Purpose: This document is a compilation of notes generated for discussion in MATH 131B with reference credit due to Terrence Tao’s text *Analysis II*. If the reader finds any errors/typos, please feel free to email me at heaton@math.ucla.edu and I will address these and post an updated set of notes to my webpage.

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**INTRODUCTION**

These notes are provided to compliment the TA discussion sessions on Tuesdays for MATH 131B. Typically, more detail is provided here than on the board during discussions since portions of solutions are given orally in class. The examples provided here are meant to be a constructive reference for students. These illustrate how to use certain logical quantifiers, how to guide the reader through your proofs, and the level of rigor desired from students this quarter. Before reading each solution, I highly encourage students to first seriously attempt the problems on their own. I cannot overstate the value of struggling through these problems before comparing your attempts to the example solutions.

These notes will be updated weekly (if not more often) on Mondays or Tuesdays, reflecting the current discussion material.
Example 1: Let $X$ be a set.

a) State the four axioms of a metric $d : X \times X \to \mathbb{R}$.

b) Let $X = \{1, 2, 3\}$ and define $d : X \times X \to \mathbb{R}$ by

$$d(x, y) := \begin{cases} 
0 & \text{if } x = y, \\
1/3 & \text{if } x > y, \\
2/3 & \text{if } x < y.
\end{cases} \quad (1)$$

Is $(X, d)$ a metric space? Prove your answer.

Solution:

a) The four axioms are as follows.

i) For any $x \in X$, $d(x, x) = 0$.

ii) (positivity) For any distinct $x, y \in X$, $d(x, y) > 0$.

iii) (triangle inequality) For any $x, y, z \in X$, $d(x, z) \leq d(x, y) + d(y, z)$.

iv) (symmetry) For any $x, y \in X$, $d(x, y) = d(y, x)$.

b) We claim $(X, d)$ does not form a metric space. Indeed, observe

$$d(1, 2) = \frac{2}{3} \neq \frac{1}{3} = d(2, 1), \quad (2)$$

and so the symmetry axiom iv) for a metric $d$ does not hold.

\[ \square \]

Remark 1: Note in the above example the other three axioms hold. However, to prove $(X, d)$ is not a metric space one only needs to provide a single counter example.
Remark 2: For $p \in [1, \infty)$, we can define the $p$-norm (also called the $\ell^p$ norm) of vectors in $\mathbb{R}^n$ by

$$
\|x\|_p := \left( \sum_{i=1}^{n} |x_i|^p \right)^{1/p}.
$$

This can in turn be used to define a metric $d_{\ell^p} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ by

$$
d_{\ell^p}(x, y) := \|x - y\|_p = \left( \sum_{i=1}^{n} |x_i - y_i|^p \right)^{1/p}.
$$

We also define the sup norm (or called “max” norm) on vectors in $\mathbb{R}^n$ by

$$
\|x\|_\infty := \max_{1 \leq i \leq n} |x_i|,
$$

which leads to the sup metric $d_{\ell^\infty}$ defined by

$$
d_{\ell^\infty}(x, y) := \|x - y\|_\infty = \max_{1 \leq i \leq n} |x_i - y_i|.
$$

Consequently, $(\mathbb{R}^n, d_{\ell^\infty})$ is a metric space for $p \in [1, \infty]$. If we simply say the space $\mathbb{R}^n$, then we implicitly mean the space $(\mathbb{R}^n, d_{\ell^2})$, with which we are well-familiar.

Example 2: Let $(x^{(n)})_{n=1}^{\infty}$ be a sequence in $\mathbb{R}^m$ converging to $z$ with respect to the $d_{\ell^\infty}$ metric. Prove $x^{(n)} \longrightarrow z$ with respect to the $d_{\ell^1}$ metric.

Proof:

Let $\varepsilon > 0$ be given. We must show there is $N > 0$ such that

$$
d_{\ell^1}(x^{(n)}, z) < \varepsilon \ \forall \ n > N.
$$

First observe

$$
d_{\ell^1}(x^{(n)}, z) = \sum_{i=1}^{m} |x_i^{(n)} - z| \leq \sum_{i=1}^{m} \max_{1 \leq i \leq m} |x_i^{(n)} - z| = m \cdot \max_{1 \leq i \leq m} |x_i^{(n)} - z| = m \cdot d_{\ell^\infty}(x^{(n)}, z).
$$

By the convergence given in our hypothesis, there is $N > 0$ such that

$$
d_{\ell^\infty}(x^{(n)}, z) < \frac{\varepsilon}{m} \ \forall \ n > N,
$$

and so

$$
d_{\ell^1}(x^{(n)}, z) \leq m \cdot d_{\ell^\infty}(x^{(n)}, z) < m \cdot \frac{\varepsilon}{m} = \varepsilon \ \text{for all} \ n > N.
$$

This verifies (7) and completes the proof.
Remark 3: We can also talk about metric spaces in similar fashion to \((\mathbb{R}^n, d_{\ell^p})\), but that are of infinite dimension. Indeed, let \(p \in [1, \infty)\) Then for each sequence \(x = (x_i)_{i=1}^{\infty}\) of numbers in \(\mathbb{R}\) we can take\(^1\)

\[
\|x\|_p := \left( \sum_{i=1}^{\infty} |x_i|^p \right)^{1/p} \quad \text{and} \quad d_{\ell^p}(x, y) := \|x - y\|_p = \left( \sum_{i=1}^{\infty} |x_i - y_i|^p \right)^{1/p}. \tag{11}
\]

Note the only difference here is that our sums from before now turn into infinite series. Then let \(X\) be the space consisting of all sequences with finite \(p\)-norm, i.e.,

\[
X := \left\{ x = (x_i)_{i=1}^{\infty} : \|x\|_{\ell^p} := \left( \sum_{i=1}^{\infty} |x_i|^p \right)^{1/p} < \infty \right\}. \tag{12}
\]

Then \((X, d_{\ell^p})\) forms a metric space. This metric space is commonly written simply as \(\ell^p\).

In similar fashion to above, now consider \(p = \infty\). For a sequence \(x = (x_i)_{i=1}^{\infty}\) we set

\[
\|x\|_{\infty} := \sup_{i \in \mathbb{N}} |x_i| \quad \text{and} \quad d_{\ell^\infty}(x, y) := \sup_{i \in \mathbb{N}} |x_i - y_i|. \tag{13}
\]

When

\[
X := \left\{ x = (x_i)_{i=1}^{\infty} : \|x\|_{\ell^\infty} = \sup_{i \in \mathbb{N}} |x_i| < \infty \right\}, \tag{14}
\]

the combination \((X, d_{\ell^\infty})\) forms a metric space, denoted \(\ell^\infty\).

Remark 4: The sequence space \(\ell^2\) is complete. That is, if a sequence \((x^{(n)})_{n=1}^{\infty}\) of sequences is in \(\ell^2\) and is Cauchy, then it converges to some limit in \(\ell^2\).

