Digest 9

(A compilation of emailed homework questions, answered around Wednesday.)

**Question.** (From a student): Does the divergence test only apply to series that begin at n=1?

**Answer.** No. For example, let \( \{a_n\} \) be a sequence and suppose we start at \( n = 0 \) instead of \( n = 1 \). Then \( \sum_{n=0}^{\infty} a_n \) diverges if and only if \( \sum_{n=1}^{\infty} a_n \). And Then \( \sum_{n=0}^{\infty} a_n \) diverges if and only if \( \sum_{n=53}^{\infty} a_n \). And so on. To be even more specific, consider \( a_n = n \). Then \( \sum_{n=0}^{\infty} \frac{1}{n(n+1)} \) diverges if and only if \( \sum_{n=20}^{\infty} a_n = 20 + 21 + 22 + 23 + \cdots \).

As far as convergence and divergence are concerned, the place where we start the sum does not matter.

**Question.** (From a student): This question refers to number 9 from 11.3 in the book. It asks to use the integral test for the infinite series:

\[
\sum_{n=1}^{\infty} \frac{1}{n(n+1)}.
\]

So I took its integral and I solved that it would be

\[
\lim_{R \to \infty} \int_{1}^{R} \frac{1}{x(x+1)} \, dx = \ln(R) \ln(R+1) - \ln(1) \ln(2).
\]

That equals infinity and diverges yes? Perhaps I’m doing it wrong, the book say that it converges.

**Answer.** The integration is incorrect. You do want to compute the integral of \( \frac{1}{x(x+1)} \). However, you might want to use something like \( \frac{1}{x(x+1)} \leq \frac{1}{x^2} \) when \( x \geq 1 \). Then

\[
\lim_{R \to \infty} \int_{1}^{R} \frac{1}{x(x+1)} \, dx \leq \lim_{R \to \infty} \int_{1}^{R} \frac{1}{x^2} \, dx = \lim_{R \to \infty} [-x^{-1}]_{1}^{R} = \lim_{R \to \infty} [-R^{-1} + 1] = 1.
\]

So, we see that the integral converges. So the sum converges.

**Question.** [From a student] In Example 1 of 11.7 (pg. 598), it says: Find the Taylor series for \( f(x) = x^{-3} \), where \( c = 1 \). So, \( f'(x) = -3x^{-4} \), and \( f''(x) = 12x^{-5} \), and in general \( f^{(n)}(x) = (-1)^n (3)(4) \cdots (n+2)x^{-3-n} \).

Why is the last term multiplied by \( (n+2) \)?

**Answer.** We know that \( f'(x) = -3x^{-4} \) and \( f''(x) = 12x^{-5} \). In general, we claim that

\[
f^{(n)}(x) = (-1)^n 3 \cdot 4 \cdot 5 \cdots n(n+1)(n+2)x^{-3-n}. \tag{*}
\]
We can prove this formula holds by induction on $n$. The case $n = 1$ holds, since $f'(x) = f^{(1)}(x) = (-1)^1 3x^{-3-1} = -3x^{-4}$. (Note that $n + 2 = 3$ when $n = 1$.) So, to prove that the equation $(\ast)$ holds, we consider the inductive step. Suppose $(\ast)$ holds for $n$. We must then prove that it holds for the case $n + 1$. To see this, we use that $(\ast)$ holds for the case $n$ to get

$$f^{(n+1)}(x) = \frac{d}{dx} f^{(n)}(x) = (-1)^n 3 \cdot 4 \cdot 5 \cdots n(n+1)(n+2) \frac{d}{dx} x^{-3-n}$$

$$= (-1)^n 3 \cdot 4 \cdot 5 \cdots n(n+1)(n+2)(-3-n)x^{-3-n-1}$$

$$= (-1)^n(-1) 3 \cdot 4 \cdot 5 \cdots n(n+1)(n+2)(n+3)x^{-3-(n+1)}$$

$$= (-1)^{n+1} 3 \cdot 4 \cdot 5 \cdots (n+1)((n+1)+1)((n+1)+2)x^{-3-(n+1)}.$$

So, formula $(\ast)$ holds for $n + 1$. So, it holds for all $n$. (I will not expect you guys to do proofs by induction in this class; this method can just help you understand how to find certain Taylor series.)

Finally, formula $(\ast)$ tells us the Taylor series for $f$, since $(\ast)$ implies that

$$f^{(n)}(1) = (-1)^n 3 \cdot 4 \cdot 5 \cdots n(n+1)(n+2).$$

So, the Taylor series is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!} (x - 1)^n = \sum_{n=0}^{\infty} \frac{(-1)^n 3 \cdot 4 \cdot 5 \cdots n(n+1)(n+2)}{1 \cdot 2 \cdot 3 \cdots (n-1)n} (x - 1)^n$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (n+1)(n+2)}{2} (x - 1)^n$$