrak{d}) = \omega(d)\xi^{n+2v+\mathrm{Res}(w)}$ because ω is of conductor 1. We choose φ as in Lemma 2.2. Then it is easy to see that $\xi^{n+2v+\mathrm{Res}(w)} \in \mathbb{Q}$. This shows that

$$\omega(\Delta^{-1}\mathfrak{d})^{-1}G(\varphi)/G(\psi^{-1}\alpha) \in \mathbb{Q}(\lambda).$$

Since $G(\psi^{-1}\alpha)G(\psi\alpha) \in \mathbb{Q}$, we finally get

THEOREM 7.1. *Suppose that we have an arithmetic Hecke character φ of K and ω of F such that (i) the conductor of ω is 1 and (ii) $\psi\varphi_F\omega = \alpha$. Then we have*

$$\frac{(1 \otimes \Delta^{\widehat{v}+w+\widehat{n}/2}\Gamma(1, \mathrm{Ad}(\lambda) \otimes \alpha))\mathbb{L}_{C'}(1, \mathrm{Ad}(\lambda) \otimes \alpha)}{(1 \otimes \sqrt{-1})\mathbf{G}(\psi\alpha)\underline{\Omega}_{(1,2)}(\widehat{\lambda}; \mathbb{Q}(\lambda))} \in \mathbb{Q}(\lambda) \subset \mathbb{Q}(\lambda) \otimes_{\mathbb{Q}} \mathbb{C}.$$

If φ and ω are algebraic Hecke characters satisfying (i) and (ii), then we have

$$\frac{\Gamma(1, \mathrm{Ad}(\lambda) \otimes \alpha)L_{C'}(1, \mathrm{Ad}(\lambda) \otimes \alpha)}{\Omega_{(1,2)}(\widehat{\lambda}; \mathbb{Q}(\lambda))} \in \overline{\mathbb{Q}}.$$

Since the L-value in Theorem 7.1 does not depends on the choice of φ, even if φ is not arithmetic, presumably, the $\mathbb{Q}(\lambda)$-rationality would hold in general. Further study has to be done to prove this.

The period $\Omega_{(1,2)}$ is only defined for the base change lift $\widehat{\lambda}$, while we have defined similar periods for any system μ of Hecke eigenvalues in the previous sections when F is totally real. When $\mu \neq \mu_\sigma$, the action of σ on $H^3(\mathbb{Q}(\mu))$ does not preserve $H^3(\mathbb{Q}(\mu))[\mu]$, and this causes a trouble. If F has complex places, the same problem shows up as will be seen in the following section.

8. General quadratic extensions

Let K/F be a semi-simple quadratic extension of a number field F. Thus we allow here $K = F \oplus F$. When $K = F \oplus F$, we regard $D_{K/F} = 1$ and $\alpha = \mathrm{id}$; otherwise, α denotes the quadratic character of $F_{\mathbb{A}}^{\times}/F^{\times}$ corresponding to K/F. We shall prove the rationality theorem of $L(1, \mathrm{Ad}(\lambda) \otimes \alpha)$ in this general case. Computation is the same as in the previous sections. Since the definition of $f \mapsto \delta(f)$ is given in [**Hi94**] by a procedure which is basically a tensor product of definitions over archimedean places, the computation of the pull back $\delta(f)|_F$ is again essentially the tensor product of the pull back over archimedean places of K. Thus the computation is the same as in Section 3 for a complex place of K over a real place of F, the same as in Section 4 for two real places of K over a real place of F and the same as in Section 7 for two complex places of K over a complex place of F. After computing $\delta(f)$, the computation of the Rankin product is fairly standard. Thus our exposition of the computation will be brief, but we state the result in a precise form. When $K = F \oplus F$, we regard that every archimedean place of F splits in K.

We decompose $I = I(\mathbb{C}) \sqcup I(\mathbb{R})$ and $I(\mathbb{C}) = \Sigma(\mathbb{C}) \sqcup \Sigma(\mathbb{C})c$. Similarly we decompose $I_K = I_K(\mathbb{R}) \sqcup I_K(\mathbb{C})$. We decompose $I_K = \Psi \sqcup \sigma\Psi$ for the generator $\sigma \in \mathrm{Gal}(K/F)$ and $\Psi = \Psi(\mathbb{R}) \sqcup \Psi(\mathbb{C})$, where $\Psi(\mathbb{R})$ is the subset of all real embeddings of Ψ. When $K = F \oplus F$, we have $\sigma(x \oplus y) = y \oplus x$. We write J' (resp. J'') for $\{\sigma\tau \in \sigma\Psi(\mathbb{C}) | \mathrm{Res}_{K/F}(\tau) \in \Sigma(\mathbb{C})\}$ (resp. $\{\tau \in I(\mathbb{R}) | \tau$ extends to a complex place of $K\}$) and put $J = \Psi(\mathbb{R})$. When $K = F \oplus F$, we identify I_K with $I \sqcup I$ through the right and left projections of K to F. Then we identify Ψ with the left component I in I_K. Then we shall make use of the isomorphism

$$\delta_{J,J'} : S_{\widehat{\kappa},J}(N,\chi) \hookrightarrow H^q(Y_0(N), \mathcal{L}(\widehat{\kappa},\chi;\mathbb{C})) \quad (\kappa = (n,v))$$

defined in [**Hi94**, Proposition 2.1], where $q = [F:\mathbb{Q}] + r_1(F) + r_2(F) = \dim(\mathfrak{H})$. When $K = F \oplus F$, then $Y_{0,K}(N) = Y_{0,F}(N_0) \times Y_{0,F}(N_0)$ for the ideal $N = N_0 \oplus N_0 \subset \mathfrak{r} \oplus \mathfrak{r} = \mathcal{R}$. In this case,

$$L((\widehat{n},\widehat{v}),\chi;\mathbb{C}) \cong L((n,v),\psi;\mathbb{C}) \otimes_{\mathbb{C}} L((n,v),\psi;\mathbb{C}).$$