\(^1\)Note here we use different notation to denote the sequences in order to avoid confusion in the example below.
Example 3: For each \( n \in \mathbb{N} \), let \( e^{(n)} = (e^{(n)}_i)_{i=1}^{\infty} \) be the sequence of real numbers satisfying
\[
e^{(n)}_i = \begin{cases} 
1 & \text{if } n = i, \\
0 & \text{otherwise.} 
\end{cases}
\] (15)

Note \( e^{(n)} \) is itself a sequence for fixed \( n \in \mathbb{N} \).

a) Show \( e^{(n)} \in \ell^2 \) for each \( n \in \mathbb{N} \).

b) Show the sequence \( (e^{(n)})_{n=1}^{\infty} \) is not Cauchy in \( \ell^2 \).

c) Define the sequence \( (x^{(n)})_{n=1}^{\infty} \) by \( x^{(n)} := n \cdot e^{(n)} \) for each \( n \in \mathbb{N} \). Is this sequence \( (x^{(n)})_{n=1}^{\infty} \) in \( \ell^2 \)?

d) Is \( (x^{(n)})_{n=1}^{\infty} \) bounded?

Solution:

a) Observe for each \( n \in \mathbb{N} \) we have
\[
\|e^{(n)}\|_{\ell^2} = \left( \sum_{i=1}^{\infty} |e^{(n)}_i|^2 \right)^{1/2} = \left( |e^{(n)}_n|^2 \right)^{1/2} = (1^2)^{1/2} = 1 < \infty. \] (16)

Consequently, \( e^{(n)} \in \ell^2 \) for each \( n \in \mathbb{N} \).

b) It suffices to provide a single counter example. Let \( \varepsilon = 1 \). Then for \( m \neq n \) we have
\[
d_{\ell^2}(e^{(n)}, e^{(m)}) = \|e^{(n)} - e^{(m)}\|_{\ell^2} = \left( \sum_{i=1}^{\infty} |e^{(n)}_i - e^{(m)}_i|^2 \right)^{1/2} = \left( |e^{(n)}_n - e^{(m)}_n|^2 + |e^{(n)}_m - e^{(m)}_m|^2 \right)^{1/2} = \left( |0 - 1|^2 + |1 - 0|^2 \right)^{1/2} = \sqrt{2}. \] (17)

Consequently,
\[
d_{\ell^2}(e^{(n)}, e^{(m)}) = \sqrt{2} > \varepsilon \quad \text{whenever } n \neq m. \] (18)

This shows there does not exist \( N > 0 \) such that
\[
d_{\ell^2}(e^{(n)}, e^{(m)}) < \varepsilon \quad \forall m, n > N, \] (19)

and so the sequence \( (e^{(n)})_{n=1}^{\infty} \) is not Cauchy.
c) We claim the sequence \( (x^{(n)})_{n=1}^\infty \) is in \( \ell^2 \). Indeed, for \( n \in \mathbb{N} \) we discover
\[
\|x^{(n)}\|_{\ell^2} = \left( \sum_{i=1}^{\infty} |x_{i}^{(n)}|^2 \right)^{1/2} = \left( |x_{n}^{(n)}|^2 \right)^{1/2} = (n^2)^{1/2} = n < \infty. \tag{20}
\]

\[\text{d) We claim the sequence } (x^{(n)})_{n=1}^\infty \text{ is unbounded. Indeed, by way of contradiction, suppose this sequence is bounded by some } M > 0. \text{ Then by the Archimedean property of } \mathbb{R} \text{ there is } k \in \mathbb{N} \text{ such that } k > M. \text{ Consequently, (20) shows}
\]
\[
\|x^{(k)}\|_{\ell^2} = k > M, \tag{21}
\]
contradicting the fact we assumed \( M \) was an upper bound. Thus the initial assumption was false and we conclude \( (x^{(n)})_{n=1}^\infty \) is unbounded.

\[\square\]

Remark 5: The above example contains a sequence in a sequence space. In other words, \( (e^{(n)})_{i=1}^\infty \) is a sequence of sequences.

Remark 6: The above example shows results we know in \( \mathbb{R}^n \) do not necessarily hold in infinite dimensional spaces. Indeed, recall the Bolzano-Weierstrass theorem discussed in MATH 131A last quarter, which tells us every bounded sequence in \( \mathbb{R}^n \) has a convergent subsequence, which in turn is Cauchy. However, in the above example, the sequence \( (e^{(n)})_{n=1}^\infty \) is bounded, but it does not have a Cauchy subsequence.
Lemma: Let $\alpha \in (0, 1)$. Then $\alpha^n \longrightarrow 0$ as $n \longrightarrow \infty$. \hfill $\triangle$

Proof:
We claim this sequence is decreasing. Indeed, for each $n \in \mathbb{N}$ we see $\alpha^{n+1} = \alpha \alpha^n < \alpha^n$. And the sequence is nonnegative. Consequently, the monotone convergence theorem implies this sequence converges to a limit $z$. Moreover,

$$z = \lim_{n \to \infty} \alpha^n = \lim_{n \to \infty} \alpha^{n+1} = \alpha \lim_{n \to \infty} \alpha^n = \alpha z,$$

(22)

which holds precisely when $z = 0$. This completes the proof. \hfill $\square$

Example 4: Let $(X, d)$ be a metric space. Suppose the sequence $(x^{(n)})_{n=1}^\infty$ satisfies the inequality

$$d(x^{(n+1)}, x^{(n)}) \leq \alpha^{n+1} \text{ for } n \in \mathbb{N},$$

(23)

where $\alpha \in (0, 1)$. Is this sequence Cauchy? Prove your answer.

Proof:
We claim this sequence is Cauchy. Let $\varepsilon > 0$ be given. We must show there is $N > 0$ such that

$$d(x^{(m)}, x^{(n)}) < \varepsilon \ \forall \ m, n > N.$$  

(24)

Observe for $m, n \in \mathbb{N}$ with $n \geq m$ the triangle inequality yields

$$d(x^{(m)}, x^{(n)}) \leq \sum_{j=1}^{n-m} d(x^{(m-1+j)}, x^{(m+j)}) \leq \sum_{j=1}^{n-m} \alpha^{m+j}.$$  

(25)

Evaluating this geometric sum gives

$$d(x^{(m)}, x^{(n)}) \leq \alpha^m \sum_{j=1}^{n-m} \alpha^j = \alpha^m \left( \frac{1 - \alpha^{n-m+1}}{1 - \alpha} \right) \leq \frac{\alpha^m}{1 - \alpha}.$$  

(26)

From the above lemma, we know $\alpha^m/(1 - \alpha) \longrightarrow 0$ as $m \longrightarrow \infty$. This implies there exists $N > 0$ such that

$$\left| \frac{\alpha^m}{1 - \alpha} \right| < \varepsilon \ \forall \ m > N.$$  

(27)

Equations (26) and (27) together imply (24) holds, and we are done. \hfill $\square$
**Example 5:** Let \((X,d)\) be a metric space and fix \(z \in X\). The define the function \(f : X \to \mathbb{R}\) by 
\[ f(x) := d(x, z) .\] Prove \(f\) is continuous.

*Proof:* 
This is currently left to the reader. A solution will be posted at a later date. ■

**Example 6:** Let \((X,d)\) be a metric space and \(x \in X\). Prove the set \(S := \{x\} \subset X\) is closed.

*Proof:* 
We use the fact a set is closed if and only if its complement is open. It thus suffices to show 
\[S^c = X - \{x\}\] is open. Pick \(y \in S^c\). Then \(y \neq x\), and so \(d(x, y) > 0\). Let \(r = d(x, z)/2\) and \(z \in B(y, r)\). Then the triangle inequality yields
\[d(x, y) \leq d(x, z) + d(z, y) \implies d(x, z) \geq d(x, y) - d(y, z) = s - d(y, z) > s - \frac{s}{2} = \frac{s}{2} > 0,\] (28)
and so \(z \neq x\), which implies \(z \in S^c\). This reveals \(B(y, r) \subset S^c\), and so \(y \in \text{int}(S^c)\). Because \(y\) was chosen arbitrarily in \(S^c\), it follows that \(S^c\) is open, from which we conclude \(S = \{x\}\) is closed. ■
Remark 7: For the following example, it may be helpful to draw several pictures, visualizing $E$ and the multiple open balls inside it, each centered at $x \in E$.

