I am interested in applying analytic techniques to probabilistic problems. These problems are sometimes motivated by hardness results in theoretical computer science. A recurring theme of our research is that mathematical objects that were initially motivated by physics (such as semigroups of operators, hypercontractivity, Fourier analysis, soap bubbles and minimal surfaces, Bell’s inequality from quantum mechanics, optimization of energy functionals, the calculus of variations, etc.) now have renewed motivation from important problems in computer science.

For example, we recently proved a noncommutative version of a nonlinear Central Limit Theorem [HV16], using some techniques from Fourier analysis [MOO10], random matrix theory, and functional analysis [Gro72, CL93]. Our result then implies that a very general computational problem, the noncommutative Grothendieck problem, cannot be solved efficiently (assuming the Unique Games Conjecture; see Section 1.6). That is, our Central Limit-type theorem implies that computers cannot run quickly when solving a wide class of problems. Put another way, our result in analysis in Euclidean space has ramifications for the analysis and probability of discrete functions. Or put another way, the best constant in Bell’s inequality from quantum mechanics provides a precise quantitative barrier for computation.

For another example, one main focus of my research has been isoperimetric problems in Euclidean space equipped with the Gaussian measure [HJN13, Hei14, HMN16, Hei15, Hei16b, Hei16a]. These optimization problems ask for sets of fixed Gaussian volume with smallest Gaussian perimeter. Some additional constraints are added to the sets, and various notions of perimeter are used, some of which depend on the entire set and not just its boundary. We have proved some special cases and provided counterexamples for some of these conjectures. My long term goal is to develop general methods for approaching these problems, since such general methods do not exist. Most recently, I have been developing calculus of variations techniques [Hei16b, Hei16a]. The “classical” calculus of variations does not apply to these problems.

Currently, there are many unsolved Gaussian isoperimetric problems, and their resolution will yield countless dividends. The Gaussian measure is most interesting since it is almost interchangeable with the uniform measure on the discrete hypercube, via Central Limit Theorems [Rot79, Cha06, MOO10, Mos10, IM12]. These Gaussian isoperimetric results therefore imply inequalities on the discrete hypercube. The discrete inequalities are then applied to machine learning and Grothendieck inequalities [KN09, KN13, HV16], to the Unique Games Conjecture [KKMO07, MOO10, KM16], to semidefinite programming algorithms such as MAX-CUT [KKMO07, IM12], to social choice theory [MOO10, IM12], to learning theory [FGRW12], to communication complexity [CR11], etc. So, solving these isoperimetric problems can tell us how quickly computers can run, and how to design elections so that erroneous tabulation of votes does not affect the outcome of the election.

I have also studied \( L_p \) Poincaré inequalities for \( 1 < p < \infty \) [HMO14] on general measure spaces that do not seem to be provable using standard techniques, such as Littlewood-Paley theory. In particular, we proved the degree one case of a conjecture of [MN14]. The conjectured Poincaré inequalities [MN14] can be used to construct graphs which are expanders in a very general sense [MN14]. The explicit construction of expander graphs are then used in computer science [HLW06].

1. Noncommutative Majorization Principle

1.1. Commutative Invariance Principles. The invariance principles of [Rot79, Cha06, MOO10] are nonlinear versions of the Central Limit Theorem, with error bounds. That is, the invariance principle implies the Berry-Esséen Central Limit Theorem, which we now recall.

Let \( n \) be a positive integer. Let \( x_1, \ldots, x_n \) be commutative indeterminate variables, and let

\[
Q(x_1, \ldots, x_n) := \frac{x_1 + \cdots + x_n}{\sqrt{n}}.
\]
Let \( b_1, \ldots, b_n \) be independent identically distributed (i.i.d.) uniform random variables in \( \{-1, 1\} \), and let \( g_1, \ldots, g_n \) be i.i.d. standard Gaussian random variables. Letting \( E \) denote expected value, we then define the 2-norm of \( Q \) to be

\[
\|Q\|_2 := (\langle E |Q(b_1, \ldots, b_n)\rangle)^{1/2}.
\]

The Berry-Esséen Central Limit Theorem then says

\[
\sup_{t \in \mathbb{R}} |\mathbb{P}(Q(b_1, \ldots, b_n) \leq t) - \mathbb{P}(Q(g_1, \ldots, g_n) \leq t)| \leq 3 \max_{i=1, \ldots, n} \left\| \frac{\partial}{\partial x_i} Q \right\|_2.
\]

If the rightmost expression looks unfamiliar, note that \( Q(g_1, \ldots, g_n) \) has a standard Gaussian distribution, and \( \left\| \frac{\partial}{\partial x_i} Q \right\|_2 = 1/\sqrt{n} \) for all \( i \in \{1, \ldots, n\} \). The proof of (2) can also be extended to moments of \( Q \):

\[
\left| \mathbb{E} |Q(b_1, \ldots, b_n)|^4 - \mathbb{E} |Q(g_1, \ldots, g_n)|^4 \right| \leq 240 \max_{i=1, \ldots, n} \left\| \frac{\partial}{\partial x_i} Q \right\|_2.
\]

A similar statement can be made for higher moments of \( Q \). The commutative invariance principle \[Rot79, Cha06, MOO10\] implies, among other things, that (3) holds for multilinear polynomials.

Let \( d, \nu \in \mathbb{N} \). Let \( Q(x_1, \ldots, x_n) \) be a multilinear polynomial of degree \( d \), so that

\[
Q(x_1, \ldots, x_n) = \sum_{S \subseteq \{1, \ldots, n\} : |S| \leq d} c_S \prod_{i \in S} x_i, \quad c_S \in \mathbb{R}, \forall S \subseteq \{1, \ldots, n\}.
\]

Assume that \( \|Q\|_2 \leq 1 \). Then the \textbf{commutative invariance principle} \[MOO10\] says that

\[
\left| \mathbb{E} |Q(b_1, \ldots, b_n)|^4 - \mathbb{E} |Q(g_1, \ldots, g_n)|^4 \right| \leq 24 \cdot 10^d \max_{i=1, \ldots, n} \left\| \frac{\partial}{\partial x_i} Q \right\|_2.
\]

The commutative invariance principle (4) in \[MOO10\] is proven by a combination of the Lindeberg replacement argument and the hypercontractive inequality \[Sta59, Fed69, Bon70, Nel73, Gro73, Bec75\] (see (17) below). That is, one replaces one argument of \( Q \) at a time, adding up the resulting errors and controlling them via the hypercontractive inequality. The invariance principle (4) has seen many applications \[QD14\] in recent years. Here is a small sample of such applications and references: isoperimetric problems in Gaussian space and in the hypercube \[MOO10, IM12\], social choice theory, Unique Games hardness results \[KKMO07, IM12\], analysis of algorithms \[BR15\], random matrix theory \[MP14\], free probability \[NPR10\], optimization of noise sensitivity \[Kan14\], etc. We anticipate that our noncommutative version of (4) (see (7) below) could have similar applications as well.

