Discussion Notes for Methods of Applied Math (MATH 146)

UCLA
Winter 2018

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Purpose: This document is a compilation of notes generated for discussion in MATH 146 with reference credit due John L. Troutman’s text *Variational Calculus and Optimal Control* [2]. If the reader finds any errors/typos, please feel free to email me at heaton@math.ucla.edu and I will address these and post an updated set of notes to my webpage.

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INTRODUCTION

These notes are provided to compliment the TA discussion sessions on Thursdays for MATH 146. Typically, more detail is provided here than on the board during discussions since portions of solutions are given orally in class. The examples provided here are meant to be a constructive reference for students. These illustrate how to use set up the variational problems we will see this quarter, what details are important to include, and provide example of acceptable solution presentation. Before reading each solution, I highly encourage students to first seriously attempt the problems on their own. I cannot overstate the value of struggling through these problems before comparing your attempts to the example solutions.

These notes will be updated weekly (if not more often), reflecting the current discussion material.
**Definition:** Define $f : [a, b] \rightarrow \mathbb{R}$. For any $x \in [a, b]$, define the quotient

$$
\phi(t) := \frac{f(t) - f(x)}{t - x} \quad (a < t < b, \ t \neq x), \quad (1)
$$

and define

$$
f'(x) := \lim_{t \to x} \phi(t), \quad (2)
$$

provided the limit exists. We associate the function $f'$ with $f$ at the points where the limit (2) exists. The function $f'$ is called the **derivative** of $f$. If $f'$ is defined at a point $x$, we say $f$ is **differentiable** at $x$. And if $f'$ is defined at every point in a set $I \subset [a, b]$, then we say $f$ is **differentiable on** $I$. △

**Example 1:** Use the above definition to compute $f'(1)$ for the function $f(x) = x^2$.

**Solution:**

Through direct computation, we find

$$
f'(1) = \lim_{t \to 1} \frac{f(t) - f(1)}{t - 1}
= \lim_{h \to 0} \frac{f(1 + h) - f(1)}{(1 + h) - 1}
= \lim_{h \to 0} \frac{(1 + h)^2 - 1}{h}
= \lim_{h \to 0} \frac{1 + 2h + h^2 - 1}{h}
= \lim_{h \to 0} \frac{2h + h^2}{h}
= \lim_{h \to 0} 2 + h
= 2 + 0
= 2.
$$

□
Taylor’s Theorem: Let $I \subset \mathbb{R}$ be a neighborhood of $x_0$ and $n$ be a nonnegative integer. Suppose the function $f : I \to \mathbb{R}$ has $n + 1$ derivatives. Then for each point $x \neq x_0$ in $I$ there is a point $\xi$ strictly between $x$ and $x_0$ such that

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k + \frac{f^{(n+1)}(\xi)}{(n+1)!} (x - x_0)^{n+1}. \quad (4)$$

Remark 1: The second term in (4) is known as the Lagrange Remainder. \hfill \bigtriangleup

Consider using Taylor’s theorem when $n = 1$. That is, suppose $f$ is twice differentiable at $x$ and define

$$\varepsilon(h) := \frac{f^{(2)}(\xi(h))}{2} h^2 \quad (5)$$

where $\xi(h)$ is the point strictly between $x$ and $x + h$ such that

$$f(x + h) = f(x) + f'(x)h + \varepsilon(h), \quad (6)$$

which we know exists by Taylor’s theorem. This form of expansion will be useful for us to remember when we look at differentiation of more abstract quantities known as functionals. Furthermore, this shows

$$f'(x) = \lim_{h \to 0} f'(x) = \lim_{h \to 0} \left( \frac{f(x + h) - f(x) - \varepsilon(h)}{h} \right) = \lim_{h \to 0} \left( \frac{f(x + h) - f(x)}{h} - \frac{\varepsilon(h)}{h} \right) = f'(x) - \lim_{h \to 0} \frac{\varepsilon(h)}{h}. \quad (7)$$

Thus $\lim_{h \to 0} \varepsilon(h)/h = 0$. Using little-oh notation (defined below), we write this as $\varepsilon(h) = o(h)$.

Definition: Assume $g(x)$ is nonzero. Then we say $f(x) = o(g(x))$ as $x \to x^*$ provided

$$\lim_{x \to x^*} \left| \frac{f(x)}{g(x)} \right| = 0. \quad (8)$$

This notation is referred to as little-oh notation. \hfill \bigtriangleup
Example 2: Define \( f(x) := x^2 \). Express \( f(x + h) \) explicitly in the form of (6).