Example 7: Is every point of every open set $E \subset \mathbb{R}^n$ an adherent point of $E$? (Use the Euclidean norm.)

Proof:
We claim the answer is yes. Let $x = (x_1, x_2, \ldots, x_n) \in E$. To verify $x$ is a limit point of $E$, we must show for each neighborhood of $x$ there exists $y \in E$ distinct from $x$ (i.e., $y \neq x$) contained in the neighborhood. Let $r > 0$ be given. Then it suffices to find $y \in E \cap B(x, r)$ with $y \neq x$. We do this as follows. Since $E$ is open and $x \in E$, there is $s > 0$ such that $B(x, s) \subset E$. Let $d = \frac{1}{2} \min\{s, r\}$ so that $B(x, d) \subset B(x, s) \cap B(x, r) \subset E \cap B(x, r)$. Then choose

$$y = (y_1, y_2, \ldots, y_n) = (x_1 + d/2, x_2, \ldots, x_n),$$

noting $y \neq x$, and observe

$$\|x - y\|_2 = \left( \sum_{i=1}^{n} (y_i - x_i)^2 \right)^{1/2} = \left( (d/2)^2 \right)^{1/2} = d/2 < d.$$ (30)

Consequently, $y \in B(x, d) \subset E \cap B(x, r)$, and we are done.

Example 8: Is every point of every closed set $E \subset \mathbb{R}^n$ an adherent point of $E$?

Proof:
We claim the answer is no. Let $x \in \mathbb{R}$. Then the singleton set $E = \{x\}$ is closed. And, there are no elements in $E$ distinct from $x$ contained in the open ball $B(x, r)$ for any $r > 0$, and we are done.
Relatively Open/Closed Sets:

Example 9: Define $X = \mathbb{R}$. Let $Y = [0, 2]$ and $E = (0, 1/2)$. Prove

a) $E$ is not open with respect to $X$;

b) $E$ is not closed with respect to $X$;

c) $E$ is open with respect to $Y$.

Proof:

a) We now show $E$ is not open in $X$. It suffices consider $y = -1$. For each $r > 0$, the open ball $B(y, r) \subset X$ contains the point $y - r/2 = -1 - r/2 < -1 \notin E$. Thus $-1$ is not an interior point of $E$ and so $E$ is not open.

b) We now show $E$ is not closed. Recall by Proposition 1.2.15b, $E$ is closed if and only if $E$ contains all of its adherent points. So, it suffices to find an adherent point of $E$ not contained in $E$. We claim $1$ is such a point. Indeed, define the sequence $(x^{(n)})_{n=1}^{\infty}$ by $x^{(n)} = 1 - 1/n$. Then

$$0 = 1 - 1/1 \leq 1 - 1/n = x^{(n)} < 1,$$

and so $x^{(n)} \in E$ for each $n \in \mathbb{N}$. Let $\varepsilon > 0$ be given. Then the Archimedean property of $\mathbb{R}$ asserts there is $N \in \mathbb{N}$ such that $1/N < \varepsilon$, and so

$$|1 - x^{(n)}| = \frac{1}{n} = \frac{1}{n} \leq \frac{1}{N} < \varepsilon \quad \forall \ n > N. \quad (32)$$

This shows $x^{(n)} \longrightarrow 1 \notin E$. Thus $1$ is an adherent point of $E$ not contained in $E$. This completes the proof.

c) By Proposition 1.3.4 in our text, a subset $E \subseteq Y \subseteq X$ is open with respect to $Y$ if and only if there is a subset $V \subset X$ such that $V$ is open with respect to $X$ and $E = V \cap Y$. Let $V = (-1/2, 1/2)$ and observe $V$ is open since this is precisely an open ball about $0$ of radius $1/2$, i.e., $V = B(0, 1/2)$. Furthermore,

$$V \cap Y = (-1/2, 1/2) \cap [0, 2] = [0, 1/2) = E. \quad (33)$$

Therefore $E$ is open with respect to $Y$. ■
Example 10: Let $X = \mathbb{R}^2$, $d = d_\ell^2$, $Y = (0, 1) \times [0, 1]$, and $E = (0, 1) \times [0, 1/2]$. Note $(X, d)$ forms a metric space. Is $E$ relatively closed in $Y$?

Proof:
We claim the answer is yes. Proposition 1.3.4b asserts $E$ is relatively closed in $Y$ if and only if there is a subset $K \subseteq X$ that is closed in $X$ and satisfies $E = K \cap Y$. Let $K = [0, 1] \times [0, 1/2]$. Since $K$ is the Cartesian product of two closed intervals, it is closed in $X$. Moreover,

$$
K \cap Y = ([0, 1] \times [0, 1/2]) \cap ((0, 1) \times [0, 1])
$$

$$
= ([0, 1] \cap (0, 1)) \times ([0, 1/2] \cap [0, 1])
$$

$$
= (0, 1) \times [0, 1/2]
$$

(34)

$$
= E.
$$

The claim then follows from Proposition 1.3.4b. ■
Cauchy Sequences:

**Example 11:** Let \((X, d)\) be a metric space. Prove every Cauchy sequence in \((X, d)\) has at most one limit point.

*Proof:*

Let \((x^{(n)})_{n=1}^{\infty} \subseteq X\) be a Cauchy sequence. If this sequence has zero limit points, then we are done (since \(0 < 1\)). Now suppose this is not the case, i.e., the sequence has at least one limit point. Then let \(y\) and \(z\) be limits points of \((x^{(n)})\). It suffices to show \(y = z\), which we do as follows.

Let \(\varepsilon > 0\). Since \((x^{(n)})\) is Cauchy, there is \(N_1\) such that \(m, n > N_1\) implies \(d(x^{(m)}, x^{(n)}) < \varepsilon\). Also because \(y\) is a limit point of \((x^{(n)})\), there is a subsequence \((x^{(n_k)}) \subseteq (x^{(n)})\) and \(N_2\) such that \(k > N_2\) implies \(d(x^{(n_k)}, y) < \varepsilon\). Similarly, there is a subsequence \((x^{(m_k)}) \subseteq (x^{(n)})\) and \(N_3\) such that \(k > N_3\) implies \(d(x^{(m_k)}, z) < \varepsilon\). Now let \(N = \max\{N_1, N_2, N_3\}\). Noting \(m_k \geq k\) and \(n_k \geq k\), it follows that \(k > N\) implies

\[
\begin{align*}
    d(y, z) & \leq d(y, x^{(n_k)}) + d(x^{(n_k)}, z) \\
    & \leq d(y, x^{(n_k)}) + d(x^{(n_k)}, x^{(m_k)}) + d(x^{(m_k)}, z) \\
    & < \varepsilon + \varepsilon + \varepsilon \\
    &= 3\varepsilon.
\end{align*}
\]

Consequently, \(0 \leq d(y, z) < 3\varepsilon\) for each \(\varepsilon > 0\). Taking the limit as \(\varepsilon \longrightarrow 0\), the squeeze theorem asserts \(d(y, z) = 0\), which implies \(y = z\), as desired.

\[\blacksquare\]
Example 12: Let \((X, d)\) be a metric space. Suppose the sequence \((y^{(n)})_{n=1}^\infty \subset X\) converges to a limit \(y^*\) and the sequence \((x^{(n)})_{n=1}^\infty\) is Cauchy. Prove the sequence \((d(x^{(n)}, y^{(n)}))_{n=1}^\infty\) is Cauchy in \(\mathbb{R}\).