### 1.2. Grothendieck’s Inequality

The first application of (4) we will describe is computational hardness for the commutative Grothendieck inequality \[RS09\]. For any \( x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \mathbb{C}^n \), define \( \langle x, y \rangle := \sum_{i=1}^n x_i \overline{y_i} \) and define \( \|x\|_2 := \sqrt{\langle x, x \rangle} \). Let \( \{a_{ij}\}_{i,j=1}^n \) be a real matrix. Proven first in 1953, \textbf{Grothendieck’s Inequality} \[Gro53, LP68, AN06, BMMN13\] says there exists a constant \( K > 0 \) which does not depend on \( n \) or on \( \{a_{ij}\}_{i,j=1}^n \) such that

\[
\sup_{w_1, \ldots, w_n, r_1, \ldots, r_n \in \mathbb{R}^{2n-1}} \sum_{i,j=1}^n a_{ij} \langle w_i, r_j \rangle \leq K \cdot \sup_{u_1, \ldots, u_n, v_1, \ldots, v_n \in \{-1, 1\}} \sum_{i,j=1}^n a_{ij} u_i v_j.
\]

That is, for a general optimization problem (corresponding to the left side of (5)), it is possible to “round” the unit vectors \( u_i, v_i \), \( i = 1, \ldots, n \) to a one-dimensional set of unit vectors \( u_i, v_i \), \( i = 1, \ldots, n \). And the weighted sum of inner products of the vectors does not decrease very much after we perform this rounding procedure. It is known that \( K < \pi/(2 \log(1 + \sqrt{2})) \) \[BMMN13\], and that a rounding procedure can establish the best constant in Grothendieck’s inequality \[NR14\].
However, it remains a major open problem to find this optimal rounding procedure and to find the smallest possible constant $K$ in Grothendieck’s inequality.

Finding the smallest possible constant $K$ in (5) has several interpretations beyond mathematics. From the physics perspective, the best constant $K$ in (5) is also the smallest constant in certain Bell inequalities in quantum mechanics [Pis12]. More specifically, Bell’s inequality says that the smallest $K$ possible in (5) satisfies $K > 1$. From the computer science perspective, assuming the Unique Games Conjecture (see Section 1.6), it is impossible to approximate the right side of (5), in time polynomial in $n$, within a multiplicative factor smaller than $K$, where $K$ is the smallest possible constant in the inequality (5) [RS09]. Mathematically, (5) can be rewritten as a ratio between two tensor product norms. One could consider this breadth of interpretation as evidence for the difficulty and importance of finding the smallest possible value of $K$ in (5).

1.3. The Noncommutative Grothendieck Inequality. Since (5) is very general, and since rounding procedures appear often in theoretical computer science [AN06, LOGT12, ABS10], inequality (5) has many applications. Similarly, the noncommutative version of (5) has many applications [NRV14]. The noncommutative Grothendieck inequality was conjectured in [Gro53] and proven in [Pis78, Kai83]. The noncommutative Grothendieck inequality was also given an algorithmic interpretation in [NRV14]. This algorithmic interpretation gives constant factor approximation algorithms for variants of the Principle Component Analysis problem and a generalized Procrustes problem.

Let $n,m$ be positive integers and let $S$ be a set. We denote $M_m(S)$ as the set of $m \times m$ matrices $(U_{ij})_{i,j=1}^m$ such that $U_{ij} \in S$ for all $i,j \in \{1, \ldots, m\}$. We write $U \in U_m(\mathbb{C}^n)$ if $U \in M_m(\mathbb{C}^n)$ and $\sum_{k=1}^m (U_{ik}, U_{jk}) = \sum_{k=1}^m (U_{ki}, U_{kj}) = 1_{i=j}$. In particular, $U_m(\mathbb{C})$ is the set of $m \times m$ complex unitary matrices. Let $M_{ijkl} \in M_m(M_m(\mathbb{C}))$. Then the Noncommutative Grothendieck Inequality for Complex Scalars [Pis78, Kai83] says

$$
\sup_{U,V \in U_m(\mathbb{C}^n)} \left| \sum_{i,j,k,\ell=1}^m M_{ijkl} (U_{ij}, V_{k\ell}) \right| \leq 2 \cdot \sup_{X, Y \in U_m(\mathbb{C})} \left| \sum_{i,j,k,\ell=1}^m M_{ijkl} X_{ij} Y_{k\ell} \right|.
$$

If $M_{ijkl} = 0$ for all $i,j,k,\ell \in \{1, \ldots, m\}$ such that either $i \neq j$ or $k \neq \ell$, then (6) becomes (5) (replacing $\mathbb{R}$ by $\mathbb{C}$) [NRV14]. In this way, (6) generalizes (5). Also, (6) can be proven by somehow “rounding” the vector-valued matrices $U, V$ to scalar-valued matrices $X, Y$.

The constant 2 is the smallest possible constant in (6), in stark contrast to (5), where the smallest constant is still unknown. As in [RS09], we seek an algorithmic interpretation of the sharp constant 2 in (6) in terms of the Unique Games Conjecture.

1.4. Our Contribution: Noncommutative Moment Majorization. The technical heart of the computational hardness result of [RS09] for (5) is the invariance principle [4]. So, in order to generalize the result of [RS09] to the noncommutative inequality (6), we need to prove a noncommutative version of [4]. That is, we need to allow the polynomials $Q$ to be matrix-valued, and we need to replace the random variables in (4) with random matrices.

We now describe our noncommutative version of the invariance principle [4] [HIV16], which then implies computational hardness for (6). If $Y$ is a complex matrix, we write $|Y| := (YY^*)^{1/2}$.

