Version 1

1. Compute the following numbers, explain how you get the answer and write your answer in the following places as indicated:

a. 6	b. 35	c. 4	d. 10	e. 2

a. The index in S_4 of the subgroup generated by $\left(\begin{smallmatrix}1&2&3&4\\2&3&4&1\end{smallmatrix}\right)$.

$\sigma = \left(\begin{smallmatrix}1&2&3&4\\2&3&4&1\end{smallmatrix}\right)$ is a cyclic permutation $(1,2,3,4)$, which generates a subgroup C of order 4, that is, $|C| = 4$. Then the index is $|S_4|/|C| = \frac{4!}{4} = 6$.

b. The order $|xy|$ for elements x, y of an abelian group G if $|x| = 7$ and $|y| = 5$.

Let m be the order of xy. Since $(xy)^{35} = (x^7)^5(y^5)^7 = e$, we find $m|35$ (Theorem 7.8 (3)). Thus possibilities are therefore $m = 1, 5, 7, 35$. We have $(xy)^m = x^m y^m = e$, because G is abelian. Then $x^m = y^{-m}$; so, m cannot be 1, because if it is, $7 = |x| = |y^{-1}| = |y| = 5$. m cannot be 5, because $e \neq x^5 = y^{-5} = (y^5)^{-1} = e$. Similarly m cannot be 7; so, $m = 35$.

c. The number of subgroups in the additive group $\mathbb{Z}_6$.

Every subgroup of a cyclic subgroup is cyclic; so, the subgroup $H \subset \mathbb{Z}_6$ is determined by its index i, since H is generated by $i = i \cdot 1$. The index is a factor of 6. Since $6 = 2 \times 3$, there is only 4 factors: 1,2,3,6. Thus there are 4 subgroups.

d. The order of the group U_{11} made up of all units of the ring $\mathbb{Z}_{11}$.

Since 11 is a prime, $\mathbb{Z}_{11}$ is a field; so, $U_{11} = \mathbb{Z}_{11} - \{0\}$. Thus $|U_{11}| = 10$.

e. The order $|h \circ r_1|$ for the $90°$ rotation r_1 and the horizontal reflection h in the dihedral group D_4.

All rotations form a cyclic subgroup C of D_4; so, all reflections form a coset hC, and rotation followed by a reflection is a reflection. The order is 2.

4. Here are the three axioms of a group G with operation $(a, b) \mapsto ab$:

(a) $(ab)c = a(bc)$ for any $a, b, c \in G$ (associative law);
(b) There exists $e \in G$ such that $ea = ae = a$ for all $a \in G$;
(c) For each $a \in G$, there is an element $d \in G$ such that $ad = da = e$.

Using only the above axioms, prove the following theorems in the text:
(i) The identity of G is unique.

If e and e' are the identities, replace a by e in $ae' = a$, we find $ee' = e$. Replacing a by e' in $ea = a$, we find $ee' = e'$; so, $e' = ee' = e$.

(ii) If $ab = ac$ in G, then $b = c$.

Multiplying $ab = ac$ by a^{-1} from the right, we get

$$b \overset{(b)}{=} eb \overset{(c)}{=} (a^{-1}a)b \overset{(a)}{=} a^{-1}(ab) = a^{-1}(ac) \overset{(a)}{=} (a^{-1}a)c \overset{(c)}{=} ec \overset{(b)}{=} c.$$

(iii) The inverse of $a \in G$ is unique.

If d and d' are inverses of a, we find $ad = e = ad'$. Then by (ii), we have $d = d'$.

(iv) $(ab)^{-1}$ for $a, b \in G$ is given by $b^{-1}a^{-1}$.

We need to check by (iii) that $(b^{-1}a^{-1})ab = e$, which follows from:

$$(b^{-1}a^{-1})ab \overset{(a)}{=} b^{-1}(a^{-1}a)b = b^{-1}eb \overset{(b)}{=} b^{-1}b \overset{(c)}{=} e.$$

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2. Label the following statements as being true or false. In the following statements, G and H are groups, e is the identity element of G and a, b are elements of G.

Statements	Label
If $a^7 = e$, then for every $b \in G$, $(bab^{-1})^7 = e$.	T
In a group G, the equation $ax = b$ for $a, b \in G$ has a solution.	T
Define a map $\phi_a : G \to G$ by $\phi_a(x) = axa^{-1}$. Then $\phi_a \circ \phi_b = \phi_{ab}$.	T
The subset $X = \{a^m \mid m \in \mathbb{Z}\}$ of a group G is an abelian subgroup of G.	T
There is a subgroup of order 3 in a group G of order 10.	F
The centralizer of an element $a \in G$ is always abelian.	F
If H and K are subgroups of a group G, $H \cup K$ is always a subgroup of G.	F
In a dihedral group, a rotation after a reflection is a rotation.	F
$\mathbb{Z}$ with subtraction "$-$" is a group.	F
The order of ab is always the product of the orders of a and b.	F
The center of a group is an abelian group.	T
Every finite group is isomorphic to a subgroup of S_n for an integer n.	T
Any two cyclic groups of order 4 are isomorphic.	T
If H and K are subgroups of a group G, $H \cap K$ is always a subgroup of G.	T
If $X \subset G$, $X \neq \emptyset$ and $ab^{-1} \in X$ for all $a, b \in X$, then X is a subgroup of G.	T
For an abelian group G, $\{a^3 \mid a \in G\}$ is always a subgroup distinct from G.	F

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3. Which of the following five functions $f : H \to G$ are isomorphism of groups? Explain briefly your reason.

f, G, H	Yes/No	Reason
$G = \mathrm{Aut}(\mathbb{Z}_3)$ with composition $H = U_3$: multiplicative group $f(m)$ sends $x \in \mathbb{Z}_3$ to mx. where $\mathrm{Aut}(\mathbb{Z}_3)$ is the automorphism group of the additive group $\mathbb{Z}_3$.	Yes	an automorphism brings 1 to another generator: Exercise 19 Section 7.4; so, it has to be $x \mapsto mx$ with $m \in U_3$
$H = \mathbb{Z}_6$: additive groups $G = \mathbb{Z}_2 \times \mathbb{Z}_3$: product of additive groups $f(x) = ([x]_2, [x]_3)$ where $[x]_p$ is the congruence class of x modulo p.	Yes	If $f(x) = f(y)$, then $(x - y)$ is divisible; by 2 and 3 so; $6 \mid (x - y)$, and f is injective. Injectivity implies surjectivity, because $\lvert G\rvert = 6 = \lvert\mathbb{Z}_6\rvert$. f being a homomorphism is easy.
$f(i) = 2^i$ $H = \mathbb{Z}_4$: additive group $G = U_5$: multiplicative group	Yes	$U_5 = \{1, 2, 3, 4\}$ so, f is one-to-one and onto $f(i + j) = 2^{i+j} = f(i)f(j)$
$H = \{x \in \mathbb{Q} \mid x \neq 1\}$ with operation $a * b = a + b - ab$ $G = \mathbb{Q}^{\times}$: multiplicative group $f(x) = 1 - x$	Yes	See homework solution key Exercise 4 Section 7.1
$H = D_3$ with composition $G = S_3$ with composition $f(x) = \begin{pmatrix} 1 & 2 & 3 \\ x(1) & x(2) & x(3) \end{pmatrix}$ where the numbers: 1, 2, 3 indicate three vertices of the triangle on which D_3 operates.	Yes	$x \in D_3$ induces a permutation of vertices, which determines x; so, injective. D_3 and S_3 have the same order; so, onto.