Proof:

First note the metric on \(\mathbb{R}\) is implicitly given to be the absolute value of the difference of two numbers. Then let \(\varepsilon > 0\) be given. We must show there is \(N > 0\) such that

\[
|d(x^{(m)}, y^{(m)}) - d(x^{(n)}, y^{(n)})| < \varepsilon \quad \forall \ m, n > N.
\]

(36)

Observe application of the triangle inequality yields

\[
d(x^{(m)}, y^{(m)}) - d(x^{(n)}, y^{(n)}) \leq d(x^{(m)}, y^{(n)}) + d(y^{(n)}, y^{(m)}) - d(x^{(n)}, y^{(n)})
\]

\[
\leq d(x^{(m)}, x^{(n)}) + d(x^{(n)}, y^{(n)}) + d(y^{(n)}, y^{(m)}) - d(x^{(n)}, y^{(n)})
\]

\[
= d(x^{(m)}, x^{(n)}) + d(y^{(n)}, y^{(m)})
\]

\[
= d(x^{(m)}, x^{(n)}) + d(y^{(n)}, y^*) + d(y^*, y^{(m)}).
\]

(37)

Swapping the positions of \(m\) and \(n\) and using the symmetry of the metric \(d\), we obtain

\[
d(x^{(n)}, y^{(n)}) - d(x^{(m)}, y^{(m)}) \leq d(x^{(m)}, x^{(n)}) + d(y^{(n)}, y^*) + d(y^*, y^{(m)}),
\]

(38)

from which we deduce

\[
|d(x^{(m)}, y^{(m)}) - d(x^{(n)}, y^{(n)})| \leq d(x^{(m)}, x^{(n)}) + d(y^{(n)}, y^*) + d(y^*, y^{(m)}).
\]

(39)

Now since \((x^{(n)})_{n=1}^\infty\) is Cauchy, there is \(N_1 > 0\) such that

\[
d(x^{(n)}, x^{(m)}) < \frac{\varepsilon}{3} \quad \forall \ m, n > N_1.
\]

(40)

Similarly, by the convergence of our other sequence, there is \(N_2 > 0\) such that

\[
d(y^{(n)}, y^*) < \frac{\varepsilon}{3} \quad \forall \ n > N_2.
\]

(41)

Letting \(N := \max\{N_1, N_2\}\), we deduce \(m, n > N\) implies

\[
|d(x^{(m)}, y^{(m)}) - d(x^{(n)}, y^{(n)})| \leq d(x^{(m)}, x^{(n)}) + d(y^{(n)}, y^*) + d(y^*, y^{(m)}) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon,
\]

(42)

and we are done.

\[\blacksquare\]
**Example 13:** Let \((x^{(n)})_{n=1}^\infty\) be a sequence in \(\mathbb{R}^2\). Suppose there is \(y \in \mathbb{R}^2\) such that \(d_{\ell^2}(y, x)\) is monotonically decreasing. If \((x^{(n)})_{n=1}^\infty\) has a convergent subsequence, does the entire sequence \((x^{(n)})_{n=1}^\infty\) necessarily converge in \((\mathbb{R}^2, d_{\ell^2})\)? Prove your answer.

**Proof:**
We claim the sequence \((x^{(n)})_{n=1}^\infty\) need not converge. For example, consider the sequence defined by

\[
x^{(n)} := ((-1)^n, (1)^{n+1}) \quad \forall \ n \in \mathbb{N}.
\]

(43)

Now let \(y = (0, 0)\) and observe

\[
d_{\ell^2}(x^{(n)}, y) = \sqrt{((-1)^n - 0)^2 + ((1)^{n+1} - 0)^2} = \sqrt{2} \quad \forall \ n \in \mathbb{N},
\]

(44)

and so \(d_{\ell^2}(x^{(n+1)}, y) \leq d_{\ell^2}(x^{(n)}, y)\) for each \(n \in \mathbb{N}\). Next note \(x^{(2n)} = x^{(n)}\) for each \(n \in \mathbb{N}\), and so the subsequence \((x^{(4n)})_{n=1}^\infty\) converges to \(x^1\). However, \((x^{(n)})_{n=1}^\infty\) does *not* converge. Indeed,

\[
d_{\ell^2}(x^{(n)}, x^{(m)}) = 2\sqrt{2} \quad \text{whenever} \ m \text{ mod} 2 \neq n \text{ mod} 2.
\]

(45)

From this, we deduce the sequence \((x^{(n)})_{n=1}^\infty\) cannot be Cauchy. Indeed, if it were, then there would exist \(N > 0\) such that \(m, n > N\) implies

\[
d_{\ell^2}(x^{(n)}, x^{(m)}) < 1,
\]

(46)

a contradiction to (45). Because every convergent sequence is Cauchy and \((x^{(n)})_{n=1}^\infty\) is not Cauchy, this sequence does not converge. ■
Example 14: Let \( C[a,b] \) denote the collection of all continuous functions on \([a,b]\), i.e.,

\[
C[a,b] := \{ f : [a,b] \to \mathbb{R} : f \text{ is continuous} \}.
\] (47)

Define the sup metric \( d : C[a,b] \times C[a,b] \to \mathbb{R} \) by

\[
d(f,g) := \sup_{x \in [a,b]} |f(x) - g(x)|.
\] (48)

Prove \( d \) is in fact a metric on \( C[a,b] \).

Proof:
First observe the sup metric is well-defined since every continuous function obtains its extreme values on a closed interval \([a,b]\) by the extreme value theorem. (Recall the sum of continuous functions is continuous.) Then we see

\[
\forall f \in C[a,b], \quad d(f,f) = \sup_{x \in [a,b]} |f(x) - f(x)| = \sup_{x \in [a,b]} 0 = 0.
\] (49)

We also see the metric is symmetric since

\[
\forall f, g \in C[a,b], \quad d(f,g) = \sup_{x \in [a,b]} |f(x) - g(x)| = \sup_{x \in [a,b]} |g(x) - f(x)| = d(g, f).
\] (50)

Additionally, the triangle inequality holds since

\[
\forall f, g, h \in C[a,b], \quad d(f,g) = \sup_{x \in [a,b]} |f(x) - g(x)|
\]

\[
\leq \sup_{x \in [a,b]} (|f(x) - h(x)| + |h(x) - g(x)|)
\]

\[
\leq \sup_{x \in [a,b]} |f(x) - h(x)| + \sup_{x \in [a,b]} |h(x) - g(x)|
\]

\[
= d(f,h) + d(h,g).
\] (51)

Lastly, suppose \( f \neq g \). Then there is \( z \in [a,b] \) such that \( f(z) \neq g(z) \), which implies \( f(z) - g(z) \neq 0 \). Consequently,

\[
d(f,g) = \sup_{x \in [a,b]} |f(x) - g(x)| \geq |f(z) - g(z)| > 0.
\] (52)

