MIT 18.01SC FALL 2010 PRACTICE FINAL SOLUTIONS

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Contents

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1. Question 1

Compute the following derivatives

(a) $f'(x)$ where $f(x) = x^3 e^x$.

From the product rule, $f'(x) = x^3 e^x + 3x^2 e^x$.

(b) $f^{(7)}(x)$ where $f(x) = \sin(2x)$.

$f'(x) = 2\cos(2x)$, $f''(x) = -4\sin(2x)$, $f'''(x) = -8\cos(2x)$, $f^{(4)}(x) = 16\sin(2x)$, $f^{(5)}(x) = 32\cos(2x)$, $f^{(6)}(x) = -64\sin(2x)$, $f^{(7)}(x) = -128\cos(2x)$.

2. Question 2

(a) Find the tangent line to $y = 3x^2 - 5x + 2$ at $x = 2$.

$y'(x) = 6x - 5$, so $y'(2) = 7$. Also, $y(2) = 4$, so the tangent line has equation $(y - 4) = 7(x - 2)$.

(b) Show that the curve defined by $xy^3 + x^3y = 4$ has no horizontal tangent.

For (x, y) with $xy^3 + x^3y = 4$, we have $3xy^2y' + y^3 + x^3y' + 3x^2y = 0$, so $y'(3xy^2 + x^3) = -3x^2y - y^3$. We prove that no horizontal tangent occurs by contradiction. Assume that

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$y'(x) = 0$. Then $-3x^2y - y^3 = 0$, so $-3x^3y - xy^3 = 0$. Since $xy^3 + x^3y = 4$, adding our two equations shows that $-2x^3y = 4$. Since $xy^3 + x^3y = 4$, $x \neq 0$, so our previous equality says $y = -2x^{-3}$. Substituting this equality into $xy^3 + x^3y = 4$ gives $-8x^{-8} - 2 = 4$, so $x^{-8} = -3/4$. However, no x satisfies $x^{-8} = -3/4$, since $x^{-8} > 0$, but $-3/4 < 0$. Since we have achieved a contradiction, we conclude that $y'(x)$ is never zero.

3. Question 3

(a) We compute $\dfrac{d}{dx}\left(\dfrac{x}{x+1}\right)$ from the definition of the derivative.

Let $f(x) = x/(x+1)$, and let $x \neq -1$, $h \in \mathbb{R}$. Then

$$\begin{aligned}\frac{f(x+h)-f(x)}{h} &= \frac{\frac{x+h}{x+h+1} - \frac{x}{x+1}}{h} = \frac{(x+h)(x+1) - x(x+h+1)}{h(x+h+1)(x+1)} \\ &= \frac{x^2 + x + hx + h - x^2 - xh - x}{h(x+h+1)(x+1)} \\ &= \frac{h}{h(x+h+1)(x+1)} = \frac{1}{(x+h+1)(x+1)}.\end{aligned}$$

So,

$$\begin{aligned}f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{1}{(x+h+1)(x+1)} \\ &= \frac{1}{\lim_{h\to 0}(x+h+1)(x+1)} = \frac{1}{(x+1)^2}.\end{aligned}$$

(b) Compute $\displaystyle\lim_{x\to\sqrt{3}}\frac{\tan^{-1}(x) - \pi/3}{x - \sqrt{3}}$.

Since $\tan^{-1}(\sqrt{3}) = \pi/3$, L'Hopital's rule applies. We therefore have

$$\lim_{x\to\sqrt{3}}\frac{\tan^{-1}(x) - \pi/3}{x - \sqrt{3}} = \lim_{x\to\sqrt{3}}\frac{1}{1+x^2} = \frac{1}{4}.$$

4. Question 4

Describe the graph of the function $y(x) = \dfrac{x}{x^2+1}$.

First, note that $y(-x) = (-x)/(x^2+1) = -(x/(x^2+1)) = -y(x)$, so y is an odd function. We now check the derivative of y. Observe

$$y'(x) = \frac{x^2 + 1 - 2x^2}{(x^2+1)^2} = \frac{-x^2+1}{(x^2+1)^2}.$$

Suppose x satisfies $y'(x) = 0$. Then $x^2 = 1$, so $x = 1, -1$. Since $y'(x)$ is a continuous function on $(-\infty, \infty)$, it therefore only touches the x-axis twice. For $-\infty < x < -1$, $y'(x) < 0$, so y is decreasing on this interval. For $-1 < x < 1$, $y'(x) > 0$, so y is increasing on this interval. And for $1 < x < \infty$, $y'(x) < 0$, so y is decreasing on this interval.