For normalized Hecke eigenforms $f_0 \in S_{\kappa,I(\mathbb{R})}(N_0,\psi;\mathbb{C})[\lambda]$ and $f'_0 \in S_{\kappa,\emptyset}(N_0,\psi;\mathbb{C})[\lambda]$, the base change lift $f \in S_{\widehat{\kappa},J}(N,\chi;\mathbb{C})$ to $G = H \times H$ is just $f(z,z') = f_0(z) f'_0(z')$ for $(z,z') \in \mathfrak{H} \times \mathfrak{H} = \mathfrak{Z}$ and

$$\delta_{J,J'}(f) = \delta_{I(\mathbb{R}),I(\mathbb{C})}(f_0)(z) \wedge \delta_{\emptyset,\emptyset}(f'_0)(z'),$$

which gives a cohomology class in

$$\begin{aligned} H^q(Y_{0,N}(N_0),&\mathcal{L}((n,v),\psi;\mathbb{C})) \otimes_{\mathbb{C}} H^{q'}(Y_{0,F}(N_0),\mathcal{L}((n,v),\psi;\mathbb{C})) \\ &\subset H^Q(Y_{0,K}(N),\mathcal{L}((\widehat{n},\widehat{v}),\chi;\mathbb{C})) \end{aligned}$$

for $Q = q + q' = \dim \mathfrak{H}$ as real manifolds, $q' = r_1(F) + 2r_2(F)$ and $q = r_1(F) + r_2(F)$. In this case, we have a standard choice of φ: $\varphi = \mathrm{id} \times \psi^{-1}$.

We compute the integral of $\pi_\varepsilon(\nabla^n \delta(g))$ on $Y_{0,F}(N')$ for $g = f|R(\varphi)$. At each real place ρ of K, we follow the computation done in the real quadratic case, at each complex place ρ over a real place of F, we follow the computation in the imaginary quadratic case, and for each ρ over complex place of F, we follow the computation done in the case of quadratic extension of an imaginary quadratic field.

We have

$$f^{(a)}\left(y^{-1/2}\begin{pmatrix} y & x \\ 0 & 1\end{pmatrix}\right) = y^{\widehat{n}/2}|ay|_{K_{\mathbb{A}}}\left\{\sum_{\xi \in K^\times} \widehat{\lambda}(T(\xi a \mathfrak{d}))\xi^{-\widehat{v}} W(\xi y)\mathbf{e}(\xi x)\right\},$$

where, writing $n^*_\tau = 2n_\tau + 2$ for $\tau \in \Sigma_K(\mathbb{C})$,

$$\begin{aligned} W(y) &= \prod_{\tau \in \Sigma_K} W_\tau(y_\tau), \\ W_\tau(y) &= \begin{cases} \sum_{\alpha=0}^{n^*_\tau} \binom{n^*_\tau}{\alpha}\left(\frac{y}{i|y|}\right)^{n_\tau+1-\alpha} K_{n_\tau+1-\alpha}(4\pi|y|) S_\tau^{n^*_\tau-\alpha} T_\tau^\alpha & \text{if } \tau \in \Sigma_K(\mathbb{C}), \\ \exp(-2\pi|y|) & \text{if } \tau \in \Sigma_K(\mathbb{R}). \end{cases} \end{aligned}$$

We consider $\nabla_\tau = \partial^2/\partial X_{\sigma\tau}\partial Y_\tau - \partial^2/\partial X_\tau \partial Y_{\sigma\tau}$ and write $\nabla^n = \prod_{\tau \in \Psi} \nabla_\tau^{n_\tau}$.

Take an algebraic Hecke character φ of K and an everywhere unramified Hecke character ω of F with $\infty(\varphi) = -w \in 1/2\Xi_K$ such that $\psi\omega\varphi_F = \alpha$. We then compute the pull back $(n!)^{-2}(\nabla^n \delta_{J,J'}(g^{(a)})|_F) = \iota^*((n!)^{-2}\nabla^n \delta_{J,J'}(g^{(a)}))$ for $g = f|R(\varphi)$ and $\iota : \mathfrak{H} \hookrightarrow \mathfrak{Z}$ induced by the inclusion $H \subset G$. Here we identify J' with a subset of I by Res^K_F and J'' with $\{\tau \in I(\mathbb{R}) |\ \tau$ extends to a complex place of $K\}$. We put $\widehat{n}^* = \sum_{\tau \in \Sigma_K(\mathbb{C})}(2n_\tau + 2)\tau$. We write I' for $I(\mathbb{C}) \cup J''$. For each $j \in \mathbb{Z}[I']$, we also write n' (resp. j') for $n(I')$ (resp. $\widehat{n}(\Sigma_K(\mathbb{C})) + 2j(J'') + j^\#$), where $j^\# =$

$\sum_{\tau\in J'}(j_\tau - j_{\tau c})\tau + (j_{\tau c} - j_\tau)\tau c$. Then we have

$$\begin{aligned}
&(n!)^{-2}\nabla^n\delta_{J,J'}(g^{(a)})|_F \\
&\quad = (-1)^{n(J\cup J')}(2i)^{n(J)+J} \sum_{j\in\mathbb{Z}[I'],0\le j(I')\le n(I')} (-1)^{j(I(\mathbb{C}))}\binom{n'}{j} d\mu(z) \\
&\quad\quad \cdot \big\{ - g^{(a)}_{j'+2J'} + g^{(a)}_{j'+2J''+J'+\sigma J'} - g^{(a)}_{j'+2\sigma J'} - g^{(a)}_{j'+2J''+2J'} \\
&\quad\quad + g^{(a)}_{j'+2J''+J'+\sigma J'} - g^{(a)}_{j'+2J''+2\sigma J'}\big\},
\end{aligned}$$

where

$$\begin{aligned}
d\mu &= \left\{\bigwedge_{\tau\in I(\mathbb{R})} y_\tau^{-2}dy_\tau\wedge dx_\tau\right\}\wedge\left\{\bigwedge_{\tau\in\Xi} y_\tau^{-3}dy_\tau\wedge dx_\tau\wedge d\overline{x}_\tau\right\}, \\
g^{(a)} &= \sum_{0\le\alpha\le\widehat{n}^*} g^{(a)}_\alpha \binom{\widehat{n}^*}{\alpha} S^{\widehat{n}^*-\alpha}T^\alpha.
\end{aligned}$$