We have now verified each for the four axioms of a metric and conclude. ■
Example 15: Define the sequence of functions \( (f_n : [0, 2] \to \mathbb{R})_{n=1}^\infty \) by

\[
f_n(x) := \begin{cases} 
1 - nx & \text{if } x \in [0, 1/n], \\
0 & \text{if } x \in (1/n, 2].
\end{cases}
\]

(53)

Prove this sequence converges point-wise to the function \( f : [0, 2] \to \mathbb{R} \) defined by

\[
f(x) := \begin{cases} 
1 & \text{if } x = 0, \\
0 & \text{if } x \in (0, 2].
\end{cases}
\]

(54)

Proof:
Let \( \varepsilon > 0 \) be given and choose any \( x \in (0, 2] \). Then by the Archimedean property of \( \mathbb{R} \) there is \( N \in \mathbb{N} \) such that \( 1/N < x \). This implies \( x \in (1/n, 2] \) for \( n > N \), and so

\[
|f_n(x) - 0| = |0 - 0| = 0 < \varepsilon \quad \forall \ n > N.
\]

(55)

This shows \( \lim_{n \to \infty} f_n(x) = 0 = f(x) \) for \( x \in (0, 2] \). Now observe

\[
|f_n(0) - 1| = |1 - n0 - 1| = 0 < \varepsilon \quad \forall \ n \in \mathbb{N}.
\]

(56)

This shows \( \lim_{n \to \infty} f_n(0) = 1 = f(0) \) and completes the proof. \( \blacksquare \)

Remark 8: The above result will be used as a point of reference in the following examples. Remember if a sequence converges, then it converges to a unique limit. In this case, if \( (f_n)_{n=1}^\infty \) converges, then \( f_n \to f \). However, such convergence is dependent upon the metric space in which we speak of this convergence, as the following examples illustrate.
Example 16: Let $X = C[0, 2]$ and $d$ be the sup norm metric defined on $C[0, 2]$. Prove the sequence of functions defined in Example 15 is not Cauchy in the metric space $(X, d)$.

Proof:
Let $\varepsilon > 0$ be given and, by way of contradiction, suppose $(f_n)_{n=1}^{\infty}$ is Cauchy. Then there is $N > 0$ such that

$$d(f_m, f_n) < \varepsilon \quad \forall \ m, n > N. \quad (57)$$

Now let $m, n \in \mathbb{N}$ and assume $m > n$ without loss of generality. Then

$$d(f_m, f_n) = \sup_{x \in [0, 2]} |f_m(x) - f_n(x)| = \sup_{x \in [0, 2]} |f_m(1/m) - f_n(1/m)| = f_n(1/m) = 1 - \frac{n}{m}. \quad (58)$$

The second equality follows from the fact

$$f_n(x) - f_m(x) = \begin{cases} (m - n)x & \text{for } x \in [0, 1/m], \\ 1 - nx & \text{for } x \in (1/m, 1/n], \\ 0 & \text{for } x \in (1/n, 2]. \end{cases} \quad (59)$$

(The reader is also encouraged to make a simple plot of $f_n$ and $f_m$ to gain more intuition.) The result in (58) implies

$$d(f_n, f_{3n}) = 1 - \frac{n}{3n} = 1 - \frac{1}{3} = \frac{2}{3} > \frac{1}{2} \quad \forall \ n \in \mathbb{N}, \quad (60)$$

a contradiction to (57). Thus $(f_n)_{n=1}^{\infty}$ is not Cauchy in $(X, d)$. $\blacksquare$

Remark 9: The above result should not be too surprising. Why? Remember that if $(f_n)_{n=1}^{\infty}$ converges in $X$, then its limit is in $X$ (i.e., $f \in X$). However, the function $f$ is not continuous, and so $f \notin X$. Also, the space $(X, d)$ in the above example is complete. So, a sequence converges in $X$ if and only if it is Cauchy. Thus, having the result of Example 15, we should be able to intuitively guess the sequence $(f_n)_{n=1}^{\infty}$ is not Cauchy. $\Diamond$
Example 17: Define the space of functions $X$ by

$$X := \left\{ f : [0, 2] \to \mathbb{R} \mid \int_0^1 |f(x)| \, dx < \infty \right\}.$$  

(61)

Then define the metric $d : X \times X \to \mathbb{R}$ by

$$d(f, g) = \int_0^2 |f(x) - g(x)| \, dx.$$  

(62)

Then $(X, d)$ forms a metric space (which we shall not prove here). Show the sequence of functions defined in Example 15 converges in the metric space $(X, d)$.

Proof:

First note each bounded function defined on $[0, 2]$ is contained in $X$. This implies $f_n \in X$ for each $n \in \mathbb{N}$ and $f \in X$ where $f$ is defined as in (54). Let $\varepsilon > 0$ be given. We claim $f_n \longrightarrow f$ and to show this it suffices to verify there is $N > 0$ such that

$$d(f_n, f) < \varepsilon \quad \forall \ n > N.$$  

(63)

Observe we have\(^2\)

$$d(f_n, f) = \int_0^2 |f_n(x) - f(x)| \, dx = \int_0^2 |f_n(x)| \, dx = \frac{1}{2n}.$$  

(64)

Note the single point $x = 0$ where $f(x) = 1$ does not affect the value of the integral.\(^3\) By the Archimedean property of $\mathbb{R}$, there is $N > 0$ such that $1/N < \varepsilon$. Consequently, we deduce

$$d(f_n, f) = \frac{1}{2n} < \frac{1}{n} < \frac{1}{N} < \varepsilon \quad \forall \ n > N,$$

(65)

which completes the proof. \hfill \blacksquare

Remark 10: This example shows we the sequence $(f_n)_{n=1}^\infty$ converges in one metric space, but does not converge in another. \hfill ◦

\(^2\)Here we do not provide the details of this calculation. The reader should note the area of a triangle is half its width multiplied by its height, and the integral yields precisely the integral of triangle of width $1/n$ and height 1.

\(^3\)This can be shown by looking at the upper and lower Darboux sums.
Compactness:
Here we provide a few examples related to Section 1.5 of our text.

Example 18: Let $S = [0, 1]$. Show there exists a finite open cover of $S$ and give an example.

Proof:
First note $S$ is a closed interval and is bounded since $[0, 1] \subset B(0, 1/2) = (-1/2, 1/2)$. By the Heine-Borel theorem, it follows that $S$ is compact. Then Theorem 1.5.8 asserts for each open cover $\{V_\alpha\}_{\alpha \in I}$ of $S$, there exits a finite subcover of $S$. In other words, there is a finite subset $F \subset I$ such that

$$[0, 1] = S \subset \bigcup_{i \in F} V_i. \quad (66)$$

Now let us provide an explicit example. For each $x \in [0, 1]$, let $V_x = B(x, 1/2)$. Then the collection of all $V_x$ with $x \in [0, 1]$ forms an open cover of $[0, 1]$. Our result above tells us there is some finite subset of $[0, 1]$, denoted $F$, such that

$$[0, 1] \subset \bigcup_{x \in F} V_x. \quad (67)$$

For example, observe we could use $F = \{0, 1/2, 1\}$ and obtain

$$B(0, 1/2) \cup B(1/2, 1/2) \cup B(1, 1/2) = (-1/2, 1/2) \cup (0, 1) \cup (1/2, 3/2) = (-1/2, 3/2) \supset [0, 1]. \quad (68)$$

\[\blacksquare\]

Example 19: Explicitly show the set $Y \subset \mathbb{R}$ defined by $Y := \{(a, b) : a \in [-3, 5], b \in [3, 12]\}$ is bounded in $(\mathbb{R}^2, d_{\ell^2})$.