Solution:
First observe \( f'(x) = 2x \) and \( f''(x) = 2 \). Then we see

\[
f(x + h) = (x + h)^2 = x^2 + 2xh + h^2 = f(x) + f'(x)h + \varepsilon(h) \tag{9}
\]

where \( \varepsilon(h) := h^2 \).

We now turn our attention to a necessary condition for a point \( x \) to be a local minimizer of \( f \).

Theorem: If \( f : \mathbb{R} \to \mathbb{R} \) is a continuously differentiable function and \( x \) is a local minimizer of \( f \), then \( f'(x) = 0 \).

Proof:
Let \( x \) be a minimizer of \( f \), i.e., there is a \( \delta^* > 0 \) such that \( f(x) \leq f(x) \) for all \( x \in (x - \delta^*, x + \delta^*) \). We proceed by way of contradiction, i.e., suppose \( f'(x) \neq 0 \). By hypothesis \( f' \) is continuous, and so there is a \( \delta > 0 \) such that

\[
|z - x| < \delta \implies |f'(z) - f'(x)| < \frac{|f'(x)|}{2}. \tag{10}
\]

But, using the reverse triangle inequality, we see

\[
|f'(x)| - |f'(z)| \leq |f'(z) - f'(x)| < \frac{|f'(x)|}{2} \implies \frac{|f'(x)|}{2} < |f'(z)|. \tag{11}
\]

Suppose \( f'(x) > 0 \) and pick \( z \in (x - \delta/2, x) \). Taylor’s theorem asserts there is \( \xi \in (z, x) \) such that

\[
f(z) = f(x) + f'(\xi)(z - x) = f(x) - f'(\xi)|z - x| < f(x) - \frac{|f'(x)|}{2}|z - x| < f(x). \tag{12}
\]

This shows \( f(z) < f(x) \) for all \( z \in (x - \delta/2, x) \). Thus \( x \) cannot be a local minimizer of \( f \), contradicting our initial assumption. Whence \( f'(x) \leq 0 \). By analogous argument to above, if instead \( f'(x) < 0 \), we pick \( z \in (x, x + \delta/2) \) to deduce

\[
f(z) = f(x) + f'(\xi)(z - x) = f(x) + f'(\xi)|z - x| < f(x) - \frac{|f'(x)|}{2}|z - x| < f(x), \tag{13}
\]

again giving a contradiction. This shows \( f'(x) \geq 0 \). Therefore, combining our results, we conclude \( f'(x) = 0 \), as desired.
Remark 2: The above theorem shows that a necessary condition for \( \bar{x} \) to be a local minimizer of \( f \) is that \( f'(\bar{x}) = 0 \). Below we provide several examples illustrating the use and limitations of this theorem.

Example 3: Define \( f(x) = (x - 3)^2 + 5x + 3 \). Solve the optimization problem

\[
\min_{x \in \mathbb{R}} f(x),
\]

using only the above theorem and definition of a minimizer.

Solution:

First note \( f \) is continuously differentiable since it is a polynomial. And,

\[
f'(x) = 2(x - 3) + 5 + 0 = 2x - 1.
\]

The single critical point of \( f \) is at \( x = 1/2 \). The above theorem shows this is the only candidate solution to the optimization problem.

All that remains is to verify \( x = 1/2 \) is, in fact, a minimizer. We can rewrite \( f \) as \( f(x) = x^2 - x + 12 \). Pick any \( z \in \mathbb{R} \) and set \( \delta := z - 1/2 \) so that \( z = 1/2 + \delta \). Then

\[
f(z) = f \left( \frac{1}{2} + \delta \right) = \left( \frac{1}{2} + \delta \right)^2 - \left( \frac{1}{2} + \delta \right) + 12
\]

\[
= \left( \frac{1}{4} + \delta + \delta^2 \right) - \left( \frac{1}{2} + \delta \right) + 12
\]

\[
= \left( \frac{1}{4} + \frac{1}{2} + 12 \right) + \delta^2
\]

\[
= f \left( \frac{1}{2} \right) + \delta^2
\]

\[
\geq f \left( \frac{1}{2} \right).
\]

This shows \( f(1/2) \leq f(z) \) for all \( z \in \mathbb{R} \), i.e., \( 1/2 \) is the global minimizer of \( f \), and we are done. \( \square \)
Example 4: Define $f(x) = x^3$. Can the above theorem be applied to find a local minimum?