We choose $\Delta\in K^\times$ such that

$$(\Delta)\qquad \Delta^\sigma = -\Delta \quad\text{and}\quad \Psi(\mathbb{R}) = \{\tau\in I_K(\mathbb{R})|\Delta^\tau > 0\}.$$

If $K = F\oplus F$, as already explained, there is a standard choice of J, J'. In this case, $\Delta = (1,-1)\in K$ is an optimal choice of Δ. Anyway we see

$$\xi\Delta^{-1}\in\{x\in a^{-1}\mathfrak{d}^{-1}|x^\sigma = -x\} \iff \xi\in a^{-1}\Delta\mathfrak{d}^{-1}\bigcap F.$$

Thus we have

$$\sum_{a\in Cl_F}\omega(a)\int_{\Phi^{(a)}_\infty\backslash\mathfrak{H}}(n!)^{-2}(\nabla^n\delta_{J,J'}(g^{(a)})|_F)|a|^s_{F_\mathbb{A}}y^{s\mathbf{1}} = c_1(s)L(s,\lambda,\varphi_F\omega),$$

where

$$\begin{aligned}
c_1(s) &= \omega(\Delta^{-1}\mathfrak{d})^{-1}\Delta^{\widehat{v}+w+\widehat{n}/2}|D_F|^{1/2}G(\varphi)\left(|D_F|N_{F/\mathbb{Q}}(D_{K/F})^{1/2}\right)^{1+s} \\
&\quad\cdot 2^{n+sI(\mathbb{C})+sJ''+J}(-1)^{n(J\cup J')+J'}\sqrt{-1}^{n(J)+J+J'}(4\pi)^{-n-(s+1)\mathbf{1}} \\
&\quad\cdot 2^{-J''}\prod_{\tau\in J''}\left(1+(-1)^{n_\tau+1+2v_\tau+\mathrm{Res}(w)_\tau}\right) \\
&\quad\cdot\left(\frac{(5s+1)\Gamma(s)\Gamma(s+1)}{\Gamma(2+2s)}\right)^{r_2(F)}\left(\frac{\Gamma(\frac{s}{2})^2}{\Gamma(s)}\right)^{|J''|}\prod_{\tau\in I}\Gamma(n_\tau+1+s).
\end{aligned}$$

We apply the Rankin convolution method. For that we put

$$E(s) = \sum_{a\in Cl_F}\omega(a)|a|^s_{F_\mathbb{A}}\mathcal{E}^{(a)}(s),$$

where

$$\mathcal{E}^{(a)}(s) = \sum_{\gamma\in\Phi^{(a)}/\Phi^{(a)}_\infty}{}' \varphi^2_{N'}\psi^2_{N'}(\gamma)y^{s\mathbf{1}}\circ\gamma.$$

Write $Y = Y_{0,F}(N') = \bigsqcup_{a \in Cl_F} Y^{(a)}$, if $(\omega\psi\varphi_F)^2 = \mathrm{id}$, by (RES3) in Appendix, we get in exactly the same manner as in the previous sections,

$$\mathrm{Res}_{s=1}\zeta_{F,N'}(2s)\int_Y E(s)(n!)^{-2}(\nabla^n\delta_{J,J'}(g)|_F)$$
$$= \frac{\phi(N')2^{[F:\mathbb{Q}]-1}\pi^{[F:\mathbb{Q}]}R_\infty}{N_{F/\mathbb{Q}}(N')^2 w(F)|D_F|}\sum_{a\in Cl_F}\omega(a)\int_{Y^{(a)}}(n!)^{-2}(\nabla^n\delta_{J,J'}(g^{(a)})|_F).$$

Then by choosing a character $\varepsilon : C_\infty/C_{\infty+} \to \{\pm 1\}$ with $\varepsilon((-1,-1)_\tau) = (-1)^{n_\tau+1}$ for all $\tau \in J$ as in Section 5, we compute the residue using the Euler factorization of the zeta function of the right hand side, and we get, choosing φ so that $\omega\varphi_F\psi = \alpha$

$$\Gamma(1, \mathrm{Ad}(\lambda)\otimes\alpha)L_{C'}(1,\mathrm{Ad}(\lambda)\otimes\alpha)$$
$$= c_0 G(\varphi)^{-1}\omega(\Delta^{-1}\mathfrak{d})^{-1}\Delta^{-\widehat{v}-w-\widehat{n}/2}\sqrt{-1}^{-n(J)+J+J'}$$
$$\cdot\sum_{a\in Cl_F}\omega(a)\int_{Y^{(a)}}(n!)^{-2}(\nabla^n\pi_\varepsilon\delta_{J,J'}(g^{(a)})|_F),$$

where c_0 is an explicit non-zero constant in $\mathbb{Q}$.

We now define a modular transcendental factor of the above L-value. When F has more than one complex places, we cannot define the transcendental factor solely using the data of cohomology group for $Y_{0,K}(N)$ as in the previous sections. Instead we need to use cohomology groups for $Y_{0,K}(N)$ and $Y_{0,F}(N')$. Because of this, we can define the period only for $\widehat{\lambda}$ and not for general non-base-change lift. Let $\varepsilon : S = C_\infty/C_{\infty+} \to \{\pm 1\}$ be a character satisfying $\varepsilon((-1)_\tau, (-1)_{\sigma\tau}) = -1$ for all $\tau \in J$. We start with a system of Hecke eigenvalues λ for H with character ψ and write its base change lift to G as $\widehat{\lambda}$, whose conductor is N. Let $q = \dim(\mathfrak{H}) = 2r_1(F) + 3r_2(F)$. We write

$$H(A)[\widehat{\lambda},\varepsilon] = H^q_{\mathrm{cusp}}(Y_{0,K}(N), \mathcal{L}((\widehat{n},\widehat{v}),\chi;A))[\widehat{\lambda},\varepsilon]$$

for the subspace on which $T(\mathfrak{n})$ acts via $\widehat{\lambda}$ and S acts via ε. Note that $\{\pi_\varepsilon\delta_{J,J'}(f)\}_{J'}$, J' running through all subsets of $\Sigma_K(\mathbb{C})$ of cardinality $\#(\Sigma(\mathbb{C}))$, forms a base of $H(\mathbb{C})[\widehat{\lambda},\varepsilon]$. Let $N_\varphi = N \cap C^2$ for the conductor C of φ. We have the following sequence of maps:

$$R(\varphi) : H(A)[\lambda,\varepsilon] \longrightarrow H^q_{\mathrm{cusp}}(Y_0(N_\varphi), \mathcal{L}((\widehat{n},\widehat{v}+w),\chi\varphi^2;A))[\lambda,\varepsilon\varphi_\infty];$$

$$i^* : H^q_{\mathrm{cusp}}(Y_0(N_\varphi),\mathcal{L}((\widehat{n},\widehat{v}+w),\chi\varphi^2;A)) \longrightarrow H^q_{\mathrm{cusp}}(Y,\mathcal{L}((\widehat{n},\widehat{v}+w),(\chi\varphi^2);A)|_Y);$$

$$\pi_* : H^q_{\mathrm{cusp}}(Y,\mathcal{L}((\widehat{n},\widehat{v}+w),(\chi\varphi^2)_F;A)) \longrightarrow H^q_c(Y,\mathcal{L}(0,\omega^{-2};A));$$

$$\omega_* : H^q_c(Y,\mathcal{L}(0,\omega^{-2};A)) \underset{\cup[\det(\omega)]}{\longrightarrow} H^q_c(Y,A) \cong A.$$

The map i^* is induced by the inclusion $i : Y \hookrightarrow Y_{0,K}(N_\varphi)$, and π_* is induced by the morphism of sheaves $\pi : \mathcal{L}((\widehat{n},\widehat{v}+w),\chi\varphi^2;A)|_Y \to \mathcal{L}(0,\omega^{-2};A)$ (here $\chi\varphi^2$ is well chosen under the condition that $\omega\varphi_F\psi = \alpha$ so that the map π exists). The last map is induced by the cup product with the global section $\det(\omega)$ of $\mathcal{L}(0,\omega^2;A)$. We consider the composition $Ev = Ev_\varphi = \omega_* \circ \pi_* \circ i^* \circ R(\varphi)$. Then by the above computation, $Ev([\pi_\varepsilon\delta_{J,J'}(f)]) \neq 0$. We write M for the image under Ev of $H(A)[\lambda,\varepsilon]$. We suppose that $M \subset A$ is free of rank 1 over A and write its generator as $\xi_\varepsilon(f)$. Since the value $Ev([\pi_\varepsilon\delta_{J,J'}(f)])$ is independent of J', we can define $\Omega'_{1,2}(\varepsilon,\widehat{\lambda};A)\in\mathbb{C}^\times$ by the following formula:

$$Ev([\pi_\varepsilon\delta_{J,J'}(f)]) = \Omega'_{1,2}(\varepsilon,\widehat{\lambda};A)\xi_\varepsilon(f),$$

where f is the cohomological modular form whose Fourier coefficient at ideal $\mathfrak{n}$ is given by $\widehat{\lambda}(T(\mathfrak{n}))$. For the moment, we assume that φ and ω are arithmetic, and A is a Dedekind domain inside a finite extension of $\mathbb{Q}(\widehat{\lambda},\varphi,\omega)$. Here we note that

$$Ev([\pi_\varepsilon\delta_{J,J'}(f)]) = \sum_{a\in Cl_F}\omega(a)\int_{Y(a)}(n!)^{-2}(\nabla^n\pi_\varepsilon\delta_{J,J'}(f|R(\varphi))|_F).$$

We also define for $\rho\in\mathrm{Aut}(\mathbb{C})$ a constant $\Omega'_{1,2}(\varepsilon,\lambda^\rho;A)$ by

$$Ev([\pi_\varepsilon\delta_{J,J'}(f^\rho)]) = \Omega'_{1,2}(\varepsilon,\lambda^\rho;A)\xi_\varepsilon(f^\rho).$$

If we have two choices of (φ,ω), say (φ,ω) and (φ',ω'), the ratio:

$$Ev_\varphi([\pi_\varepsilon\delta_{J,J'}(f)])/Ev_{\varphi'}([\pi_\varepsilon\delta_{J,J'}(f)])$$

is just equal to the identity component of

$$(\omega(\Delta^{-1}\mathfrak{d})^{-1}\otimes 1)\mathbf{G}(\varphi)/(\omega'(\Delta^{-1}\mathfrak{d})^{-1}\otimes 1)\mathbf{G}(\varphi')$$

in $\mathbb{Q}(\lambda,\varphi,\varphi')\otimes_{\mathbb{Q}}\mathbb{C}$, and therefore, by an argument on Gauss sums close to the one given just above Theorem 7.1, $(G(\psi^\rho\alpha)^{-1}\Omega'_{1,2}(\varepsilon,\widehat{\lambda}^\rho;A))_\rho$ is a well defined element in

$$(\mathbb{Q}(\lambda)\otimes_{\mathbb{Q}}\mathbb{C})^\times/(\mathbb{Q}(\lambda)^\times\otimes 1),$$

which we write $\underline{\Omega}_{(1,2)}(\varepsilon,\widehat{\lambda};\mathbb{Q}(\lambda)) = (G(\psi^\rho\alpha)^{-1}\Omega'_{1,2}(\varepsilon,\widehat{\lambda};A))_\rho$. When φ or ω is algebraic (but not arithmetic), we write simply $\Omega_{(1,2)}(\varepsilon,\widehat{\lambda};\overline{\mathbb{Q}})$ for $\Omega'_{(1,2)}(\varepsilon,\widehat{\lambda};\overline{\mathbb{Q}})$. Then we have