We first define a noncommutative multilinear polynomial. Let $X_1, \ldots, X_n$ be noncommutative $m \times m$ indeterminate variables. Let $Q(X_1, \ldots, X_n)$ be a noncommutative multilinear polynomial of degree $d \in \mathbb{N}$. That is, for any $S \subseteq \{1, \ldots, n\}$, there exists $m \times m$ complex matrices $c_S$ such that

$$
Q(X_1, \ldots, X_n) = \sum_{S \subseteq \{1, \ldots, n\} : |S| \leq d} c_S \prod_{i \in S} X_i.
$$
Also, the product terms are in increasing order, so e.g. \( \prod_{i=1,2,3} X_i = X_1 X_2 X_3 \). If \( p > m \), we can extend \( Q \) to take as input \( p \times p \) indeterminate variables \( X_1, \ldots, X_n \) by defining
\[
Q^t(X_1, \ldots, X_n) := \sum_{S \subseteq \{1, \ldots, n\}: |S| \leq d} \left( \begin{array}{c} e^S \\ 0 \\ 0 \end{array} \right) \prod_{i \in S} X_i.
\]

Let \( p > m, p, m \in \mathbb{N} \). Let \( d \in \mathbb{N} \). Let \( I_m \) denote the \( m \times m \) identity matrix and let \( \| \cdot \| \) denote the operator norm of a matrix. We say that random matrices \( G_1, \ldots, G_n \) are i.i.d. if the matrices are independent, and the distribution of \( G_i \) is equal to the distribution of \( G_j \) for any \( 1 \leq i, j \leq n \). (This condition is more general than the matrices having i.i.d. entries.)

Let \( Q(X_1, \ldots, X_n) \) be a noncommutative multilinear polynomial of degree \( d \). Assume that \( \| Q \|_2 \leq \sqrt{m} \), where \( \| Q \|_2 \) is defined in \( \{1\} \). Let \( H_1, \ldots, H_n \) be i.i.d. uniformly random \( p \times p \) unitary matrices. Let \( G_1, \ldots, G_n \) be i.i.d. random \( m \times m \) matrices that satisfy \( \mathbb{E} G_i G_i^* = I_m \), \( \mathbb{E} G_i = 0 \), and \( \| \mathbb{E} |G_1|^4 \| \leq 10 \). Then the Noncommutative Majorization Principle for Fourth Moments [HV16] says
\[
\mathbb{E} \left\| \frac{1}{m} \text{Tr} \left( Q^t \left( \begin{array}{c} G_1 \\ 0 \\ 0 \end{array} \right) H_1, \ldots, \left( \begin{array}{c} G_n \\ 0 \\ 0 \end{array} \right) H_n \right) \right\|^4 \leq \mathbb{E} \left\| \frac{1}{m} \text{Tr} \left( Q(b_1, \ldots, b_n) \right) \right\|^4 + m^4 2^{4d} \max_{i=1, \ldots, n} \left\| \frac{\partial}{\partial x_i} Q \right\|_2 + O_{m,n} \left( \frac{1}{p} \right).
\]

Comparing (7) with (4), we see that (7) is a weaker assertion. Fortunately, (7) is sufficient for applications. Moreover, (4) is false in the generality of (7), so we cannot hope to improve (7) to (4). We can prove more general statements than (7), addressing higher moments of \( Q \) and the operator norm of \( Q \). The proof of (7) uses the Lindeberg replacement argument, Fourier analysis and noncommutative hypercontractive inequalities [Gro72, CL93].

1.5. Our Application: Hardness for the Noncommutative Grothendieck Problem. Inequality (7) implies the following computational hardness result [HV16]:

Assuming the Unique Games Conjecture, no polynomial time algorithm (in \( m \)) can approximate the quantity
\[
\sup_{U, V \in \ell^4_m(\mathbb{C})} \left\| \sum_{i,j,k,l=1}^m M_{ijkl} U_{ij} V_{kl} \right\|
\]
within a multiplicative factor smaller than 2. That is, the left side of (6) (which can be computed in polynomial time, since it is a semidefinite program) is the best way to approximate the right side of (6) in polynomial time.

We should mention that this hardness result was proven recently in [BRS15] using entirely different techniques. However, our approach also proves computational hardness for other versions of the noncommutative Grothendieck inequality [BKS16], to which the approach of [BRS15] does not seem to apply.

Over the last two decades, many sharp computational hardness results of this type have been proven [KKMO07, KR08, KN09, KN13, KN12, IM12, RS09]. And essentially all of these results used the commutative invariance principle (4). So, our hardness result appears to be the first use of a noncommutative invariance-type principle.

1.6. The Unique Games Conjecture. The Unique Games Conjecture [Kho02] is a standard assumption in theoretical computer science. This conjecture can be considered a contemporary proxy for the assumption that \( P \neq \text{NP} \). That is, proving or disproving the Unique Games Conjecture is expected to have similar significance and consequences to proving or disproving \( P \neq \text{NP} \). Moreover,
both problems are closely related. As we will describe below, the Unique Games Conjecture can be succinctly stated as: approximate linear algebra is hard.

**Definition 1.1 (Γ-MAX-2LIN(p)).** Let $p \geq 2$ be a prime number. We define the Γ-MAX-2LIN($p$) problem. In this problem, we are given $n \in \mathbb{N}$ and $2n$ variables $x_i \in \mathbb{Z}/p\mathbb{Z}$, $i \in \{1, \ldots, 2n\}$. We are also given a matrix $\{a_{ij}\}_{i,j=1}^{2n}$ with $a_{ij} \geq 0$ for all $i, j \in \{1, \ldots, 2n\}$ and a set $E \subseteq \{1, \ldots, n\} \times \{1, \ldots, n\}$ with $n$ elements. An element $(i, j) \in E$ corresponds to one of $n$ linear equations of the form $x_i - x_j = c_{ij} (\text{mod } p)$, where $c_{ij} \in \mathbb{Z}/p\mathbb{Z}$. The goal of the Γ-MAX-2LIN($p$) problem is to find the following quantity:

$$\max_{(x_1, \ldots, x_{2n}) \in (\mathbb{Z}/p\mathbb{Z})^{2n}} \sum_{(i, j) \in E: x_i - x_j = c_{ij} (\text{mod } p)} a_{ij}. \quad (8)$$

That is, we need to maximize the (weighted) number of equations $x_i - x_j = c_{ij}(\text{mod } p)$ that are satisfied.

**Conjecture 1.2 (Unique Games Conjecture, [Kho02, KKMO07]).** For every $\varepsilon \in (0, 1)$, there exists a prime number $p(\varepsilon)$ such that no polynomial time algorithm (with respect to the parameter $n$) can distinguish between the following two cases, for instances of Γ-MAX-2LIN($p(\varepsilon)$) with $a_{ij} = 1$ for all $i, j \in \{1, \ldots, 2n\}$:

(i) (8) is larger than $(1 - \varepsilon)n$, or

(ii) (8) is smaller than $\varepsilon n$.