Solution:
Observe $f'(x) = 3x^2$ and so $f'(x) = 0$ if and only if $x = 0$. But, $f(0) = 0 > -\varepsilon^3 = f(-\varepsilon)$ for every $\varepsilon > 0$ and so $0$ is not a local minimum of $f$. Thus the above theorem cannot be applied to find a local minimum. Moreover, because this was the only candidate for a minimizer, we are able to further conclude $f$ has no global minimizer over $\mathbb{R}$. □

Remark 3: The above theorem shows that the condition $f'(x) = 0$ is necessary, but not sufficient. We illustrate this again with the following example.

Example 5: Define $f(x) = -x^2$. Can the above theorem be applied to find a local minimum?

Solution:
Observe $f'(x) = -2x$ and so $f'(x) = 0$ if and only if $x = 0$. But, $f(z) = -z^2 < 0 = f(0)$ for all $z \neq 0$. This shows $0$ is not a local minimum of $f$. Thus the above theorem cannot be applied to find a local minimum. In fact, the above shows $x = 0$ is a global maximizer of $f$. □

Example 6: Define $f(x) := 3|x - 5|$. What is the global minimizer of $f$ and can the above theorem be applied? Explain.

Solution:
The global minimizer is $x = 5$. Indeed,

$$f(5) = 0 \leq 3|x - 5| = f(x) \quad \forall \ x \in \mathbb{R}. \quad (17)$$

However, $f$ is not continuously differentiable since $f'$ is not continuous at $x = 5$. Indeed,

$$\lim_{x \to 5^{-}} f'(x) = -3 \neq 3 = \lim_{x \to 5^{+}} f'(x). \quad (18)$$

Thus a condition for the theorem does not hold and so it cannot be applied. □
Remark 4: Here we review integration by parts. Let $f, g \in C^1[a,b]$. Then using the product rule we write
\[
\frac{d}{dx} [f(x)g(x)] = f'(x)g(x) + f(x)g'(x).
\] (19)
So,
\[
\int_a^b \frac{d}{dx} [f(x)g(x)] \, dx = \int_a^b f'(x)g(x) + f(x)g'(x) \, dx.
\] (20)
But, the left hand side can be rewritten as
\[
\int_a^b \frac{d}{dx} [f(x)g(x)] \, dx = \int_{f(a)g(a)}^{f(b)g(b)} d(fg) = [f(x)g(x)]_{x=a}^{x=b} = f(b)g(b) - f(a)g(a).
\] (21)
Thus the integration by parts formula becomes
\[
\int_a^b f(x)g'(x) \, dx = -\int_a^b f'(x)g(x) \, dx + [f(x)g(x)]_{x=a}^{x=b}.
\] (22)
This will be especially useful tool for us and is important to have at our disposal. \diamond
Simple Methods for Finding Minimizers of $J : V \to \mathbb{R}$

Remark 5: For the following problems, we proceed roughly by taking the following steps.

1. Find a lower bound $\ell$ for $J(y)$ (the tightest lower bound we can establish).

2. Find a collection of candidates $y$ for which $J(y)$ equals this lower bound, i.e., $J(y) = \ell$.

3. Any functions in the collection of candidates contained in $\mathcal{A}$ are minimizers. If at least one candidate is contained in $\mathcal{A}$, then $\ell$ is the minimum.

This approach works when the minimizer is obtained. However, what can we do when it isn’t obtained? In this case, we might attempt to do as follows.

1. Find a lower bound $\ell$ for $J(y)$ (the tightest lower bound we can establish).

2. Find a collection of candidates $y$ for which $J(y)$ equals this lower bound.

3. If none of the candidates $y$ is admissible (i.e., $y \notin \mathcal{A}$), we must find a sequence $\{y^n\}_{n=1}^{\infty}$ of functions in $\mathcal{A}$ for which

$$\lim_{n \to \infty} J(y^n) = \ell. \quad (23)$$

Then $\ell$ is the infimum and it is not obtained.
Definition: Let $V$ be a vector space and $J: V \to \mathbb{R}$ be a mapping. Let $\mathcal{A}$ be a subset of $V$, i.e., $\mathcal{A} \subset V$. Then we say $y \in \mathcal{A}$ is a global minimizer of $J$ over $\mathcal{A}$ provided

$$J(y) \leq J(z) \quad \forall \ z \in \mathcal{A}. \quad (24)$$

\[\triangle\]

Example 7: Define the admissibility class $\mathcal{A} := C[0,2]$ and let $J: C[0,2] \to \mathbb{R}$ be the functional defined by

$$J(y) := \int_{0}^{2} [y(x) - 9]^{2} + 7 \, dx. \quad (25)$$

Find the minimum of $J$ over $\mathcal{A}$. What is the minimizer?