We now check the second derivative of y. Observe

$$y''(x) = \frac{(x^2+1)^2(-2x) - (1-x^2)2(x^2+1)2x}{(x^2+1)^4} = \frac{2x(-(x^2+1) - 2(1-x^2))}{(x^2+1)^3} = \frac{2x(x^2-3)}{(x^2+1)^3}.$$

Suppose $y''(x) = 0$. Then $x = -\sqrt{3}, 0, \sqrt{3}$. Since $y''(x)$ is a continuous function on $(-\infty, \infty)$, it therefore only touches the x-axis three times. For $-\infty < x < -\sqrt{3}$, $y''(x) < 0$, so y is concave down on this interval. For $-\sqrt{3} < x < 0$, $y''(x) > 0$, so y is concave up on this interval. For $0 < x < \sqrt{3}$, $y''(x) < 0$, so y is concave down on this interval. And for $\sqrt{3} < x < \infty$, $y''(x) > 0$, so y is concave up on this interval. In conclusion, $x = -\sqrt{3}, 0, \sqrt{3}$ are all inflection points of y.

Lastly, we check the asymptotes of y. Since y is continuous on $(-\infty, \infty)$, it has no vertical asymptotes. Also,

$$\lim_{x\to\infty} \frac{x}{x^2+1} = \lim_{x\to\infty} \frac{1}{x + 1/x} = 0.$$

$$\lim_{x\to-\infty} \frac{x}{x^2+1} = \lim_{x\to-\infty} \frac{1}{x + 1/x} = 0.$$

5. Question 5

A rectangular poster is designed with 50 in^2 of printed type, 4 inch margins on the top and bottom, and 2 inch margins on the left and right side. Find the dimensions of the poster that minimize the amount of paper used.

Suppose the poster has width W and height H, both in inches. It is given that $(W - 4)(H - 8) = 50$, and we want to minimize the area of the poster WH for $W > 4$ and $H > 8$. Since $(W - 4)(H - 8) = 50$, $H = 8 + 50/(W - 4)$. So, we need to minimize the function $f(W) = W(8 + 50/(W - 4))$ for $W \geq 4$. The function $f(W)$ is a differentiable function for $W > 4$, so we apply the closed interval method. We first check for critical numbers of f. Observe

$$f'(W) = W\frac{-50}{(W-4)^2} + 8 + \frac{50}{W-4}.$$

Suppose $f'(W) = 0$. Then $W(-50) + 8(W-4)^2 + 50(W-4) = 0$, so $-50W + 8(W^2 - 8W + 16) + 50W - 200 = 0$, so $8W^2 - 64W - 72 = 0$, so $W^2 - 8W - 9 = 0$, so $(W-9)(W+1) = 0$. So, for $W > 4$, the only critical point occurs at $W = 9$.

So, to minimize f, it suffices to check f at the endpoints of the interval $(-4, \infty)$, and at the point $W = 9$. Note that $\lim_{W\to 4} f(W) = \infty$, $\lim_{W\to\infty} f(W) = \infty$, and $f(9) = 9(8 + 10) = 162 < \infty$. So, the absolute minimum of the function f occurs when $W = 9$. Since $(W - 4)(H - 8) = 50$, we also have $H = 18$. So, the most efficient poster has width 9 in and height 18 in.

6. Question 6

A highway patrol plane is flying 1 mile above a long, straight road, with constant ground speed of 120 mph. Using radar, the pilot detects a car ahead whose distance from the plane is 1.5 miles and decreasing at a rate of 136 mph. How fast is the car traveling along the highway?

Let x denote the distance along the road between the plane and the car. From the Pythagorean Theorem, the distance from the plane to the car is $f = \sqrt{x^2+1}$. It is given that $df/dt = -136$. From implicit differentiation

$$\frac{df}{dt} = \frac{df}{dx}\frac{dx}{dt} = \frac{x}{\sqrt{x^2+1}}\frac{dx}{dt}$$

Since $f = 1.5 = 3/2 = \sqrt{1+x^2}$, we conclude that $x = \sqrt{5}/2$. So, for $x = \sqrt{5}/2$, $dx/dt = -136(3/2)(2/\sqrt{5}) = -136(3/\sqrt{5})$. Note that dx/dt is the rate of change of the ground distance between the plane and the car. That is, dx/dt is the ground speed of the car, relative to the plane. So, to get the ground speed of the car, we need to add the ground speed of the plane. That is, the ground speed of the car is $120 - 136(3/\sqrt{5}) \approx -62.5$. So, the car is moving towards the plane with a ground speed of 62.5 mph.