If (8) were equal to $n$, then we could find $(x_1, \ldots, x_{2n})$ achieving the maximum in (8) by Gaussian elimination. One can therefore interpret the Unique Games Conjecture as an assertion that approximate linear algebra is hard. Recently, it has been shown that there is a subexponential time algorithm that can distinguish case (i) from case (ii) above [ABS10], leading some to believe that the Unique Games Conjecture could be false. On the other hand, there is recent evidence that the conjecture could still be true [KM16, KS16]. The truth or falsity of this conjecture remains a major open problem.

2. The Standard Simplex Conjecture

2.1. Euclidean Isoperimetry. Classical isoperimetry can be traced to ancient times, though its full understanding still remains incomplete. Generally speaking, we look for an object with least perimeter among all objects of fixed volume. And we expect that the smallest perimeter object has a simple structure.

Isoperimetric problems and minimal surface theory have a long history in differential geometry [Ste38, Sch72, Min96, Wei27, Hur02, Lev51, Sim68, Law70, Bor75, Alm76, Gro83, Sim83, BdC84, EH89, Tal95, BL96, HMRR02, CM12, CIMW13], with some motivation from the physics of soap bubbles [Pla73, Tay76]. In particular, soap bubbles we encounter in the real world are minimal surfaces. Many deep results have come from these investigations, and we hope to continue this tradition.

To see that our knowledge is still limited, note that we still do not know the three disjoint sets of fixed Euclidean volume with minimum total surface area. This problem is known as a triple bubble, or multi bubble problem [HMRR02, Rei08, CCH+08]. Further below, we will discuss a Gaussian version of this problem, known as the Standard Simplex Conjecture, which is also still unsolved. Gaussian isoperimetric problems have gained recent interest due to their applications in theoretical computer science. For background motivation, we begin with the usual Euclidean isoperimetric inequality.

Let $n \geq 1$ be an integer, let $A \subseteq \mathbb{R}^n$ be a Borel set with smooth boundary $\partial A$. Let $\text{vol}_n(A)$ denote the Euclidean volume of $A$, and let $\text{vol}_{n-1}(\partial A)$ denote the Euclidean surface area of $\partial A$. \[5\]
Let $r > 0$ and let $B(0, r) := \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1^2 + \cdots + x_n^2 \leq r^2\}$ be a Euclidean ball such that $\text{vol}_n(A) = \text{vol}_n(B(0, r))$. The **Classical Isoperimetric Inequality** says that the Euclidean ball has the smallest boundary among all sets of fixed volume:

$$\text{(9)} \quad \text{vol}_n(A) = \text{vol}_n(B(0, r)) \implies \text{vol}_{n-1}(\partial A) \geq \text{vol}_{n-1}(\partial B(0, r)).$$

Let $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, and recall that $\|x\|_2 := (x_1^2 + \cdots + x_n^2)^{1/2} = \langle x, x \rangle^{1/2}$. Let $dx$ be Lebesgue measure on $\mathbb{R}^n$, and let $d\gamma_n(x) := e^{-\|x\|_2^2/2(2\pi)^{-n/2}} dx$. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a bounded function and let $t \geq 0$. Let $\Delta := -\sum_{i=1}^n \partial^2 / \partial x_i^2$. Let $e^{-t\Delta} f(x)$ denote the classical heat semigroup applied to $f$. That is, for any $x \in \mathbb{R}^n$,

$$e^{-t\Delta} f(x) := \int_{\mathbb{R}^n} f(x + y\sqrt{2t}) d\gamma_n(y).$$

The classical isoperimetric inequality \textbf{(9)} is a consequence of the following inequality for the heat semigroup $e^{-t\Delta}$, which can be proven via symmetrization \textbf{[Led96]}.

$$\text{(10)} \quad \text{vol}_n(A) = \text{vol}_n(B(0, r)) \implies \forall t \geq 0, \quad \int_{\mathbb{R}^n} 1_A \cdot e^{-t\Delta} 1_A dx \leq \int_{\mathbb{R}^n} 1_{B(0, r)} \cdot e^{-t\Delta} 1_{B(0, r)} dx.$$  

So, as heat flows out of a given set, the one that preserves the most of its heat is the ball. To get \textbf{(9)} from \textbf{(10)}, let $t \to 0$ in \textbf{(10)}. The quantity $\int_{\mathbb{R}^n} 1_A \cdot e^{-t\Delta} 1_A dx$ is sometimes called the heat content of $A$. We will focus on statements of the form \textbf{(10)} below.

### 2.2. Gaussian Isoperimetry

For applications to theoretical computer science, it is more useful to have a Gaussian version of \textbf{(10)}. To this end, we first replace the heat semigroup by the Ornstein-Uhlenbeck semigroup. Define the operator $L$ by $Lf(x) := \langle x, \nabla f(x) \rangle + \Delta f(x)$, for any $x \in \mathbb{R}^n$. For any $x \in \mathbb{R}^n$ and $t \geq 0$, define the **Ornstein-Uhlenbeck semigroup** applied to $f : \mathbb{R}^n \to \mathbb{R}$ by

$$e^{-tL} f(x) := \int_{\mathbb{R}^n} f(x - t + y\sqrt{2}) d\gamma_n(y).$$

Let $A \subseteq \mathbb{R}^n$ be a Borel set. Denote $\gamma_n(A) := \int_A d\gamma_n(x)$. Let $H \subseteq \mathbb{R}^n$ be a half space. That is, $H$ is the region that lies on one side of a hyperplane. For brevity, we will not state the Gaussian analogue of \textbf{(9)}, and we instead state the Gaussian analogue of \textbf{(10)} \textbf{[Bor85]}, since the latter implies the former.

$$\text{(11)} \quad \gamma_n(A) = \gamma_n(H) \implies \forall t \geq 0, \quad \int_{\mathbb{R}^n} 1_A \cdot e^{-tL} 1_A d\gamma_n \leq \int_{\mathbb{R}^n} 1_H \cdot e^{-tL} 1_H d\gamma_n.$$  

The quantity $\int_{\mathbb{R}^n} 1_A \cdot e^{-tL} 1_A d\gamma_n$ is referred to as the **noise stability** of the set $A$ with parameter $e^{-t}$. Borell’s original result \textbf{(11)} from \textbf{[Bor85]} has been highly influential, and we wish to duplicate his success by generalizing \textbf{(11)}.

### 2.3. Social Choice Theory

By combining \textbf{(11)} with the invariance principle \textbf{(4)}, the work \textbf{[MOO10]} solved the Majority is StableView problem. This problem says that the most noise-stable way to determine the winner of an election between two candidates is to take the majority. This result assumes that no one person has too much influence over the election’s outcome. The rigorous statement of this problem uses functions with domain $\{-1, 1\}^n$ (see Section 3.1). Finally, inequality \textbf{(11)} should be considered well-understood, due to recent proofs \textbf{[MN15, Eld15]} of stability versions of \textbf{(11)}.