Proof:
We say $Y \subset \mathbb{R}^2$ is bounded if and only if there exists a ball $B(x, r) \subseteq \mathbb{R}^2$ which contains $Y$. Take $x = (0, 0) \in \mathbb{R}^2$. Then observe

$$\forall (a, b) \in Y, \quad d_{\ell^2}((a, b), (0, 0)) = \sqrt{a^2 + b^2} \leq \sqrt{5^2 + 12^2} = 13. \quad (69)$$

This implies for each $(a, b) \in Y$, $(a, b) \in B(x, 14)$. Letting $r = 14$, we deduce $Y \subset B(x, r) \subset \mathbb{R}^2$ and thus we conclude $Y$ is bounded.

\[\blacksquare\]
Example 20: Give an example of an open cover of $(-1, 1)$ for which there is no finite subcover.

Solution:
Consider the collection of open intervals \( \{V_n\}_{n=1}^{\infty} \) with \( V_n := (-1, 1 - 1/n) \). We claim

\[
(-1, 1) = \bigcup_{n=1}^{\infty} (-1, 1 - 1/n).
\]  \hspace{1cm} (70)

For each \( n \), it directly follows that \((-1, 1 - 1/n) \subseteq (-1, 1)\). Conversely, suppose \( x \in (-1, 1) \). Then there is \( r > 0 \) such that \( B(x, r) \subseteq (-1, 1) \). This implies \( x + r < 1 \), and so \( 1 - (x + r) > 0 \). By the Archimedean property of \( \mathbb{R} \), we can pick \( N \in \mathbb{N} \) such that \( 1/N < 1 - (x + r) \). For such \( N \), we see

\[
x \in B(x, r) \subseteq (-1, 1 - 1/N) \subseteq \bigcup_{n=1}^{\infty} (-1, 1 - 1/n).
\]  \hspace{1cm} (71)

This shows \((-1, 1)\) is contained in the open cover.

Now note that for each finite subcollection \( \{V_{n_j}\}_{j=1}^{J} \) of \( \{V_n\}_{n=1}^{\infty} \) we have

\[
\bigcup_{j=1}^{J} V_{n_j} = (-1, 1 - 1/n_J).
\]  \hspace{1cm} (72)

Consequently,

\[
\left(1 - \frac{1}{2n_J}\right) \notin (-1, 1 - 1/n_J) = \bigcup_{j=1}^{J} V_{n_j}.
\]  \hspace{1cm} (73)

Thus there is no finite subcover of \( \{V_n\}_{n=1}^{\infty} \) containing \((-1, 1)\). \( \square \)
Example 21: Prove if \((X, d)\) is a compact metric space, then \(X\) is complete.

Proof:
Let \((x^{(n)})_{n=1}^{\infty} \subseteq X\) be Cauchy. We most prove this sequence converges. Because \(X\) is compact, every sequence in \(X\) has a convergent subsequence. This implies there is a subsequence \((x^{(n_k)k})_{k=1}^{\infty} \subseteq (x^{(n)})_{n=1}^{\infty}\) that converges to a limit \(z \in X\). We claim \(x^{(n)} \longrightarrow z\) and verify this as follows.

Let \(\varepsilon > 0\) be given. We must show there is \(N > 0\) such that

\[
d(x^{(n)}, z) < \varepsilon \quad \forall \ n > N. \tag{74}
\]

By hypothesis, there is \(N_1 > 0\) such that

\[
d(x^{(n_k)}, z) < \frac{\varepsilon}{2} \quad \forall \ k > N_1. \tag{75}
\]

Similarly, there is \(N_2 > 0\) such that

\[
d(x^{(n)}, x^{(m)}) < \frac{\varepsilon}{2} \quad \forall \ n, m > N_2. \tag{76}
\]

Now let \(N = \max\{N_1, N_2\}\) and note \(n_{N+1} \geq N + 1 > N\). Together with (75) and (76), this implies

\[
d(x^{(m)}, z) \leq d(x^{(m)}, x^{(n_{N+1})}) + d(x^{(n_{N+1})}, z) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \forall \ m > N. \tag{77}
\]

This verifies (74) and completes the proof. \(\blacksquare\)
Example 22: We say a metric space \((X, d)\) is \textit{totally bounded} provided for each \(\varepsilon > 0\), there is \(n \in \mathbb{N}\) and a finite collection \(\{B(x_i, \varepsilon)\}_{i=1}^n\) which covers \(X\), i.e.,

\[
X = \bigcup_{i=1}^n B(x_i, \varepsilon). \tag{78}
\]

Prove every totally bounded space \((X, d)\) is bounded.

\textit{Proof:}
Suppose \((X, d)\) is totally bounded. We must show there is a ball \(B(x, r) \subseteq X\) such that \(X \subseteq B(x, r)\).

By hypothesis, there is \(n \in \mathbb{N}\) and a collection \(\{B(x_i, 1)\}_{i=1}^n\) such that

\[
X = \bigcup_{i=1}^n B(x_i, 1). \tag{79}
\]

This implies for each \(z \in X\), there is an index \(j_z \in \{1, 2, \ldots, n\}\) such that \(z \in B(x_{j_z}, 1)\). Now set

\[
\alpha := \max_{1 \leq i \leq n} d(x_1, x_i). \tag{80}
\]

Then application of the triangle inequality yields

\[
d(x_1, z) \leq d(x_1, x_{j_z}) + d(x_{j_z}, z) \leq \alpha + d(x_{j_z}, z) < \alpha + 1. \tag{81}
\]

Consequently, \(z \in B(x_1, \alpha + 1)\). Because \(z\) was chosen arbitrarily in \(X\), we deduce \(X \subseteq B(x_1, \alpha + 1)\).

Thus we conclude \(X\) is bounded. \[\blacksquare\]
Continuity:

**Example 23:** Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x, y) = \begin{cases} 
0 & \text{if } x = y = 0, \\
\frac{2xy}{x^2 + y^2} & \text{otherwise.}
\end{cases} \quad (82)$$

Prove $f$ is not continuous with respect to the Euclidean metric $d_{\ell^2}$.

**Proof:**

By Theorem 2.1.5, $f$ is continuous if and only if whenever a sequence $(x^{(n)}) \subseteq \mathbb{R}^2$ converges to a point $x \in \mathbb{R}^2$ with respect to $d_{\ell^2}$ the sequence $(f(x^{(n)}))$ converges to $f(x)$. Consequently, to verify $f$ is not continuous, it suffices to construct a sequence converging to $(0, 0) \in \mathbb{R}^2$ for which the limit of the function values do not converge to $f(0, 0) = 0$. For the sequence $((1/n, 1/n))_{n=1}^{\infty} \subseteq \mathbb{R}^2$ observe

$$d_{\ell^2} \left( \left( \frac{1}{n}, \frac{1}{n} \right) \right) = \sqrt{\left( \frac{1}{n} \right)^2 + \left( \frac{1}{n} \right)^2} = \frac{\sqrt{2}}{n}. \quad (83)$$

Thus

$$\lim_{n \to \infty} d_{\ell^2} \left( \left( \frac{1}{n}, \frac{1}{n} \right) \right) = \lim_{n \to \infty} \frac{\sqrt{2}}{n} = 0, \quad (84)$$

which implies

$$\lim_{n \to \infty} \left( \frac{1}{n}, \frac{1}{n} \right) = (0, 0). \quad (85)$$

Moreover,

$$f \left( \frac{1}{n}, \frac{1}{n} \right) = \frac{2(1/n)^2}{(1/n)^2 + (1/n)^2} = 1 \quad \text{for each } n \in \mathbb{N}. \quad (86)$$

This shows

$$\lim_{n \to \infty} f \left( \frac{1}{n}, \frac{1}{n} \right) = \lim_{n \to \infty} 1 = 1 \neq 0 = f(0, 0), \quad (87)$$

and we conclude $f$ is not continuous. ■
**Example 24:** Let \((X, d_X)\) and \((Y, d_Y)\) be metric spaces. Suppose \(f : X \to Y\) is uniformly continuous. Prove if the sequence \((x^{(n)}) \subseteq X\) is Cauchy, then \((f(x^{(n)})) \subseteq Y\) is Cauchy.