Solution:

Let $y \in \mathcal{A}$. Then

$$J(y) = \int_{0}^{2} [y(x) - 9]^{2} + 7 \, dx \geq \int_{0}^{2} 0 + 7 \, dx = 14, \quad (26)$$

where the inequality holds since $[y(x) - 9]^{2} \geq 0$ for all possible values of $y(x)$. This shows 14 is a lower bound for $J(y)$. To verify this is the minimum for $J(y)$, it suffices to find $f \in \mathcal{A}$ such that $J(f) = 14$. This is accomplished if and only if the inequality in (26) is a strict equality. The only candidate is $f(x) = 9$ since this would give

$$[f(x) - 9]^{2} = [9 - 9]^{2} = 0^{2} = 0. \quad (27)$$

Indeed, this implies

$$J(y) \geq 14 = J(f) \quad \forall \ y \in \mathcal{A}. \quad (28)$$

Because $f$ is constant, it is continuous on $[a, b]$, and so $f \in \mathcal{A}$. Thus we conclude $\boxed{f(x) = 9}$ is the minimizer of $J(y)$ over $\mathcal{A}$ and $\boxed{14}$ is the minimum of $J(y)$ over $\mathcal{A}$. \[\square\]
**Example 8:** Define the admissibility class $\mathcal{A} := \{ f \in C[a,b] : f(x) \geq 5 \}$ and let $J : C[a,b] \to \mathbb{R}$ be the functional defined by

$$J(y) := \int_a^b y(x)^2 - 8y(x) + 20 \, dx. \tag{29}$$

Find the minimum of $J$ over $\mathcal{A}$. What is the minimizer?

**Solution:**

Let $y \in \mathcal{A}$. Then

$$J(y) = \int_a^b y(x)^2 - 8y(x) + 20 \, dx$$

$$= \int_a^b (y(x)^2 - 8y(x) + 16) + 4 \, dx$$

$$= \int_a^b (y(x) - 4)^2 + 4 \, dx$$

$$\geq \int_a^b (5 - 4)^2 + 4 \, dx$$

$$= \int_a^b 5 \, dx$$

$$= 5(b-a). \tag{30}$$

This shows $5(b-a)$ is a lower bound for $J(y)$. To verify this is the minimum for $J(y)$, it suffices to find $f \in \mathcal{A}$ such that $J(f) = 5(b-a)$. This is accomplished if and only if the inequality in (30) is a strict equality. The only candidate is $f(x) = 5$. Since $f$ is continuous on $[a,b]$ and $f \geq 5$, we see $f \in \mathcal{A}$. Thus we conclude $\boxed{f(x) = 5}$ is the minimizer of $J(y)$ over $\mathcal{A}$ and $\boxed{5(b-a)}$ is the minimum of $J(y)$ over $\mathcal{A}$. \hfill \square

**Remark 6:** Note in the above example we say $f(x) = 5$ is “the” minimizer. This is because the is the only function in $\mathcal{A}$ that gives $J(f) = 5(b-a)$. In the next example, multiple minimizers exist. \hfill ◊

**Remark 7:** Note $\mathcal{A}$ does not form a vector space in the following example. This follows from the fact it is not closed under scalar multiplication. For example, if $f \in \mathcal{A}$, then $-f \notin \mathcal{A}$. \hfill ◊
Example 9: Define the admissibility class $A := \{ f \in C[0, 1] : f(x) \geq x^2 - 10x + 28 \}$. Then let $J : C[0, 1] \to \mathbb{R}$ be the functional defined by

$$J(f) := \inf_{x \in [0, 1]} f(x).$$

(31)

Find $\inf_{f \in A} J(f)$. Does $J(f)$ attain its infimum?