7. Question 7

(a) Evaluate $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n}\sqrt{1+\frac{2i}{n}}$.

Let $f(x) = \sqrt{1+x}$. For $n \geq 1$, suppose we partition the interval $[0,2]$ into intervals of length $2/n$. Then the Riemann sum evaluated at the right endpoint of each rectangle has value

$$S = \sum_{i=1}^{n} \frac{2}{n}\sqrt{1+\frac{2i}{n}}.$$

Since f is continuous on $[0,2]$, any Riemann sum of rectangles with width decreasing to zero approaches $\int_0^2 f(x)dx$. In this case, this means that

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n}\sqrt{1+\frac{2i}{n}} = \int_0^2 \sqrt{1+x}\,dx.$$

Finally, using the fundamental theorem of calculus, we have

$$\int_0^2 \sqrt{1+x}\,dx = \int_0^2 \frac{d}{dx}(2/3)(1+x)^{3/2} = (2/3)(1+x)^{3/2}|_{x=0}^{x=2} = \frac{2}{3}(3^{3/2}-1).$$

(b) Evaluate $\lim_{h\to 0} \frac{1}{h}\int_2^{2+h} \sin(x^2)dx$.

From the Fundamental theorem of calculus,

$$\lim_{h\to 0} \frac{1}{h}\int_2^{2+h} \sin(x^2)dx = \sin(2^2) = \sin(4).$$

8. Question 8

(a) Compute $\int_0^{\pi/4} \tan(x)\sec^2(x)dx$.

Let $f(u) = u$, and let $g(x) = \tan(x)$. So, by changing variables, we have

$$\begin{aligned}\int_0^{\pi/4} \tan(x)\sec^2(x)dx &= \int_0^{\pi/4} f(g(x))g'(x)dx \\ &= \int_{g(0)}^{g(\pi/4)} f(u)du = \int_0^1 udu = (1/2)u^2|_{u=0}^{u=1} = 1/2.\end{aligned}$$

(b) Compute $\int_1^2 x\ln(x)dx$.

For $x > 0$, note that $(d/dx)(x^2 \ln(x)) = x + 2x\ln(x)$. Using this equality and the Fundamental theorem of calculus,

$$\int_1^2 x\ln(x)dx = \int_1^2 (-x/2)dx + \int_1^2 (d/dx)(x^2\ln(x))dx = [-x^2/4]_{x=1}^{x=2} + [x^2\ln(x)]_{x=1}^{x=2}$$
$$= -1 + 1/4 + 4\ln(2).$$

9. Question 9

Calculate $\int \frac{x^2}{\sqrt{9-x^2}}dx$.

Let $-3 \leq a, b \leq 3$. We use the substitution $x = 3\sin(\theta)$, so $dx = 3\cos(\theta)d\theta$. Alternatively, let $f(\theta) = 9\sin^2(\theta)$, and let $g(x) = \sin^{-1}(x/3)$. Then for $-3 \leq x \leq 3$, $f(g(x)) = x^2$, and $g'(x) = (1/3)(1 - x^2/9)^{-1/2} = (9 - x^2)^{-1/2}$, so a change of variables says that

$$\int_a^b \frac{x^2}{\sqrt{9-x^2}}dx = \int_a^b f(g(x))g'(x) = \int_{g(a)}^{g(b)} f(\theta)d\theta = \int_{\sin^{-1}(a/3)}^{\sin^{-1}(b/3)} 9\sin^2(\theta)d\theta$$

Since $\sin^2(\theta) = (1 - \cos(2\theta))/2$, we have

$$\begin{aligned}\int_{\sin^{-1}(a/3)}^{\sin^{-1}(b/3)} 9\sin^2(\theta)d\theta &= \int_{\sin^{-1}(a/3)}^{\sin^{-1}(b/3)} (9/2)(1 - \cos(2\theta))d\theta \\ &= (9/2)(\sin^{-1}(b/3) - \sin^{-1}(a/3)) + \int_{\sin^{-1}(a/3)}^{\sin^{-1}(b/3)} (9/4)\frac{d}{d\theta}(-\sin(2\theta))d\theta \\ &= (9/2)(\sin^{-1}(b/3) - \sin^{-1}(a/3)) \\ &\qquad + (9/4)(-\sin(2\sin^{-1}(b/3)) - (-\sin(2\sin^{-1}(a/3)))) \\ &= \int_a^b \frac{d}{dx}\left((9/2)\sin^{-1}(x/3) - (9/4)\sin(2\sin^{-1}(x/3))\right)dx.\end{aligned}$$

In the last line, we used the Fundamental theorem of calculus.