THEOREM 8.1. *Let α be a Hecke character of $F_{\mathbb{A}}^\times/F^\times$ with $\alpha^2=1$. We allow $\alpha=\mathrm{id}$. Suppose that we have an arithmetic Hecke character φ of K and ω of F such that (i) the conductor of ω is 1 and (ii) $\psi\varphi_F\omega=\alpha$. Then we have*

$$\frac{(1\otimes\Delta^{\widehat{v}+w+\widehat{n}/2}\Gamma(1,\mathrm{Ad}(\lambda)\otimes\alpha))\mathbb{L}_{C'}(1,\mathrm{Ad}(\lambda)\otimes\alpha)}{(1\otimes\sqrt{-1}^{n(J)+J+J'})\mathbf{G}(\psi\alpha)\underline{\Omega}_{(1,2)}(\varepsilon,\widehat{\lambda};\mathbb{Q}(\lambda))}\in\mathbb{Q}(\lambda)\subset\mathbb{Q}(\lambda)\otimes_{\mathbb{Q}}\mathbb{C}$$

for each character $\varepsilon:S\to\{\pm1\}$ with $\varepsilon((-1,-1)_\tau)=-1$ for all $\tau\in J$. If φ is an algebraic Hecke character satisfying (i) and (ii), then we have

$$\frac{\Gamma(1,\mathrm{Ad}(\lambda)\otimes\alpha)L_{C'}(1,\mathrm{Ad}(\lambda)\otimes\alpha)}{\Omega_{(1,2)}(\varepsilon,\widehat{\lambda};\overline{\mathbb{Q}})}\in\overline{\mathbb{Q}}.$$

9. Period relation

We shall list here several period relations which follow easily from the main theorems. Some of them is a partial generalization of such relations for totally real extensions studied by Shimura, Harris and Yoshida (cf. [**Y95**], [**Y94**]). For simplicity, we suppose that F is totally imaginary. Let ϕ^ρ be a unique cohomological form on H whose Mellin transform is the standard L-function of λ^ρ. We write C_0 for the conductor of ϕ. Then we choose a generator $\xi_m(\phi^\rho)$ $(m=1,2)$ of

$$H^{mq}_{\mathrm{cusp}}(Y_{0,F}(C_0),\mathcal{L}((n,v),\psi;\mathbb{Q}(\lambda^\rho))[\lambda^\rho]$$

for $q = [F : \mathbb{Q}]/2$ so that $\xi_m(\phi^\rho) = \xi_m(\phi)^\rho$, where $\rho \in \mathrm{Aut}(\mathbb{C})$ acts on the cohomology group through covariant functoriality of its action on $L((n,v),\psi;\mathbb{Q}(\lambda))$. We then define a constant $\Omega_m(\lambda^\rho;\mathbb{Q}(\lambda))$ $(m = 1,2)$ by

$$[\delta(\phi^\rho)] = \Omega_m(\lambda^\rho;\mathbb{Q}(\lambda^\rho))\xi_m(\phi^\rho).$$

Of course, we can give more integral definition of $\Omega_m(\lambda^\rho;A)$ for, say, a discrete valuation ring A of $\mathbb{Q}(\lambda^\rho)$. However, here we just want to discuss rational and algebraic relations among modular periods Ω_j and $\Omega_{1,2}$. In other words, we like to study relations among periods of harmonic forms of different degree but belonging to the same system of Hecke eigenvalues.

First we apply Theorem 8.1 to $F \oplus F$ with the standard choice of $\varphi = \mathrm{id} \times \psi^{-1}$. Then it is easy to conclude from an argument given in Urban [**U**] for imaginary quadratic fields F that

$$\frac{\Gamma(1,\mathrm{Ad}(\lambda))L(1,\mathrm{Ad}(\lambda))}{\Omega_1(\lambda;\mathbb{Q}(\lambda))\Omega_2(\lambda;\mathbb{Q}(\lambda))} \in \overline{\mathbb{Q}}^\times.$$

Thus writing $a \sim b$ if $a/b \in \overline{\mathbb{Q}}^\times$, we see that

$$\Omega_{1,2}(\widehat{\lambda};\mathbb{Q}(\lambda)) \sim \Omega_1(\lambda;\mathbb{Q}(\lambda))\Omega_2(\lambda;\mathbb{Q}(\lambda)).$$

This is one of the reasons why we have written the period as $\Omega_{1,2}$. This type of relation can be shown in a little more general case. These quantities Ω_j are defined depending only on F (and λ), but $\Omega_{1,2}$ is defined relative to a general quadratic extension K/F (and λ). Thus we fix λ and write $\Omega_j^F(\lambda)$ (resp. $\Omega_{1,2}^{K/F}(\lambda)$) for $\Omega_j(\lambda;\mathbb{Q}(\lambda))$ (resp. $\Omega_{1,2}(\widehat{\lambda};\mathbb{Q}(\lambda))$ relative to K/F). When $\alpha = \mathrm{id}$, we get from the above identity

(P1) $$\Omega_{1,2}^{F\oplus F/F}(\lambda) \sim \Omega_1^F(\lambda)\Omega_2^F(\lambda).$$

Applying $(P1)$ to $\widehat{\lambda}$ in place of λ, we get

$$\begin{aligned}\Omega_1^K(\widehat{\lambda})\Omega_2^K(\widehat{\lambda}) &\sim \Gamma(1,\mathrm{Ad}(\widehat{\lambda}))L(1,\mathrm{Ad}(\widehat{\lambda})) \\ &= \Gamma(1,\mathrm{Ad}(\lambda))L(1,\mathrm{Ad}(\lambda))\Gamma(1,\mathrm{Ad}(\lambda)\otimes\alpha)L(1,\mathrm{Ad}(\lambda)\otimes\alpha) \\ &\sim \Omega_1^F(\lambda)\Omega_2^F(\lambda)\Omega_{1,2}^{K/F}(\lambda),\end{aligned}$$

On the other hand, it is proven in [**Hi94**, Theorem 8.1] that

$$\begin{aligned}(2\pi)^{2j+2\mathbf{1}}\Omega_1^K(\lambda) &\sim L(0,\lambda\otimes\eta\circ N_{K/F}) \\ &= L(0,\lambda\otimes\eta)L(0,\lambda\otimes\eta\alpha) \sim (2\pi)^{2j+2\mathbf{1}}\Omega_1^F(\lambda)^2\end{aligned}$$

if $0 \leq j \leq n$ and $\infty(\eta) = j + v + \mathbf{1}$, as long as the modular standard L-values are non-zero for some j, for example, if $n_\tau > 2$ for all τ. Thus we conclude (see (P) of Conjecture 5.1), if $n_\tau > 2$ for all τ,