**Solution:**

Let $f \in A$. Then, for each $x \in [0, 1],$

$$f(x) \geq x^2 - 10x + 28 = (x^2 - 10x + 25) + 3 = (x - 5)^2 + 3.$$  

(32)

Set $g(x) := (x - 5)^2 + 3$. Also note $g'(x) = 2(x - 5) < 0$ for $x < 5$, and so $g$ is strictly decreasing on $[0, 1]$. This implies $\inf_{x \in [0,1]} g(x) = g(1)$. Using this fact, we see

$$J(f) = \inf_{x \in [0, 1]} f(x) \geq \inf_{x \in [0,1]} g(x) = g(1) = (1 - 5)^2 + 3 = 19.$$  

(33)

This shows $J(f) \geq 19$, i.e., 19 is a lower bound. Moreover, because $g$ is a polynomial, it is continuous. Whence $g \in A$ and

$$J(f) \geq J(g) = 19 \quad \forall \ f \in A.$$  

(34)

Thus $g$ is a minimizer of $J$ over $A$ and so $\boxed{\inf_{f \in A} J(f) = 19}$. Yes, $J(f)$ attains its infimum. □

**Remark 8:** Note in the above example we say $g$ is “a” minimizer. In general, there may be multiple minimizers. For instance, in the above example consider defining $q(x) := g(x) + (x - 1)^2$. Then $q \in C[0, 1]$ and $q(x) = g(x) + (x - 1)^2 \geq g(x)$, which implies $q \in A$. Moreover,

$$q'(x) = g'(x) + 2(x - 1) = 2(x - 5) + 2(x - 1) \leq 2(x - 5) + 0 < 0 \quad \forall \ x \in [0, 1].$$  

(35)

This shows $q$ is strictly decreasing on $[0, 1]$. Thus

$$J(q) = \inf_{x \in [0,1]} q(x) = q(1) = g(1) + (1 - 1)^2 = g(1) = 19.$$  

(36)

This shows $g$ and $q$ are minimizers of $J$ over $A$. ◊
Example 10: Define the function $h : \mathbb{R} \to \mathbb{R}$ by
\[
h(x) := \begin{cases} 
0 & \text{if } |x| < 1, \\
1 & \text{if } |x| \geq 1.
\end{cases}
\] (37)

Define the admissibility class $\mathcal{A} := \{ f \in C^1(\mathbb{R}) : f(x) \geq h(x) \}$. Then let $J : C^1(\mathbb{R}) \to \mathbb{R}$ be the functional
\[
J(y) := \int_{-1}^{1} y(x) \, dx.
\] (38)

Compute $\inf_{y \in \mathcal{A}} J(y)$. Does $J$ attain its infimum?

Solution:

We proceed as follows. First we find a lower bound for $J$ over $\mathcal{A}$. Then we show this is the greatest lower bound for $J$ over $\mathcal{A}$. Lastly, we remark why $J$ does not attain its infimum, i.e., there is no minimizer in $\mathcal{A}$. Note, for $y \in \mathcal{A}$,
\[
J(y) = \int_{-1}^{1} y(x) \, dx \geq \int_{-1}^{1} h(x) \, dx = \int_{-1}^{1} 0 \, dx = 0.
\] (39)

This shows 0 is a lower bound for $J(y)$. We claim there is a sequence of functions $\{f_n\}_{n=1}^{\infty}$ contained in $\mathcal{A}$ such that $J(f_n) \to 0$. This implies there is no lower bound greater than zero and, therefore, 0 must be the greatest lower bound for $J$. In other words, $0 = \inf_{y \in \mathcal{A}} J(y)$.

All that remains is to verify the claimed sequence $\{f_n\}_{n=1}^{\infty}$ exists. Define $f_n(x) := x^{2n}$ for $n \geq 1$. Then $f_n(x) = x^{2n} \geq 0 = h(x)$ for $|x| < 1$ and $f_n(x) = x^{2n} \geq 1^{2n} = 1 = h(x)$ for $|x| \geq 1$. Hence $f_n \geq h$ and, with the fact $f$ is a polynomial (and thus smooth), we see $f_n \in \mathcal{A}$ for each $n$. Then computing $J(f_n)$ gives
\[
J(f_n) = \int_{-1}^{1} f_n(x) \, dx = \int_{-1}^{1} x^{2n} \, dx = 2 \int_{0}^{1} x^{2n} \, dx = 2 \left( \frac{1^{2n+1}}{2n+1} \right) = \frac{2}{2n+1} \leq \frac{1}{n}
\] (40)

Taking the limit as $n \to \infty$, we see
\[
0 \leq \lim_{n \to \infty} J(f_n) \leq \lim_{n \to \infty} \frac{1}{n} = 0.
\] (41)

Thus $\lim_{n \to \infty} J(f_n) = 0$, as desired.

