Problem Set 1

1. (i) Prove that for any sets $A$, $B$, and $C$ we have
   \[ A \cup (B \cap C) = (A \cup B) \cap (A \cup C), \quad A \cap (B \cup C) = (A \cap B) \cup (A \cap C). \]
   These identities may be interpreted as the mutual distributivities of $\cap$ and $\cup$.
   
   Hint. You can use e.g. a Venn diagram, but you should understand why this constitutes a rigorous proof.

   (ii) Let $S$ be an index set and $A_s$ a family of subsets of some set $X$. Prove De Morgan’s laws
   \[ (\bigcup_{s \in S} A_s)^c = \bigcap_{s \in S} A_s^c, \quad (\bigcap_{s \in S} A_s)^c = \bigcup_{s \in S} A_s^c, \]
   where $A^c := X \setminus A$ denotes the complement of $A$.

2. Show that the scalar product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^n$ satisfies the parallelogram law or polarization identity
   \[ |x + y|^2 + |x - y|^2 = 2|x|^2 + 2|y|^2. \]
   What does this identity mean geometrically for the parallelogram spanned by $x$ and $y$ (i.e. the parallelogram with vertices 0, $x$, $y$, and $x+y$)?

3. (i) Prove that if $A$ and $B$ are closed then $A \cup B$ is also closed. Find an example that shows that infinite unions of closed sets are not necessarily closed.

   (ii) Prove that if for each $s \in S$ the set $A_s$ is closed then $\bigcap_{s \in S} A_s$ is also closed.

4. For any function $f : X \to Y$ and $A \subset Y$ recall the preimage map $f^{-1}(A) := \{ x \in X : f(x) \in A \}$.
   (i) Show that
   \[ f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B), \quad f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B). \]

   (ii) Show that
   \[ f(A \cup B) = f(A) \cup f(B), \quad f(A \cap B) \subset f(A) \cap f(B). \]
   Find an example to show that in general $f(A \cap B)$ is not equal to $f(A) \cap f(B)$. Show that if $f$ is injective (or one-to-one) then we have
   \[ f(A \cap B) = f(A) \cap f(B). \]
   
   Hint. It may be helpful to write for instance
   \[ f(A \cap B) = \{ y \in Y : \exists x \in A \cap B : y = f(x) \}. \]
5. (i) Let \(A_1, A_2, A_3, \ldots\) be subsets of some set \(X\), and define
\[
U := \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k, \quad V := \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k.
\]
Which one of \(U \subset V\) and \(V \subset U\) is true? Prove your claim.

(ii) Let \((f_k)_{k \in \mathbb{N}}\) be a sequence of functions on some set \(X\), and \(f\) a function on \(X\). Show that the set of convergence
\[
C := \left\{ x \in X : \lim_{k \to \infty} f_k(x) = f(x) \right\}
\]
may be written as
\[
C = \bigcap_{l=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} \left\{ x \in X : |f_k(x) - f(x)| \leq \frac{1}{l} \right\}.
\]

6. Prove that a sequence in \(\mathbb{R}^n\) converges if and only if all of its components converge.

7. The closure of a set \(A\) is by definition
\[
\overline{A} := \bigcap_{B \supseteq A; \ B \text{ closed}} B.
\]

(i) Prove that \(\overline{A}\) is closed. Hence, \(\overline{A}\) is the smallest closed set containing \(A\).

(ii) Recall that by definition \(x\) is a limit point of \(A\) if for all \(r > 0\) the set \(B_r(x)\) contains a point of \(A\) that is not \(x\). Show that
\[
\overline{A} = A \cup \{\text{limit points of } A\}.
\]

Hints. This is the same as proving \(\overline{A} = L\) where
\[
L := \left\{ \lim_{k \to \infty} x_k : (x_k)_{k \in \mathbb{N}} \text{ is a convergent sequence with } x_k \in A \ \forall k \in \mathbb{N} \right\}
\]
(why?). In order to prove the equality, you have to prove that \(\overline{A} \subset L\) and \(L \subset \overline{A}\). For the first inclusion, take \(y \in \overline{A}\) and prove, by contradiction (i.e. using the method of indirect proof) that for all \(r > 0\) we have \(B_r(y) \cap A \neq \emptyset\). Once this is proved, it will easily follow that \(y \in L\).

(iii) Show that \(\overline{A \cup B} = \overline{A} \cup \overline{B}\) and \(\overline{A \cap B} \subset \overline{A} \cap \overline{B}\).

Hint. Use the characterization of (ii).

(iv) Find an example that shows that \(\overline{A \cap B} \neq \overline{A} \cap \overline{B}\) in general.
8. (i) Prove the following characterization of continuity, which is a (stronger) variant of the one given in class: a function $f$ is continuous at $a$ if and only if every sequence $(x_k)_{k \in \mathbb{N}}$ that converges to $a$ has a subsequence $(x_{k_j})_{j \in \mathbb{N}}$ such that $\lim_{j \to \infty} f(x_{k_j}) = f(a)$.

*Hint.* The “only if” direction follows from the characterization of continuity given in class. In order to prove the “if” direction, do an indirect proof by assuming that $f$ is not continuous at $a$.

(ii) Suppose that $f$ is a continuous, bijective function defined on a compact set $A$. Show that $f^{-1}$ is also continuous. (A bijective continuous function whose inverse is also continuous is called a *Homeomorphism*.)

*Hint.* By (i), it suffices to show (why?) that if $(x_k)_{k \in \mathbb{N}}$ is a sequence in $A$ satisfying $f(x_k) \to f(x)$ for some $x \in A$, then there exists a subsequence $(x_{k_j})_{j \in \mathbb{N}}$ such that $x_{k_j} \to x$.

(iii) Can you find an example of a continuous, bijective function $f$ such that $f^{-1}$ is not continuous?

Due: Thursday, February 14, in class.