Answer Keys to 2nd Midterm Examination: Math 110B, Version 1

1. Compute the following numbers, explain how you get the answer and write your answer in the following places as indicated:

a. 6	b. 3	c. 3	d. 6	e. 1

a. The number of transpositions in S_4.

Number of 2–subsets in $\{1,2,3,4\}$: $\{1,2\}$, $\{1,3\}$, $\{1,4\}$, $\{2,3\}$, $\{2,4\}$, $\{3,4\}$,

b. The number of subgroups of D_3 which are not normal.

Order 3 subgroup is normal. All order 2 subgroups are not normal. So there are three, each generated by a transposition.

c. The number of normal subgroup of order 4 in D_4.

Such subgroups N are of index 2. Thus for any $x \notin N$, $G = N \sqcup Nx = N \sqcup xN$, which shows $xN = Nx$ and N is normal. D_4/N is of order 2; so, cyclic (Theorem 7.28). Thus N contains the commutator subgroup $Z = \{r_0, r_2\}$ of D_4. There are only 3 subgroups of order 2 in $D_4/Z \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Then by Theorem 7.44, there are three subgroup of order 4 in D_4: $\{r_0, r_1, r_2, r_3\}$, $\{r_0, r_2, h, v\}$ $\{r_0, r_2, d, t\}$.

d. The order of $(1,2,3,4)(3,4,6)$ in S_6.
$(1,2,3,4)(3,4,6) = (1,2,3)(4,6)$; so, it has order 6.

e. The number of elements of the set $\{x \in G | x^{10} = e\}$ for a cyclic group G of order 101, where e is the identity element of G.

For any element x in the group, $x^{101} = e$ and $|x|\big|101$. If $x^{10} = e$, $|x|\big|10$. Thus $|x|$ is a common divisor of 101 and 10, which has to be 1. Thus $x = x^1 = e$, and there is only one such element, which is e.

3. Let M and N be two normal subgroups of a group G. If $M \cap N = \{e\}$ for the identity element e of G, show that $mn = nm$ for all $m \in M$ and all $n \in N$.

Consider $mnm^{-1}n^{-1}$ for $m \in M$ and $n \in N$. Since $G \triangleleft N$, $mnm^{-1} \in N$ and hence $mnm^{-1}n^{-1} \in N$. Since $G \triangleleft M$, $nm^{-1}n^{-1} \in M$, and hence $mnm^{-1}n^{-1} \in M$. This shows $mnm^{-1}n^{-1} \in M \cap N = \{e\}$. Thus $mnm^{-1}n^{-1} = e$. Multiplying by n from the right and by m from the right, we get $mn = nm$.

Answer Keys to Second Midterm Examination: Math 110B

2. Label the following statements as being true or false. In the following statements, G and H are finite groups, N is a normal subgroup of G and K is a subgroup of G. For a finite set X, $|X|$ is the number of elements in X.

Statements	Label
$\lvert\{\text{subgroups of } G/N\}\rvert = \lvert\{\text{subgroups } K \text{ of } G \mid K \supset N\}\rvert$.	T
The kernel of a group homomorphism is normal.	T
$\lvert\{Ka \mid a \in G\}\rvert = \lvert\{aK \mid a \in G\}\rvert$.	T
If G and H are both cyclic, $G \oplus H$ is cyclic.	F
S_5 has a normal subgroup N with $1 < \lvert N\rvert < \lvert S_5\rvert$	T
S_n $(n > 2)$ has the center Z with $1 < \lvert Z\rvert < \lvert S_5\rvert$	F
The centralizer of $x \in G$ is always abelian.	F
If $\lvert G\rvert$ is a prime, G is abelian.	T
For a homomorphism $f : H \to G$, $f^{-1}(N) = \{x \in H \mid f(x) \in N\}$ is normal.	T
If N is cyclic, then $N \bigcap K$ is cyclic.	T
There is an element of order 15 in A_8.	T
If G has an element of order 3, there are at least 2 elements of order 3.	T
$NK = \{nk \mid n \in N,\ k \in K\}$ is a subgroup of G.	T
G is cyclic if all subgroups of G not equal to G are cyclic.	F
$\mathbb{Z}_2 \oplus \mathbb{Z}_{30} \cong \mathbb{Z}_6 \oplus \mathbb{Z}_{10}$.	T
$N \cap K$ is a normal subgroup of G.	F
For two permutations $\alpha, \beta \in S_n$, $\alpha\beta\alpha^{-1}\beta^{-1} \in A_n$.	T
If $p \,\big\vert\, \lvert G\rvert$ for a prime p, an abelian G has an element of order p.	T
Any two groups of order 11 are isomorphic.	T
A cyclic group of prime order $p > 2$ has a unique element of order p.	F