Applications of mathematics to the analysis of elections arguably began with Marquis de Condorcet in the 1700s, with further developments by Game Theorists such as Shapley, Shubik and Banzhaf in the 1950s and 1960s \textbf{[SS54, Ban65]}. In the last three decades, discrete Fourier analysis has provided new insights into social choice theory for mathematics and computer science \textbf{[KKL88, Kal02, MOO10, Mos12]}. Yet, the analogue of the Majority is Stablest Theorem for three candidates is still unresolved \textbf{[LM12]}, since the three set version of \textbf{(11)} is still unresolved (see \textbf{(12)} below).
2.4. Gaussian Isoperimetry for multiple sets. Borell’s inequality \( \text{(11)} \) gives a sharp computational hardness result for the MAX-CUT problem \( \text{[KKMO07]} \). The MAX-CUT problem asks for the partition of the vertices of an undirected graph into two disjoint sets that maximizes the number of edges going between the two sets. One can find a partition of the vertices achieving \( 0.87856 \) times the maximum number of cut edges in polynomial time \( \text{[GW95]} \). And assuming the Unique Games Conjecture, the constant \( 0.87856 \) is the best possible. When we modify the MAX-CUT problem to allow a partition of the vertices of the graph into \( k \geq 2 \) disjoint sets, we get the MAX-k-CUT problem. And for this problem, there is only a conjecture for the best possible approximation that can be done in polynomial time. This same conjecture would also solve the analogue of the Majority is Stablist problem for more than two candidates. This conjecture, formulated by Isaksson and Mossel in \( \text{[IM12]} \), is called the Standard Simplex Conjecture.

The Standard Simplex Conjecture asks for the minimum total Gaussian perimeter of a partition of \( \mathbb{R}^n \) into \( k \geq 2 \) sets, each of Gaussian measure \( 1/k \). This conjecture, stated in \( \text{(12)} \), does not seem to follow from a symmetrization argument \( \text{[BS01, IM12]} \), or from the methods of \( \text{[MN15, Eld15]} \).

Let \( A_1, \ldots, A_k \subseteq \mathbb{R}^n \) with \( 3 \leq k \leq n+1, n \geq 2 \), \( \cup_{i=1}^{k} A_i = \mathbb{R}^n \), \( \gamma_n(A_i) = 1/k \). We now describe the conjectured minimizer of the total Gaussian perimeter. Let \( z_1, \ldots, z_k \in \mathbb{R}^n \) be the vertices of a regular \( k \)-simplex, which is centered at the origin of \( \mathbb{R}^n \). For any \( i = 1, \ldots, k \), let \( B_i := \{ x \in \mathbb{R}^n : \langle x, z_i \rangle = \max_{j=1,\ldots,k} \langle x, z_j \rangle \} \). Then \( \{B_i\}_{i=1}^{k} \) is a partition of \( \mathbb{R}^n \) into \( k \) regular simplicial cones. Generalizing \( \text{(11)} \), the Standard Simplex Conjecture says

\[
\gamma_n(A_i) = 1/k, \forall i = 1, \ldots, k \quad \& \quad \bigcup_{i=1}^{k} A_i = \mathbb{R}^n
\]

\[
\implies \forall t \geq 0, \quad \sum_{i=1}^{k} \int_{\mathbb{R}^n} 1_{A_i} \cdot e^{-tL} 1_{A_i} d\gamma_n \leq \sum_{i=1}^{k} \int_{\mathbb{R}^n} 1_{B_i} \cdot e^{-tL} 1_{B_i} d\gamma_n.
\]

Morally speaking, the results of \( \text{[Eva93]} \) imply that, if equality holds in \( \text{(12)} \), then for all \( i = 1, \ldots, k \), the set \( \partial A_i \) should have constant mean curvature, except on a negligible subset. It is difficult to turn this intuition into a proof (unless equality holds for all \( t > 0 \) \( \text{[MS02]} \)), but this intuition explains why the Standard Simplex Conjecture \( \text{(12)} \) is believed to be true.

2.5. Our Contribution. For any \( n \geq 2 \), there exists \( t(n) > 0 \) such that for any \( t(n) < t < \infty \) and \( k = 3 \), the conjecture \( \text{(12)} \) holds, as I show in \( \text{[Hei14]} \). I use geometric and Fourier analytic arguments to show the first variation of \( \text{(12)} \) defines a contractive mapping, when restricted to partitions that almost achieve equality in \( \text{(12)} \).

In \( \text{[HMN16]} \), together with Mossel and Neeman, we show that if the measure restriction of \( \text{(12)} \) is changed, then the most natural restatement of the conjecture \( \text{(12)} \) is false. Specifically, if \( a_1, \ldots, a_k \) are real numbers with \( 0 < a_i < 1 \) for all \( i = 1, \ldots, k \) and \( \sum_{i=1}^{k} a_i = 1 \) with \((a_1, \ldots, a_k) \neq (1/k, \ldots, 1/k)\), and if \( t > 0 \), then the inequality \( \text{(12)} \) does not hold if we try to replace the sets \( \{B_i\}_{i=1}^{k} \) with any set of simplicial cones that partition Euclidean space. This negative result implies that all of our known proofs \( \text{[Bor85, BS01, MN15, Eld15]} \) of the case \( k = 2 \) of the Standard Simplex Conjecture (which is equivalent to Borell’s inequality \( \text{(11)} \)) do not work in the cases \( k \geq 3 \). That is, new methods are needed.

2.6. Potential Developments. Improvements on the result \( \text{[Hei14]} \) would yield: a solution of a multi-bubble problem in Gaussian space \( \text{[CCH+08, IM12, Led96]} \), the Plurality is Stablist Conjecture \( \text{[KKMO07, IM12, Theorem 1.10]} \), and optimal computational hardness results for approximating the MAX-k-CUT problem \( \text{[IM12, Theorem 1.13]} \). The former problem follows from \( \text{(12)} \) by letting \( t \rightarrow 0 \). The latter problem assumes the Unique Games Conjecture, from Section 1.6. Also, the Plurality is Stablist Conjecture says that the most noise-stable way to determine the winner of an election between \( k \) candidates is to take the plurality. This result assumes that no one person has too much influence over the election’s outcome.
3. The Propeller Conjecture

In an effort to better understand Grothendieck’s inequality [5], and to approximate the optima of kernel clustering problems from machine learning, Khot and Naor [KN09] [KN13] investigated Grothendieck’s inequality [5] for \( \{a_{ij}\}_{i,j=1}^n \) that are positive semidefinite.