(P2) $$\Omega_1^K(\widehat{\lambda}) \sim \Omega_1^F(\lambda)^2.$$

This shows

$$\Omega_1^F(\lambda)^2\Omega_2^K(\widehat{\lambda}) \sim \Omega_1^K(\widehat{\lambda})\Omega_2^K(\widehat{\lambda}) \sim \Omega_1^F(\lambda)\Omega_2^F(\lambda)\Omega_{1,2}^{K/F}(\lambda),$$

and hence

(P3) $$\Omega_1^F(\lambda)\Omega_2^K(\widehat{\lambda}) \sim \Omega_2^F(\lambda)\Omega_{1,2}^{K/F}(\lambda).$$

We put

$$\Omega'^F_2(\lambda) = \Omega^F_1(\lambda)^{-1}\Omega^{K/F}_{1,2}(\lambda).$$

Then we have

$$(P4) \qquad \Omega^K_2(\widehat{\lambda}) \sim \Omega^F_2(\lambda)\Omega'^F_2(\lambda).$$

This formula seems to be a generalization of (P2) for Ω_2. It is an interesting problem to find out its motivic meaning. Suppose that $\lambda = \widehat{\lambda}_0$ for a system of Hecke eigenvalues λ_0 for $\mathrm{GL}(2)_{/L}$ for a totally real subfield L with $[F:L]=2$. We further assume that $K = FF'$ for two totally imaginary quadratic extensions F/L and F'/L. Then one can prove easily from the above relations that, if $n_\tau > 2$ for all τ

$$(\mathrm{P5}) \qquad \Omega'^F_2(\lambda) \sim \Omega^{F'}_2(\mu), \quad \Omega^{K/F}_{1,2}(\lambda) \sim \Omega^F_1(\lambda)\Omega^{F'}_2(\mu) \\ \text{and} \quad \Omega^K_2(\widehat{\lambda}) \sim \Omega^F_2(\lambda)\Omega^{F'}_2(\mu),$$

where μ is the base change lift of λ_0 to $\mathrm{GL}(2)_{/F'}$.

Appendix A. Eisenstein series of weight 0

Here, for the reader's convenience, we shall prove the residue formula of the Eisenstein series we used in the principal text, which seems not to be found in the literature in its exact form.

Let F be a number field. We consider the algebraic group $H = \mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}(2)_{/F}$. We use the same symbol introduced in [**Hi91**, Section 4], where F is assumed to be totally real, but symbols themselves have meaning. For a finite order character χ and θ of the idele class group $F^\times_{\mathbb{A}}/F^\times$ modulo an ideal C of $\mathfrak{r}$, we consider the following Eisenstein series $\mathcal{E}(x,s) = \mathcal{E}(x,\chi,\theta;s) : H(\mathbb{A}) \to \mathbb{C}$:

$$\mathcal{E}(x,s) = \sum_{\gamma \in \mathfrak{r}^\times B(\mathbb{Q})\backslash H(\mathbb{Q})} \chi^*(\gamma x)\theta(\gamma x)\eta(\gamma x)^s,$$

where

$$B(A) = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \middle| a \in (F \otimes_{\mathbb{Q}} A)^\times \ \text{ and } \ b \in F \otimes_{\mathbb{Q}} A \right\} \subset H(A)$$

for $\mathbb{Q}$-algebras A, and for the identity component $H(\mathbb{R})_+$ of $H(\mathbb{R})$,

$$\chi^*(x) = \begin{cases} \chi_C(d_C) \ \text{ for } x = \begin{pmatrix} a & b \\ c & d \end{pmatrix} & \text{if } x \in B(\mathbb{A})U_0(C)H(\mathbb{R})_+, \\ 0 & \text{otherwise,} \end{cases}$$

$$\theta(x) = \begin{cases} \theta(a) & \text{if } x \in \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} Z_H(\mathbb{A})H(\widehat{\mathbb{Z}})C_{\infty+} \\ 0 & \text{otherwise,} \end{cases}$$

and $\eta = \theta$ for $\theta = |\ |_{F_{\mathbb{A}}}$. We normalize the Eisenstein series in the following way:

$$E^*(x,\chi,\theta;s) = N_{F/\mathbb{Q}}(C)2^{-r_2}|D_F|^{1/2}\sum_a \chi^{-1}(a)\mathcal{E}(au,\chi,\theta;s),$$

where a runs over a complete set of representatives for $F^\times_{\mathbb{A}}/F^\times\widehat{\mathfrak{r}}^\times F^\times_\infty$ and r_2 is the number of complex places of F. We further put

$$E(x,\chi,\theta,s) = L_C(2s,\chi^{-1}\theta^2)E^*(x,\chi,\theta;s).$$

We now compute the Fourier expansion of the above Eisenstein series. We put following [**Hi94**, Appendix]

$$\zeta(y,t;s)=\int_{\mathbb{C}}\exp(-2\pi i\mathrm{Tr}(t\tau))(|\tau|^2+y^2)^{-s}d\tau.$$

Then for the modified Bessel function $K_s(t)$ as in [**Hi94**, Section 6],

$$\zeta(y,t;s)=\begin{cases}(2\pi)^s\Gamma(s)^{-1}y^{1-s}|t|^{s-1}K_{s-1}(4\pi|t|y) & \text{if } t\neq 0,\\ \pi(s-1)^{-1}y^{2-2s} & \text{if } t=0.\end{cases}$$

We take $\varepsilon=\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)$ and compute the residue at $s=1$ of $E(x,s)=\mathcal{E}(x\varepsilon^{(\infty)},s)$ via its Fourier expansion. Let w be a variable of $H(\mathbb{R})_+$. Then we write $z=w(z_0)$, where $z_{0,\sigma}=\sqrt{-1}$ if σ is real and ε if σ is complex. We write $y(z)=(y_\sigma)\in F_\infty$ when $z_\sigma=x_\sigma+\sqrt{-1}y_\sigma$ for σ real and $z_\sigma=\left(\begin{smallmatrix}x_\sigma&-y_\sigma\\y_\sigma&\overline{x}_\sigma\end{smallmatrix}\right)$ for σ complex. For $\gamma_\sigma=\left(\begin{smallmatrix}a_\sigma&b_\sigma\\c_\sigma&d_\sigma\end{smallmatrix}\right)$, we define an automorphic factor by