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Lastly, we note $J$ does not attain its infimum. This is because the infimum is obtained if and only if $y(x) = 0$ for $|x| < 1$. But, because we need $y(\pm 1) \geq 1$, such a minimizer would necessarily have a jump discontinuity, contradicting the fact $y(x)$ must be continuous to be in $\mathcal{A}$. □

**Remark 9:** After reading the above example, we may ask ourselves “But why did you pick $f_n(x) = x^{2n}$? How did you know to do that?” I encourage the reader to draw a picture. A good picture can go a long way.

We want a continuous function $f$ with $f(-1) \geq 1$ and $f(1) \geq 1$, but approaches 0 for $|x| < 1$. To keep things simple, we may restrict our consideration to even functions. Perhaps an initial guess might be to use $x^2$ to get an even function with $(-1)^2 = 1 = 1^2$. Then because $|x| < 1$, we know $|x|^n \longrightarrow 0$ as $n \longrightarrow \infty$ (see Lemma below). So, we could try $(x^2)^n = x^{2n}$. Indeed, we see graphically below this does do the trick.

![Figure 1: Plots of $x^{2n}$ on $[-1, 1]$ for $n = 1, 3, 10$.](image)

**Lemma:** Let $c \in (0, 1)$. The $\lim_{n \to \infty} c^n = 0$. □

**Proof:**
Let $n \in \mathbb{N}$. Then $c^{n+1} = cc^n < 1c^n = c^n$. This shows the sequence $\{c^n\}_{n=1}^\infty$ is decreasing. And, the fact $c^n \geq 0^n = 0$ shows it is bounded from below. The Monotone Convergence Theorem then asserts $\{c^n\}_{n=1}^\infty$ converges to some limit $\alpha \in \mathbb{R}$. Observe

$$\alpha = \lim_{n \to \infty} c^n = \lim_{n \to \infty} c^{n+1} = \lim_{n \to \infty} c^n = c\alpha.$$  

Because $c \in (0, 1)$, the above can hold if and only if $\alpha = 0$. Thus $\lim_{n \to \infty} c^n = 0$. ■
**Min-Max Problems**

In this section, we discuss how a constrained minimization problem can be turned into an equivalent min-max problem. To illustrate this, we will first work with an example of a minimization problem in $\mathbb{R}$, and then we will take the ideas from there and apply them to finding minimizers in a space of functions (e.g., $C^2[a, b]$).

**Example 11:** Suppose $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) := 5(x - 7)^2$ and $g : \mathbb{R} \to \mathbb{R}$ is defined by $g(x) := x^3$. Express the constrained minimization problem

$$\min_{x \in \mathbb{R}} f(x) \text{ s.t. } g(x) = 8 \quad (43)$$

as an unconstrained min-max problem.

**Solution:**

Observe our optimization problem can be rewritten as

$$\min_{x \in \mathbb{R}} f(x) \text{ s.t. } g(x) = 8, \quad (44)$$

which is equivalent to

$$\min_{x \in \mathbb{R}} \begin{cases} 5(x - 7)^2 & \text{if } (x^3 - 8) = 0, \\ \infty & \text{otherwise}. \end{cases} \quad (45)$$

We can then express this problem as

$$\min_{x \in \mathbb{R}} \max_{\lambda \in \mathbb{R}} 5(x - 7)^2 + \lambda(x^3 - 8). \quad (46)$$
Remark 10: The step to rewrite the constrained problem as (45) initially seems as though we are moving backwards; however, this makes the following form in (46) more clear. Indeed, if \((x^3 - 8)\) is not zero, then we can pick \(\lambda\) to make this as big as we’d like. For example, if \(x^3 - 8 = \alpha > 0\), then we may heuristically write

\[
\lim_{\lambda \to \infty} 5(x - 7)^2 + \lambda(x^3 - 8) = \lim_{\lambda \to \infty} 5(x - 7)^2 + \lambda \alpha
= 5(x - 7)^2 + \lim_{\lambda \to \infty} \lambda \alpha
= 5(x - 7)^2 + \infty
= \infty.
\]

(47)

We could do similarly taking \(\lambda \to -\infty\) if \(x^3 - 8 < 0\). This same idea will next be used for a constrained optimization problem using a functional.