3.1. Generalized Positive Semidefinite Grothendieck Inequalities. Suppose \( \{a_{ij}\}_{i,j=1}^n \) is a positive semidefinite matrix, i.e. a real symmetric matrix with all eigenvalues nonnegative. Let \( \nu_1, \ldots, \nu_k \in \mathbb{R}^k \) with \( k \geq 2 \), and let \( B = \{b_{ij}\}_{i,j=1}^k \) be the symmetric positive semidefinite matrix with \( b_{ij} := \langle \nu_i, \nu_j \rangle \). Then the Generalized Positive Semidefinite Grothendieck Inequality [KN13] Theorem 3.1, Theorem 3.3] says that there exists \( C(B) > 0 \) which does not depend on \( n \) or on \( \{a_{ij}\}_{i,j=1}^n \) such that

\[
\max_{\|w_i\|_2=1, \forall i=1, \ldots, n} \sum_{i,j=1}^n a_{ij} \langle w_i, w_j \rangle \leq \frac{1}{C(B)} \cdot \max_{\sigma: \{1,2,\ldots,n\} \rightarrow \{1,2,\ldots,k\}} \sum_{i,j=1}^n a_{ij} \langle \nu_{\sigma(i)}, \nu_{\sigma(j)} \rangle. \tag{13}
\]

Instead of rounding the vectors \( w_i, i = 1, \ldots, n \) to 1 or \(-1\) as in Grothendieck’s inequality [5], Khot and Naor prove [13] by rounding the vectors \( w_i, i = 1, \ldots, n \) to \( k \) vectors \( \nu_i, i = 1, \ldots, k \). Also, the best constant \( 1/C(B) \) is found by finding the best rounding procedure.

3.2. The Sharp Constant of Grothendieck Inequalities. Let \( I_k \) denote the \( k \times k \) identity matrix. In [KN09] [KN13], it is shown that \( C(B) \) can be found by solving a finite dimensional optimization problem, whose parameters depend on \( B \). However, this optimization problem is non-convex in general, so standard methods cannot compute \( C(B) \). The Propeller Conjecture guesses the value of \( C(B) \) for \( B = I_k, k \geq 4 \):

\[
C(I_k) := \sup_{A_1, \ldots, A_k: \cup_{i=1}^k A_i = \mathbb{R}^{k-1}, \gamma_{k-1}(A_i \cap A_j) = 0, \forall i \neq j} \sum_{i=1}^k \left\| \int_{A_i} x d\gamma_{k-1}(x) \right\|_2^2 = \frac{9}{8\pi} = C(I_3). \tag{14}
\]

The constant \( C(I_3) = 9/(8\pi) \) is computed in [KN09] using Lagrange multipliers, but \( C(I_4) \) does appear to be computable using this technique.

3.3. Our Contribution. Together with Naor and Jagannath, using some theoretical results and a brute force search, we give a computer-assisted proof of the Conjecture [14] in the case \( k = 4 \) [HJN13]. The analytic results use a connection between the maximization problem [14] and discrete harmonic maps into the sphere. Intuition derived from this connection allows several ad hoc arguments to rule out candidates for partitions that maximize [14].

3.4. Potential Developments. Solving [14] for all \( k \geq 4 \) would yield better understanding of semidefinite programming algorithms and potential insight into computing the best constant in Grothendieck’s inequality [5].

4. The Symmetric Gaussian Problem

4.1. Gaussian Perimeter of Symmetric Sets. The Symmetric Gaussian Problem [15] below was first mentioned in [Bar01] (for the Gaussian perimeter), and then restated in [CR11] [O’D12] in relation to the Gap-Hamming-Distance problem from communication complexity. The Symmetric Gaussian Problem is a conjectural variant of Borell’s Theorem [11] that says the most noise stable symmetric sets in Euclidean space are balls or their complements. In other words, when we restrict
Borell’s Theorem \cite{Borell} to symmetric sets (so that half spaces are no longer considered), then ball’s or their complements have the smallest conjectured perimeter:

\[
A = -A \quad \land \quad \gamma_n(A) = \gamma_n(B(0, r)) = \gamma_n(B(0, r')) \quad \Longrightarrow \quad \forall t \geq 0,
\]

\[\int_{\mathbb{R}^n} 1_A \cdot e^{-tL} 1_A d\gamma_n \leq \max \left( \int_{\mathbb{R}^n} 1_{B(0, r)} \cdot e^{-tL} 1_{B(0, r')} d\gamma_n, \int_{\mathbb{R}^n} 1_{B(0, r')} \cdot e^{-tL} 1_{B(0, r)} d\gamma_n \right).\]

This problem exhibits many of the difficulties of the other conjectures \cite{Hei14} and \cite{MN14}, though the present problem seems more tractable.

4.2. Our Contribution. In \cite{Hei15}, I show that \[15\] is true for \( n = 1 \) and \( t \) sufficiently large, by simplifying the argument from my work on the Standard Simplex Conjecture \cite{Hei14}. When \( n \geq 2 \), I demonstrate that the Conjecture \[15\] can sometimes be false, depending on the size of the measure \( \gamma_n(A) \). More specifically, I compute the second variation of the ball and its complement, adapting a second variation argument from \cite{CS07}. It turns out that the computation of the second variation of noise stability of the ball essentially reduces to proving an \( L_2 \) Poincaré inequality on the sphere.

The main point of this work is to successfully use a second variation argument for noise stability. Previous work on noise stability problems focused on first variation (i.e. first derivative) arguments. So, the computation of the second variation of noise stability was a largely missing piece from the literature. And Conjecture \[15\] served as a test-case for this technique. As discussed in Section 6 below, I am currently working on better understanding the second variation of noise stability for other related problems, e.g. from Sections 2 and 3.

5. Strong Contractivity and Kahn-Kalai-Linial

5.1. Discrete Analysis and Hypercontractivity. Expander graphs, i.e. graphs with bounded degrees and large spectral gaps, have been studied extensively in both pure and applied mathematics \cite{HLW06}. In the paper \cite{MN14}, the authors construct a family of graphs that have a spectral gap with respect to any uniformly convex Banach space. That is, these graphs are expander graphs in a much stronger sense than the usual definition of expander graphs. In order to improve their expander graph construction, Mendel and Naor made a conjecture concerning the decay of the heat semigroup in \( L_p \) spaces. The conjecture of \cite{MN14} can be understood as an attempt to develop Littlewood-Paley theory for non-doubling metric spaces. For simplicity, we state this conjecture only in the case of real-valued functions.