$$j(\gamma_\sigma,z_\sigma)=\begin{cases}c_\sigma z_\sigma+d_\sigma & \text{if } \sigma \text{ is real},\\ \det(\rho(c_\sigma)z_\sigma+\rho(d_\sigma)) & \text{if } \sigma \text{ is complex},\end{cases}$$

where $\rho(x)=\left(\begin{smallmatrix}x&0\\0&\overline{x}\end{smallmatrix}\right)$. Then by [**Hi91**, Lemma 6.4], we have

$$\chi^*((\varepsilon a\alpha(x)\begin{pmatrix}a'&0\\0&1\end{pmatrix}\varepsilon^{-1})^{(\infty)})=\chi(a_Ca'_C),$$

$$\eta((\varepsilon a\alpha(x)\begin{pmatrix}a'&0\\0&1\end{pmatrix}\varepsilon^{-1})^{(\infty)})=|(a^2a')^{(\infty)}|_{F_{\mathbb{A}}},$$

$$\eta((\varepsilon a\alpha(x)w)_\infty)=|a_\infty^2y(z)|_{F_{\mathbb{A}}}\prod_{\sigma\in I/\langle c\rangle}|j(\varepsilon a\alpha(x)_\sigma,z_\sigma)|^{-2},$$

$$\theta((\varepsilon a\alpha(x)\begin{pmatrix}a'&0\\0&1\end{pmatrix}\varepsilon^{-1})^{(\infty)})=\theta(a^2a'\mathfrak{r})\quad\text{if } a'_C=1,$$

$$\theta((\varepsilon a\alpha(x)w)_\infty)=1,$$

where $\alpha(x)=\left(\begin{smallmatrix}1&x\\0&1\end{smallmatrix}\right)$. Then the Fourier coefficient $b(\xi,u,s)$ at $\xi\in F$ of $E(x,s)$ given by

$$\int_{F_{\mathbb{A}}/F}E(\alpha(x)u,s)\mathbf{e}_F(-\xi x)dx,$$

we get from [**Hi91**, Lemma 6.6] that for $u=x\left(\begin{smallmatrix}a&0\\0&1\end{smallmatrix}\right)w$ with $a_\infty=a_C=1$ and $x\in Z_H(\mathbb{A}^{(\infty)})$

$$\xi\in\mathfrak{a}^{-1}C^{-1}\mathfrak{d}^{-1}\quad\text{if } b(\xi,u,s)\neq 0,$$

where $\mathfrak{d}$ is the absolute different of $F/\mathbb{Q}$ and $\mathfrak{a}=a\mathfrak{r}$. Then we see from [**Hi91**, (6.11)]

$$\begin{aligned}b(\xi,u,s)&=N_{F/\mathbb{Q}}(C)^{-1}|D_F|^{-1/2}\chi\theta^{-1}(a^{(\infty)})\chi(x)N_{F/\mathbb{Q}}(\mathfrak{a})^{s-1}\\&\cdot\int_{F_\infty}\eta(\varepsilon x\alpha(v_\infty)w)^s\mathbf{e}_F(-\xi v_\infty)dv_\infty\\&\cdot\sum_{\mathfrak{n}\sim x\mathfrak{a}}\theta^2\chi^{-1}(\mathfrak{n})N_{F/\mathbb{Q}}(\mathfrak{n})^{-2s}\sum_{\mathfrak{b}\supset\mathfrak{n}+\xi\mathfrak{a}C\mathfrak{d}}\mu(\mathfrak{n}/\mathfrak{b})N_{F/\mathbb{Q}}(\mathfrak{b}),\end{aligned}$$

where $\mathfrak{a}\sim\mathfrak{b}$ means that the two ideals belong to the same ideal class. As seen in [**Hi91**, 6.12b], from its definition combined with the above formula, the Fourier coefficient at ξ for $E^*(u,\chi,\theta;s)$ is given by

$$\chi\theta^{-1}(a^{(\infty)})N_{F/\mathbb{Q}}(\mathfrak{a})^{s-1}L_C(2s,\chi^{-1}\theta^2)^{-1}\int_{F_\infty}\eta(\varepsilon\alpha(v_\infty)w)^s\mathbf{e}_F(-\xi v_\infty)dv_\infty$$
$$\cdot\begin{cases}\sum_{\mathfrak{b}\supset\xi\mathfrak{a}C\mathfrak{b}}\chi^{-1}\theta^2(\mathfrak{b})N_{F/\mathbb{Q}}(\mathfrak{b})^{1-2s} & \text{if } \xi\neq 0,\\ L_C(2s-1,\chi^{-1}\theta^2) & \text{if } \xi=0.\end{cases}$$

Following [**Hi94**, Appendix], we get for a complex place σ

$$2^{-1}\int_{\mathbb{C}}\eta(\varepsilon\alpha(v_\sigma)_\sigma w_\sigma)^s\mathbf{e}_F(-\xi v_\sigma)dv_\sigma = |y_\sigma|^{2s}\zeta(y,\xi;2s,0)$$
$$=\begin{cases}(2\pi)^{2s}\Gamma(2s)^{-1}y_\sigma|\xi|^{2s-1}K_{2s-1}(4\pi|\xi y|) & \text{if } \xi\neq 0,\\ \pi(2s-1)^{-1}y_\sigma^{2-2s} & \text{if } \xi=0.\end{cases}$$

For real σ, we get from [**Hi91**, 6.9b]

$$\int_{\mathbb{R}}\eta(\varepsilon\alpha(v_\sigma)_\sigma w_\sigma)^s\mathbf{e}_F(-\xi v_\sigma)dv_\sigma$$
$$=\begin{cases}\pi^s\Gamma(s)^{-1}|\xi|^{s-1}\exp(-2\pi|\xi y|)\omega(4\pi|\xi y|;s,s) & \text{if } \xi\neq 0,\\ \pi\Gamma(s)^{-2}\Gamma(2s-1)(4y_\sigma)^{1-s} & \text{if } \xi=0,\end{cases}$$

where $\omega(t;s,s)$ is the hyper-geometric function defined in [**Hi91**, Section 6]. By this computation, we know that the Eisenstein series has meromorphic continuation to the whole complex s-plane. Moreover the non-constant term of $E(x,\chi,\mathrm{id},s)$ is an entire function of s, and hence the residue is constant. The constant term is given by

$$\chi(a^{(\infty)})N_{F/\mathbb{Q}}(\mathfrak{a})^{s-1}(\pi 4^{1-s}\Gamma(s)^{-2}\Gamma(2s-1))^{r_1}\cdot(\pi(2s-1)^{-1})^{r_2}|y|_{F_\mathbb{A}}^{1-s}L_C(2s-1,\chi).$$