In conclusion,

$$\int \frac{x^2dx}{\sqrt{9-x^2}} = (9/2)\sin^{-1}(x/3) - (9/4)\sin(2\sin^{-1}(x/3)) + C.$$

For convenience, we simplify this expression further. From the double angle formula, $\sin(2y) = 2\sin(y)\cos(y)$. So, for x with $-3 \leq x \leq 3$,

$$\sin(2\sin^{-1}(x/3)) = 2\sin(\sin^{-1}(x/3))\cos(\sin^{-1}(x/3)) = (2/3)x\sqrt{1 - x^2/9}.$$

In conclusion,

$$\int \frac{x^2dx}{\sqrt{9-x^2}} = (9/2)\sin^{-1}(x/3) - (1/2)x\sqrt{9-x^2} + C.$$

10. Question 10

11. Question 11

The integral $\int_1^5 (e^x/x)dx$ has no elementary derivative. Use the trapezoid rule with two trapezoids and the following table of approximate values to estimate this definite integral.

x	1	2	3	4	5
e^x/x	2.7	3.7	6.7	13.6	29.7

$$\int_1^5 \frac{e^x}{x}dx \approx (3-1)\frac{f(3)+f(1)}{2} + (5-3)\frac{f(5)+f(3)}{2} \approx 6.7+2.7+6.7+29.7$$

12. Question 12

The rate of radioactive decay dm/dt of a mass m of Radium-226 is proportional to the amount m of Radium present at time t. Suppose we begin with 100 milligrams of Radium at time $t=0$.

(a) Given that the half life of Radium-226 is roughly 1600 years, find a formula for the mass of Radium that remains after t years.

It is given that $m(t) = Ae^{-kt}$ with $m(0) = 100$, and $m(1600) = 50$. So $m(t) = 100e^{-kt}$, and $m(1600) = 100e^{-1600k} = 50$, so $\log(100) - 1600k = \log(50)$, so $1600k = \log(100) - \log(50) = \log(2)$, so $k = \log(2)/1600$. In summary,

$$m(t) = 100e^{-\log(2)t/1600} = 100 \cdot 2^{-t/1600}.$$

(b) Find the amount of Radium remaining after 1000 years. Use the approximation $2^{-10/16} \approx .65$.

$m(1000) = 100 \cdot 2^{-10/16} \approx 65$.

13. Question 13

14. Question 14

15. Question 15

Prove or disprove the following statement. For all $x > 0$,

$$\frac{x}{1+x^2} < \tan^{-1}(x) < x.$$

For $x > 0$, we want to know whether or not $\frac{x}{1+x^2} < \tan^{-1}(x) < x$.

Let $F(x) = \tan^{-1}(x) - \frac{x}{1+x^2}$. Then

$$F'(x) = \frac{1}{1+x^2} - \frac{(1+x^2) - x(2x)}{(1+x^2)^2} = \frac{1}{1+x^2} - \frac{1-x^2}{(1+x^2)^2} = \frac{1+x^2-1+x^2}{(1+x^2)^2} = \frac{2x^2}{(1+x^2)^2}.$$

For $x > 0$, $F'(x) > 0$. So, F achieves its minimum on $[0,\infty)$ at $x = 0$. Since $F(0) = 0$, we conclude that $F(x) > 0$ for all $x > 0$. That is, $\tan^{-1}(x) > \frac{x}{1+x^2}$ for all $x > 0$.

Let $G(x) = x - \tan^{-1}(x)$. Then

$$G'(x) = 1 - \frac{1}{1+x^2} = \frac{1+x^2-1}{1+x^2} = \frac{x^2}{1+x^2}.$$

For $x > 0$, $G'(x) > 0$. So, G achieves its minimum on $[0,\infty)$ at $x = 0$. Since $G(0) = 0$, we conclude that $G(x) > 0$ for all $x > 0$. That is, $x > \tan^{-1}(x)$ for all $x > 0$.

In conclusion, the original statement is true.

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