Let \( n \) be a positive integer. Let \( f : \{-1, 1\}^n \to \mathbb{R} \) be a function. Let \( \mu \) be the uniform probability measure on the hypercube, so that \( \mu(x) = 2^{-n} \) for all \( x \in \{-1, 1\}^n \). Any \( f : \{-1, 1\}^n \to \mathbb{R} \) can be written as \( f = \sum_{S \subseteq \{1, \ldots, n\}} \hat{f}(S) W_S \), where for all \( x = (x_1, \ldots, x_n) \in \{-1, 1\}^n \), \( W_S(x) := \prod_{i \in S} x_i \) and \( \hat{f}(S) := \int_{\{-1, 1\}^n} f(x) W_S(x) d\mu(x) \). For any \( t \geq 0 \), define \( e^{-tL} f := \sum_{S \subseteq \{1, \ldots, n\}} e^{-t|S|} \hat{f}(S) W_S \), \( Lf := \sum_{S \subseteq \{1, \ldots, n\}} |S| \hat{f}(S) W_S \), and \( \forall 1 \leq p < \infty \), define \( \|f\|_p := (\int_{\{-1, 1\}^n} |f(x)|^p d\mu(x))^{1/p} \). For all \( i \in \{1, \ldots, n\} \), define the influence \( I_i f \) of the \( i \)th variable on \( f \) by

\[ I_i f := \sum_{S \subseteq \{1, \ldots, n\} : i \in S} (\hat{f}(S))^2. \]

Let \( k \geq 1 \), \( k \in \mathbb{Z} \). The Conjecture \cite{MN14} Remark 5.5] says: \( \forall p > 1, \exists c(p) > 0 \) such that

\[ \hat{f}(S) = 0 \quad \forall S \subseteq \{1, \ldots, n\} \text{ with } |S| < k \quad \Longrightarrow \quad \forall t > 0, \quad \|e^{-tL} f\|_p \leq e^{-tkc(p)}\|f\|_p. \]

Equivalently, Conjecture \[16\] is a “higher order” Poincaré inequality: \( \forall p > 1, \exists c(p) > 0 \) such that

\[ \hat{f}(S) = 0 \quad \forall S \subseteq \{1, \ldots, n\} \text{ with } |S| < k \quad \Longrightarrow \quad \int |f|^{p-1} \text{sign}(f) L f d\mu \geq kc(p) \int |f|^p d\mu. \]
A weaker form of (16) with the term $e^{-tkx(p)}$ replaced by $e^{-\min(t, t^2)kc(p)}$ can be proven using Hölder’s inequality and the hypercontractive inequality [Sta59, Fed69, Bon70, Nel73, Gro75, Bec75].

(17) \[ \forall 1 < p < q < \infty, \forall t > \frac{1}{2} \log \left( \frac{q - 1}{p - 1} \right), \quad \|e^{-tL}f\|_q \leq \|f\|_p. \]

Using the hypercontractive inequality (17), Kahn, Kalai and Linial proved the following famous inequality [KKL88], resolving a conjecture of Ben-Or and Linial.

**Theorem 5.1 (Kahn-Kalai-Linial).** [KKL88, Theorem 3.1] There exists a universal constant $c > 0$ such that, for any $f : \{-1, 1\}^n \to \{-1, 1\}$, we have $\max_{i=1,\ldots, n} I_i f \geq c(\int [f - \int f \, d\mu]^2 d\mu)(\log n)/n$.

This Theorem says that a discrete function with values in $\{-1, 1\}$ must have some asymmetry in its Fourier coefficients. To see this, note that it is easy to construct a function $f : \{-1, 1\}^n \to \mathbb{R}$ such that $\max_{i=1,\ldots, n} I_i f \leq 10(\int [f - \int f \, d\mu]^2 d\mu)/n$, just by choosing $f$ such that $\widehat{f}(S) = 2/(n(n-1))$ for all $|S| = 2$ and such that $\widehat{f}(S) = 0$ for all other $S \subseteq \{1, \ldots, n\}$. Note that the Fourier coefficients of $f$ are then symmetric with respect to permutations on the set $\{1, \ldots, n\}$, but $f$ does not take values $\{-1, 1\}$, so Theorem 5.1 does not apply.

5.2. **Our Contribution.** In [HMO14], together with Mossel and Oleszkiewicz, we prove the case $k = 1$ of the Conjecture (16) of Mendel and Naor. In fact, we prove (16) for any probability space with a symmetric Markov semigroup $P_t$ whose generator $L := -\frac{\partial}{\partial t}P_t|_{t=0+}$ satisfies an $L_2$ Poincaré inequality. We then answer a question of Hatami and Kalai, showing that Theorem 5.1 cannot be strengthened unless a logarithmic number of Fourier coefficients of the function $f$ vanish. That is,

**Theorem 5.2 (HMO14).** There exists $c > 0$ such that, for all $n \in \mathbb{N}$, there exists $f : \{-1, 1\}^n \to \{-1, 1\}$ with $\widehat{f}(S) = 0$ for all $S \subseteq \{1, \ldots, n\}$ with $|S| \leq \log n$ such that $\max_{i=1,\ldots, n} I_i f \leq c(\log n)/n$.

In the case that $\widehat{f}(S) = 0$ for all $S \subseteq \{1, \ldots, n\}$ with $|S| \leq C(n) \log n$, the equality $\sum_{i=1}^n I_i f = \sum_{S \subseteq \{1, \ldots, n\}} |S| (\widehat{f}(S))^2$ shows that $\max_{i=1,\ldots, n} I_i f \geq C(n)(\log n)/n$. So, there is a phase transition in the possible behavior of the maximum influence $\max_{i=1,\ldots, n} I_i f$, which occurs when $C(n) > 0$ is bounded or unbounded as $n \to \infty$.

Finally, we demonstrated a generalization of Talagrand’s inequality for functions $f : \{-1, 1\}^n \to \{-1, 1\}$ with $\widehat{f}(S) = 0$ for all $S \subseteq \{1, \ldots, n\}$ with $|S| < k$. The usual Talagrand inequality then corresponds to the case $k = 1$.