Remark 11: Suppose we wish to find the minimizer of \(J\) over a set \(X\). Here we will consider the problem unconstrained when \(X\) is a vector space without any constraints other than those imposed on the smoothness. So, if

\[
X := \left\{ y \in C^2[a, b] : \int_a^b y \, dx = 0 \right\},
\]

(48)

and we are given a functional \(J : C^2[a, b] \to \mathbb{R}\), then the problem

\[
\min_{y \in X} J(y)
\]

(49)

is here considered a constrained problem. However,

\[
\min_{y \in C^2[a, b]} J(y)
\]

(50)

is here considered unconstrained.
Example 12: Let $f : \mathbb{R} \to \mathbb{R}$ be continuous, i.e., $f \in C(\mathbb{R})$. Also assume $q : [0, 1] \to \mathbb{R}$ is continuous and define $J : C[0, 2] \to \mathbb{R}$ by

$$J(y) := \int_0^2 f(y(x)) \, dx$$

and set

$$X := \{ y \in C^2[0, 2] : y''(x) = q(x) \; \forall \; x \in [0, 1] \}.$$  

Rewrite the constrained optimization problem

$$\min_{y \in X} J(y) = \min_{y \in C^2[0, 2]} J(y) \; \text{s.t.} \; y''(x) = q(x) \; \forall \; x \in [0, 1]$$

as an unconstrained min-max problem.

**Solution:**

Observe

$$\min_{y \in X} J(y) = \min_{y \in C^2[0, 2]} \left\{ J(y) \; \text{if} \; y''(x) - q(x) = 0 \; \text{for all} \; x \in [0, 1], \infty \; \text{otherwise.} \right\}$$

To ensure the constrained $y''(x) - q(x) = 0$ holds for all $x \in [0, 1]$, we need a function $\lambda(x) \in C[0, 1]$. Indeed, then (54) becomes

$$\min_{y \in C^2[0, 2]} \max_{\lambda \in C[0, 1]} J(y) + \int_0^1 \lambda(x) [y''(x) - q(x)] \, dx.$$  

Alternatively, we could write this problem as

$$\min_{y \in C^2[0, 2]} \max_{\lambda \in \mathbb{R}} J(y) + \lambda \int_0^1 [y''(x) - q(x)]^2 \, dx.$$  

□
Remark 12: Why do we need $\lambda$ to be a function in (55) rather than simply a number in $\mathbb{R}$? The answer is this. If we merely impose that $\lambda \in \mathbb{R}$, then

$$0 = \int_0^1 \lambda y''(x) - q(x) \, dx = \lambda \int_0^1 y''(x) - q(x) \, dx,$$

which holds whenever the average of $y'' - q$ on $[0, 1]$ is zero. For example, if $y''(x) - q(x) = x - 1/2$, then for each $\lambda$ we have

$$\lambda \int_0^1 y''(x) - q(x) \, dx = \lambda \int_0^1 x - 1/2 \, dx = \lambda \left[ \frac{x^2}{2} - \frac{x}{2} \right]_0^1 = \lambda 0 = 0.$$  

But, $x - 1/2$ is not identically zero for all $x \in [0, 1]$. This is why we must use a function $\lambda(x)$ in (55).

A simpler route, which does allow for $\lambda$ to be a scalar is given in (56). For there the integral term with $y''$ and $q$ is equal to zero if and only if $y''(x) = q(x)$ for all $x \in [0, 1]$.

Remark 13: In the next example, we impose two constraints on a minimization problem.

Example 13: For a functional $J : C[0, 1] \to \mathbb{R}$, rewrite the constrained minimization problem

$$\min_{y \in C^1[0,1]} J(y) \quad \text{s.t.} \quad \int_0^1 y^2 \, dx = 5, \quad \int_0^1 y' \, dx = 0.$$  

as an unconstrained problem.

Solution:

Here the optimization problem may be rewritten as

$$\min_{y \in C^1[0,1]} \max_{\lambda \in \mathbb{R}} J(y) + \lambda \left[ \int_0^1 y^2 \, dx - 5 \right] \quad \text{s.t.} \quad \int_0^1 y' \, dx = 0.$$  

We now have a min-max problem, but there is still a constraint involved. So, we add another parameter $\mu$ to obtain the unconstrained problem

$$\min_{y \in C^1[0,1]} \max_{\lambda \in \mathbb{R}} \max_{\mu \in \mathbb{R}} J(y) + \lambda \left[ \int_0^1 y^2 \, dx - 5 \right] + \mu \int_0^1 y' \, dx.$$  

□
Example 14: Consider the optimization problem

$$\min_{y \in C^1[0,1]} \int_0^1 (y' - 2)^2 \, dx \quad \text{s.t.} \quad \int_0^1 y \, dx = 2.$$  \hfill (62)

a) Rewrite the constrained problem as an unconstrained min-max problem.

b) Solve the constrained minimization problem. What is the minimizer? What is the minimum?