This shows

(RES1) $$\mathrm{Res}_{s=1}E(x,\chi,\mathrm{id},s)$$
$$=\begin{cases}0 & \text{if } \chi\neq\mathrm{id},\\ N_{F/\mathbb{Q}}(C)^{-1}\phi(C)\frac{2^{r_1+r_2-1}\pi^{[F:\mathbb{Q}]}R_\infty h(F)}{w(F)|D_F|^{1/2})} & \text{if } \chi=\mathrm{id},\end{cases}$$

where ϕ is the Euler function: $\phi(C)=\#(\mathfrak{r}/C)^\times$, R_∞ is the regulator of F and $w(F)$ is the number of roots of unity in F.

We now show that the Eisenstein series $\mathcal{E}(x,\chi;s;C)=\mathcal{E}(x,\chi,\mathrm{id};s)$ gives a section of the sheaf $L((0,0),\chi^{-1};\mathbb{C})$ over $Y_{0,F}(C)$. The identity

$$\begin{pmatrix}a&b\\c&d\end{pmatrix}=\gamma t\begin{pmatrix}y&x\\0&1\end{pmatrix}=\gamma\begin{pmatrix}y&x\\0&1\end{pmatrix}t\in B(\mathbb{A})U_0(C)H(\mathbb{R})_+$$

for $t=\left(\begin{smallmatrix}a&0\\0&1\end{smallmatrix}\right)$ implies that

$$\gamma\begin{pmatrix}y&x\\0&1\end{pmatrix}\in B(\mathbb{A})U_0(C)t^{-1}H(\mathbb{R})_+=B(\mathbb{A})tU_0(C)t^{-1}H(\mathbb{R})_+.$$

Thus (c,d) is the second row of a matrix in $\Phi^{(a)}=H(\mathbb{Q})\cap tU_0(C)t^{-1}H(\mathbb{R})_+$. This shows that we can choose the γ modulo $\mathfrak{r}^\times B(\mathbb{Q})_+$ inside $\Phi^{(a)}$. Note that $\Phi^{(a)}\cap\mathfrak{r}^\times B(\mathbb{Q})_+=\Phi^{(a)}_\infty$. Then for $w=\left(\begin{smallmatrix}y&x\\0&1\end{smallmatrix}\right)$

$$\gamma wt=\gamma_\infty wtt^{-1}\gamma^{(\infty)}t\in\gamma_\infty wtU_0(C).$$

This shows that, for $z = w(z_0)$ which is the image of $\left(\begin{smallmatrix} y & x \\ 0 & 1 \end{smallmatrix}\right)$ in $\mathfrak{H}$,

$$\eta(\gamma w t) = |a|_{F_{\mathbb{A}}} y^{\mathbf{1}} \circ \gamma \quad \left(\text{for } \mathbf{1} = \sum_{\sigma \in I} \sigma\right) \text{ and } \chi^*(\gamma \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} t) = \chi_C(d).$$

Thus we get

$$\mathcal{E}\left(t \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}, \chi; s, C\right) = |a|^s_{F_{\mathbb{A}}} \sum_{\gamma \in \Phi^{(a)}/\Phi^{(a)}_\infty} \chi_C(\gamma) y^{s\mathbf{1}} \circ \gamma = |a|^s_{F_{\mathbb{A}}} \mathcal{E}^{(a)}(z).$$

This shows the claim.

Out of (RES1), we compute the residue of $\mathcal{E}$. We shall make use of the following identity:

$$\begin{aligned}\sum_{\varphi} E^*(x; \chi\varphi) &= N_{F/\mathbb{Q}}(C) 2^{-r_2} |D_F|^{1/2} \sum_{\varphi} \sum_{a \in Cl_F} (\chi\varphi)^{-1}(a) \mathcal{E}(ax, \chi) \\ &= N_{F/\mathbb{Q}}(C) 2^{-r_2} |D_F|^{1/2} \sum_{a \in Cl_F} \chi(a)^{-1} \sum_{\varphi} \varphi(a^{-1}) \mathcal{E}(ax, \chi\varphi) \\ &= h(F) N_{F/\mathbb{Q}}(C) 2^{-r_2} |D_F|^{1/2} \mathcal{E}(x, \chi),\end{aligned}$$

where φ runs over all characters of the class group Cl_F of F. We know from the above residue formula that

(RES2)
$$\begin{aligned}&\mathrm{Res}_{s=1} L_C(2s, \chi^{-1}) E^*(x; \chi) \\ &\qquad = \delta_{\chi,\mathrm{id}} N_{F/\mathbb{Q}}(C)^{-1} \phi(C) \frac{2^{r_1+r_2-1} \pi^{[F:\mathbb{Q}]} R_\infty h(F)}{w(F) |D_F|^{1/2}},\end{aligned}$$

where $\delta_{\chi,\mathrm{id}} = \begin{cases} 1 & \text{if } \chi = \mathrm{id}, \\ 0 & \text{otherwise.} \end{cases}$

In the principal text, we have used the residue formula when $\chi = \omega^{-2}$ for ω chosen so that $\alpha = \psi\varphi_F\omega$ under the notation of Section 2.4. In this case, $C = N'$. Since the Euler product for $L_C(2s, \omega^2)$ converges at $s = 1$, we note that $L_C(2, \omega^2) \neq 0$. This shows that if $\chi = \omega^{-2} = (\psi\varphi_F)^2$,

(RES3)
$$\begin{aligned}&\mathrm{Res}_{s=1} |a|^s_{F_{\mathbb{A}}} \mathcal{E}^{(a)}(z) \\ &\qquad = \zeta_{F,N'}(2)^{-1} N_{F/\mathbb{Q}}(N')^{-2} \phi(N') \frac{2^{[F:\mathbb{Q}]-1} \pi^{[F:\mathbb{Q}]} R_\infty}{w(F)|D_F|}.\end{aligned}$$

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