**Theorem 5.3 (HMO14).** Let $k \geq 1$. Let $f : \{-1, 1\}^n \to \mathbb{R}$ with $\widehat{f}(S) = 0$ for all $S \subseteq \{1, \ldots, n\}$ with $|S| < k$. \(\forall i = 1, \ldots, n\), let $\frac{\partial}{\partial x_i} f(x) := [f(x_1, \ldots, x_n) - f(x_1, \ldots, x_{i-1}, -x_i, x_{i+1}, \ldots, x_n)]/2$, where $x = (x_1, \ldots, x_n) \in \{-1, 1\}^n$. Then

(18) \[ \|f\|_2^2 \leq 6 \sum_{i=1}^n \frac{k + \log (\|\frac{\partial}{\partial x_i} f\|_2/\|\frac{\partial}{\partial x_i} f\|_1)}{\|\frac{\partial}{\partial x_i} f\|_2}. \]

5.3. **Potential Developments.** Proving the case $k > 1$ of the Conjecture (16) of Mendel and Naor would give improved understanding of analysis in non-Euclidean spaces, going beyond (or supplementing) Littlewood-Paley theory in non-doubling metric measure spaces. Also, the solution of this problem would improve our understanding of expander graphs.

6. **Forthcoming Work**

6.1. **Noise Stability and the Calculus of Variations.** My first goal [Hei16b, Hei16a] is to extend the variational methods of [Hei15] and [BB16] to the periodic isoperimetric problem from [KM16]. Recently, [KM16] essentially showed that the Unique Games Conjecture for the field of
two elements (that is, using $p = p(\varepsilon) = 2$ in Section 1.6) follows from the following conjecture. Let $A \subseteq \mathbb{R}^n$ with $-A = A^c$ and such that $A + e = A^c$ for every standard basis vector $e \in \mathbb{R}^n$. That is, let $A$ be a “periodized set.” Then the noise stability of $A$ is at most the noise stability of the set

$$H := \left\{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n : \sin(\pi \sum_{i=1}^{n} x_i) \geq 0 \right\}.$$ 

Note that the boundary $\partial H$ of $H$ is a set of hyperplanes of the form

$$\{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n : \exists k \in \mathbb{Z} \text{ such that } x_1 + \cdots + x_n = k \}.$$ 

For this reason, $H$ is called a “periodized half space.”

The Unique Games Conjecture for the field of two elements is expected to contain the same difficulties as the full conjecture [KM16]. So, working on this isoperimetric problem is tantamount to working directly on the Unique Games conjecture itself.

I also hope to extend the variational methods of [Hei15] and [BBJ16] to the Standard Simplex Conjecture (12) and to the Propeller Conjecture (14). Previous investigations of Gaussian isoperimetry and Gaussian noise stability all use “global” methods [Bor85, Bor03, MN15, Eld15]. That is, these investigations all use that the translation of a half space is still a half space. Many noise stability problems lack this translational symmetry. For example, in conjecture (15), if a set $A \subseteq \mathbb{R}^n$ satisfies $A = -A$, then a translation of $A$ no longer satisfies $A = -A$. Sometimes, this lack of translational symmetry is unexpected, as we showed in [HMN16]. However, a recent work [BBJ16] managed to use the calculus of variations to prove the Gaussian isoperimetric inequality (11) as $t \to 0$). Since the calculus of variations is an inherently local method, there is hope for the work [BBJ16] to perhaps apply to other related problems.

Already, it appears that my main result [Hei14] for the Standard Simplex Conjecture can be recovered with a relatively straightforward second variation argument. That is, the rather complicated argument of [Hei14] can be replaced by a rather intuitive an automatic calculation. It remains to try to improve the range of parameters of this result.

A key aspect of [BBJ16] is explicitly writing down variations that increase the noise stability of a set. These variations happen to be the eigenfunctions of an Ornstein-Uhlenbeck operator on the boundary of the set. These eigenfunctions were identified in [CM12], where a maximal version of the Gaussian perimeter called an “entropy” was studied, for surfaces that are self-similar under the mean curvature flow. The Ornstein-Uhlenbeck operator they study comes from the second-variation formula for the Gaussian perimeter. The idea of studying the eigenfunctions of an operator restricted to the surface seems to be due to Simons [Sim68]. And the idea of studying an “entropy” associated to surfaces seems to have originated in Perelman’s proof of the Poincaré Conjecture, where he introduced his entropy functional. And Perelman’s investigation of “entropy” was influenced by the hypercontractive inequality (17).

### 6.2. Hypercontractivity for Markov Chains

My second goal is to prove hypercontractive inequalities on general finite graphs [GRST14, EM12, LV09]. Other than those methods developed in [DSC96], there is currently no reliable way to prove a reasonably sharp hypercontractive inequality on a given graph. One of the main ways to prove a hypercontractive inequality on a manifold is to use curvature bounds. Some theories of Ricci curvature for metric-measure spaces have received recent attention [Stu06, LV09, Oll09, Mie13, AGS14, JL14, GRST14, EKS15, BM15, Pan16, FS15, KKRT16], but it is unclear what role these theories can play in proving hypercontractive inequalities. For example, perhaps one might be able to show that a graph has some curvature bound, but it may be more technically feasible to just prove a hypercontractive inequality directly, instead of first proving a curvature bound. With this issue in mind, there have been a great deal of recent attempts to define and study discrete versions of curvature on graphs [Oll09]. Just as curvature on
manifolds is associated with differential operators (by Bochner’s equality and semigroup theory), curvature theories on graphs are associated with Markov chains.

As an intermediate goal, it would be desirable to prove that a certain class of random graphs, e.g. random Cayley graphs, satisfies a good hypercontractive inequality with positive probability. I am currently working on this problem with Professor Georg Menz [HM16]. Also, we demonstrate that on the discrete square, essentially all Markov chains have negative curvature, while they satisfy hypercontractive inequalities. That is, certain curvature theories seem ineffective for proving hypercontractive inequalities, even in the simple example of a square.

6.3. Noncommutative Hypercontractivity and Free Probability. My third goal is to investigate noncommutative hypercontractive inequalities in general settings. This project is an outgrowth of my previous joint work [HV16]. Noncommutative hypercontractive inequalities [Gro72, CL93, Bia97, JPP+15, JPPP15, JZ15, RX16] have received a good deal of recent attention, due to applications in quantum information theory and free probability. I am currently working on this problem with Professor Dimitri Shlyakhtenko. We are considering some noncommutative versions of Bézout’s Theorem [DTW76], and hypercontractive constants for the free group von Neumann algebra [Bia97, JPPP13, JPP+15, RX16].

References


[BKS16] Afonso S. Bandeira, Christopher Kennedy, and Amit Singer, Approximating the little Grothendieck problem over the orthogonal and unitary groups, Mathematical Programming (2016), 1–43.


Vertex cover might be hard to approximate to within $2^{-\varepsilon}$.