*Proof:*

Suppose the sequence \((x^{(n)}) \subseteq X\) is Cauchy. Let \(\varepsilon > 0\) be given. Then to show \((f(x^{(n)}))\) is Cauchy, we must show there is \(N\) such that

\[
m, n > N \implies d_Y(f(x^{(n)}), f(x^{(m)})) < \varepsilon.
\]  

(88)

By the uniform continuity of \(f\), there is \(\delta > 0\) such that for \(x, y \in X\)

\[
d_X(x, y) < \delta \implies d_Y(f(x), f(y)) < \varepsilon.
\]  

(89)

And because the sequence \((x^{(n)})\) is Cauchy, there is \(N\) such that

\[
m, n > N \implies d_X(x^{(n)}, x^{(m)}) < \delta.
\]  

(90)

Combining (89) and (90) reveals if \(m, n > N\), then

\[
d_X(x^{(n)}, x^{(m)}) < \delta \implies d_Y(f(x^{(n)}), f(x^{(m)})) < \varepsilon.
\]  

(91)

This verifies (88) and completes the proof. ■
Example 25: Let \( f : X \to Y \) be continuous for some metric spaces \((X,d_X)\) and \((Y,d_Y)\). Fix \( \alpha \in Y \). Prove the set \( S = \{ x \in X : f(x) = \alpha \} \) is closed.

Proof:
By Proposition 1.2.15b, \( S \) is closed if and only if it contains all its adherent points. So, let \( (x^{(n)}) \subseteq S \) be a sequence converging to a limit \( x \in X \). Then the continuity of \( f \) reveals

\[
\alpha = \lim_{n \to \infty} \alpha = \lim_{n \to \infty} f(x^{(n)}) = f \left( \lim_{n \to \infty} x^{(n)} \right) = f(x),
\]

which implies \( x \in S \). Because the sequence \( (x^{(n)}) \) was chosen arbitrarily in \( S \), this shows \( S \) contains its adherent points and we conclude \( S \) is closed.

We now present an alternative proof.

Proof:
Note \( S = \{ x \in X : f(x) = \alpha \} = f^{-1}(\{\alpha\}) \). Because \( \{\alpha\} \) is a singleton set, Proposition 1.2.15d asserts \( \{\alpha\} \) is closed. Then Theorem 2.1.5d asserts \( f^{-1}(\{\alpha\}) \) is closed, which completes the proof.
Example 26:

a) Prove the function \( f : \mathbb{R} \rightarrow \mathbb{R} \) defined by \( f(x) = x^2 \) is not uniformly continuous.

b) Prove the function \( g : [0, 2] \rightarrow \mathbb{R} \) defined by \( g(x) = x^2 \) is uniformly continuous.

Proof:

a) By way of contradiction, suppose \( f \) is uniformly continuous. Then there is \( \delta > 0 \) such that for \( x, y \in \mathbb{R} \)

\[
|x - y| < \delta \quad \implies \quad |f(x) - f(y)| < 1. \tag{93}
\]

Fix any \( x \in \mathbb{R} \) and choose \( y = x + \delta/3 \). Then \( |x - y| = \delta/3 < \delta \) and so

\[
1 > |f(x) - f(y)| = |x^2 - y^2| = \left| x^2 - \left( x + \frac{\delta}{3} \right)^2 \right| = \left| -\frac{2\delta x}{3} + \frac{\delta^2}{9} \right| = \frac{\delta}{3} \left| \frac{\delta}{3} - 2x \right|. \tag{94}
\]

This implies

\[
\frac{\delta}{3} \left( \frac{\delta}{3} - 2x \right) < -1 \quad \implies \quad \frac{\delta}{3} - 2x < -\frac{3}{\delta} \quad \implies \quad x > \frac{1}{2} \left( \frac{\delta}{3} + \frac{3}{\delta} \right) > 0. \tag{95}
\]

Since \( x \) was chosen arbitrarily, this implies for each \( x > 0 \) for each \( x \in \mathbb{R} \), a contradiction. This contradiction shows the initial assumption was false and the result follows.

b) Let \( \varepsilon > 0 \) be given. We must show there is \( \delta > 0 \) such that for \( x, y \in [0, 2] \)

\[
|x - y| < \delta \quad \implies \quad |g(x) - g(y)| < \varepsilon. \tag{96}
\]

Observe for \( x, y \in [0, 2] \)

\[
|g(x) - g(y)| = |x^2 - y^2| = |x - y||x + y| \leq |x - y|(|x| + |y|) \leq |x - y|(2 + 2) = 4|x - y|. \tag{97}
\]

Letting \( \delta = \varepsilon/4 \), (97) reveals

\[
|x - y| < \delta \quad \implies \quad |g(x)g(y)| \leq 4|x - y| < 4\delta = 4 \left( \frac{\varepsilon}{4} \right) = \varepsilon. \tag{98}
\]

This verifies (96) and completes the proof.
Uniform Convergence

In this section we discuss different notions of convergence. We informally introduce the material as follows and refer the reader to our text for formal definitions. Suppose we have a sequence of functions \((f_n : X \to Y)\) and there is a function \(f : X \to Y\) such that for each \(x \in X\) we have \(\lim_{n \to \infty} f_n(x) = f(x)\). Then we say \(f_n \to f\) pointwise. This is a useful notion. However, there is another form of convergence known as uniform convergence. In this case, we can effectively say all parts of the function are converging to the limit at a comparable rate. In other words, if for every \(\varepsilon > 0\) there is \(N > 0\) such that

\[
d_Y(f_n(x), f(x)) < \varepsilon \quad \forall n > N, \quad x \in X,
\]

then we say \(f_n \to f\) uniformly. This notion is stronger and preserves continuity. Below we extend an example previously shown.
Example 27: Define the sequence of functions \( (f_n : [0, 2] \to \mathbb{R})_{n=1}^{\infty} \) by

\[
f_n(x) := \begin{cases} 1 - nx & \text{if } x \in [0, 1/n], \\ 0 & \text{if } x \in (1/n, 2]. \end{cases}
\]  

Define the function \( f : [0, 2] \to \mathbb{R} \) by

\[
f(x) := \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{if } x \in (0, 2]. \end{cases}
\]

(a) Prove \( f_n \longrightarrow f \) pointwise.

(b) Prove does not converge to \( f \) uniformly.