Solution:
a) We can rewrite the constrained problem as

$$\min_{y \in C^1[0,1]} \left\{ \begin{array}{ll} \int_0^1 (y' - 2)^2 \, dx & \text{if } (\int_0^1 y \, dx - 2) = 0, \\ \infty & \text{otherwise}, \end{array} \right.$$  \hfill (63)

which in turn can be expressed as

$$\min_{y \in C^1[0,1]} \max_{\lambda \in \mathbb{R}} \int_0^1 (y' - 2)^2 \, dx + \lambda \left[ \int_0^1 y \, dx - 2 \right],$$  \hfill (64)

and then simplified as

$$\min_{y \in C^1[0,1]} \max_{\lambda \in \mathbb{R}} \int_0^1 (y' - 2)^2 + \lambda (y - 2) \, dx.$$  \hfill (65)

b) First note 0 is a lower bound for our functional since

$$\forall y \in C^1[0,1], \quad \int_0^1 (y' - 2)^2 \, dx \geq \int_0^1 0 \, dx = 0.$$  \hfill (66)

We claim 0 is the minimum and verify this as follows. Suppose $y \in C^1[0,1]$. Then the inequality in (66) is an equality if and only if $y'(x) = 2$ for all $x$ in $[0,1]$, which implies $y(x) = 2x + c$ for some $c \in \mathbb{R}$. In order for $y$ to satisfy the constraint, we need

$$2 = \int_0^2 y \, dx = \int_0^2 2x + c \, dx = [x^2 + cx]_0^1 = 1 + c \quad \iff \quad c = 1.$$  \hfill (67)

This shows the inequality in (66) holds if and only if $y(x) = 2x + 1$. All that remains is to note $y$ is smooth since it is a polynomial, and so $y \in C^1[0,1]$. Thus we conclude the minimizer is $y(x) = 2x + 1$ and the minimum is 0. □
Remark 14: Below we provide a few problems for students to try and rewrite in min-max form.

Example 15: Rewrite the constrained optimization problem

\[
\min_{y \in C^1[0,2]} \frac{1}{2} \int_0^2 y^2 + (y')^2 \, dx \quad \text{s.t.} \quad \int_0^1 y \, dx = 12
\] (68)

as an unconstrained min-max problem.

Example 16: Rewrite the constrained optimization problem

\[
\min_{y \in C^2[0,2]} \frac{1}{2} \int_0^2 y^2 + 2yy' + (y')^2 \, dx \quad \text{s.t.} \quad y''(x) - y'(x) = 5x \quad \forall \ x \in [0,1]
\] (69)

as an unconstrained min-max problem.

Example 17: Rewrite the constrained optimization problem

\[
\min_{y \in C[0,2]} y(0)^2 - 3y(0) + 7 \quad \text{s.t.} \quad \int_0^1 y' \, dx = 0.
\] (70)

as an unconstrained min-max problem.

Example 18: Rewrite the constrained optimization problem

\[
\min_{y \in C^1[0,2]} \int_0^2 (y - 2)^2 + (y - 4)^2 \, dx \quad \text{s.t.} \quad y'(x) = 1 \quad \forall \ x \in [0,2].
\] (71)

as an unconstrained min-max problem. Also find the minimizer and the minimum for the constrained problem.
Example 19: Rewrite the constrained optimization problem

\[
\min_{y \in C^1[0,2]} \int_0^2 (y - 2)^2 + (y - 4)^2 \, dx \quad \text{s.t.} \quad y'(x) = 1 \quad \forall x \in [0, 2]. \tag{72}
\]

as an unconstrained min-max problem. Also find the minimizer and the minimum for the constrained problem.

Example 20: Find all solutions to the minimization problem

\[
\min_{y \in C[0,1]} \int_0^1 \sin(y(x)) \, dx \quad \text{s.t.} \quad \int_0^1 y' \, dx = 0. \tag{73}
\]
REFERENCES