Proof:

a) Let \( \varepsilon > 0 \) be given and choose any \( x \in (0, 2] \). Then by the Archimedean property of \( \mathbb{R} \) there is \( N \in \mathbb{N} \) such that \( 1/N < x \). This implies \( x \in (1/n, 2] \) for \( n > N \), and so

\[
|f_n(x) - 0| = |0 - 0| = 0 < \varepsilon \quad \forall \ n > N.
\]

This shows \( \lim_{n \to \infty} f_n(x) = 0 = f(x) \) for \( x \in (0, 2] \). Now observe

\[
|f_n(0) - 1| = |1 - n0 - 1| = 0 < \varepsilon \quad \forall \ n \in \mathbb{N}.
\]

This shows \( \lim_{n \to \infty} f_n(0) = 1 = f(0) \) and completes the proof.

b) By way of contradiction, suppose \( f_n \longrightarrow f \) uniformly and let \( \varepsilon \in (0, 1) \). Then there is \( N \in \mathbb{N} \) such that \( n > N \) implies

\[
|f_n(x) - f(x)| < \varepsilon \quad \forall \ x \in [0, 1].
\]

In particular,

\[
|f_{N+1}(x) - f(x)| < \varepsilon \quad \forall \ x \in [0, 1].
\]

However, for \( x \in (0, (1 - \varepsilon)/N) \) we have \( 1 - \varepsilon > Nx \) and so \( 1 - Nx > \varepsilon \), which implies

\[
|f_{N+1}(x) - f(x)| = |(1 - Nx) - 0| > \varepsilon,
\]

a contradiction.

\[\blacksquare\]
**Remark 11:** To provide some intuition, we illustrate the above example below.

![Plot](image)

Figure 1: A plot $f_n$ and $\varepsilon \in (0, 1)$. Note $f_n(x) > \varepsilon$ for the choice of $x$ shown.

**Remark 12:** The following example is quite lengthy. We need not expect such a long problem to show up on an exam; however, individual tricks and ideas from individual parts of this example may be useful. Things that may be worth noting include the part about having a finite subcover since $[0, 1]$ is compact and there being an index $j$ such that $z \in B(x_j, \alpha)$. Also, note how we set $N$ to be the max of a finite number of $N_i$’s. Also note the layout of the proof. Namely, the proof starts by essentially stating the definition of what we have to prove and by providing the definitions of each of the hypotheses. Having these, we obtain a collection of inequalities and then combine these to obtain the initial equality that we set out to verify.

\[ \diamond \]
Example 28: Let \((f_n : [0, 1] \to \mathbb{R})\) be a sequence of functions converging pointwise to a continuous function \(f : [0, 1] \to \mathbb{R}\). Assume the sequence \((f_n)\) is uniformly Lipschitz. Prove \(f_n \to f\) uniformly.

Proof:

Let \(\varepsilon > 0\) be given. We must show there is \(N\) such that \(n > N\) implies

\[
|f_n(x) - f(x)| < \varepsilon \quad \forall \ x \in [0, 1].
\]  \hspace{1cm} (107)

Since \([0, 1]\) is closed and bounded, the Heine-Borel theorem asserts it is compact. Then because \(f\) is continuous on a compact set, \(f\) is uniformly continuous. Thus there is \(\delta > 0\) such that for \(x, y \in [0, 1]\)

\[
|x - y| < \delta \implies |f(x) - f(y)| < \frac{\varepsilon}{3}. \hspace{1cm} (108)
\]

Also since \((f_n)\) is uniformly Lipschitz, there is \(L > 0\) such that for \(x, y \in [0, 1]\)

\[
|f_n(x) - f_n(y)| \leq L|x - y| \quad \forall \ n \in \mathbb{N}. \hspace{1cm} (109)
\]

let \(\alpha = \min\{\varepsilon/3L, \delta\}\) and observe the collection \(\{B(x, \alpha)\}_{x \in [0, 1]}\) forms an open cover of \([0, 1]\). Since \([0, 1]\) is compact, there is a finite subcollection \(\{x_i\}_{i=1}^m \subset [0, 1]\) such that

\[
[0, 1] \subseteq \bigcup_{i=1}^m B(x_i, \alpha). \hspace{1cm} (110)
\]

Since \(f_n \to f\), for each \(i \in \{1, 2, \ldots, m\}\) there is \(N_i\) such that \(n > N_i\) implies

\[
|f_n(x_i) - f(x_i)| < \frac{\varepsilon}{3}. \hspace{1cm} (111)
\]

Now set \(N = \max\{N_1, N_2, \ldots, N_m\}\) and pick \(z \in [0, 1]\). Then there is \(j \in \{1, 2, \ldots, m\}\) such that \(z \in B(x_j, \alpha)\). This implies for \(n > N\)

\[
|f_n(z) - f(z)| \leq |f_n(z) - f_n(x_j)| + |f_n(x_j) - f(x_j)| + |f(x_j) - f(z)|
\]

\[
\leq L|z - x_j| + |f_n(x_j) - f(x_j)| + |f(x_j) - f(z)|
\]

\[
< L \left( \frac{\varepsilon}{3L} \right) + \frac{\varepsilon}{3} + \frac{\varepsilon}{3}
\]

\[
= \varepsilon,
\]  \hspace{1cm} (112)

where we have applied the above results and note \(|x_j - z| < \delta\) by choice of \(\alpha\). Since \(z \in [0, 1]\) was arbitrarily chosen, we conclude (107) holds and so \(f_n \to f\) uniformly. □
Example 29: Consider a decreasing sequence of functions \((f_n : [0, 1] \to \mathbb{R})\) obeying the uniform bound \(|f_n| \leq M\) for some \(M \in (0, 1)\). Suppose \(f_n \rightarrow f\) pointwise for some continuous function \(f : [0, 1] \to \mathbb{R}\). Prove \(f_n \rightarrow f\) uniformly.

Proof:
Let \(\varepsilon > 0\) be given. We must show there is \(N\) such that \(n > N\) implies
\[
|f_n(x) - f(x)| < \varepsilon \quad \forall \ x \in [0, 1]. \tag{113}
\]
For each \(n \in \mathbb{N}\) set \(g_n := f_n - f\), note \(g_n(x) \geq 0\) for each \(x \in [0, 1]\), and observe the set \(g_n^{-1}((\infty, \varepsilon))\) is open since it is the preimage of an open set and \(g_n = f_n - f\) is continuous. Since \(f_n \rightarrow f\) pointwise, \(g_n \rightarrow 0\) pointwise, which implies the collection \(\{g_n^{-1}((\infty, \varepsilon))\}_{n \in \mathbb{N}}\) forms an open cover of \([0, 1]\). Because \([0, 1]\) is closed and bounded, the Heine-Borel theorem implies this set is compact. Thus, there is a finite subset \(F \subset \mathbb{N}\) such that \(\{g_i^{-1}((\infty, \varepsilon))\}_{i \in F}\) forms an open cover of \([0, 1]\), i.e.,
\[
[0, 1] \subseteq \bigcup_{i \in F} g_i^{-1}((\infty, \varepsilon)). \tag{114}
\]
But, because \((f_n)\) is monotonically decreasing (i.e., \(f_{n+1}(x) \leq f_n(x)\) for each \(x \in [0, 1]\) and \(n \in \mathbb{N}\)), \((g_n)\) is also monotonically decreasing, which implies \(g_n^{-1}((\infty, \varepsilon)) \subseteq g_{n+1}^{-1}((\infty, \varepsilon))\) for each \(n \in \mathbb{N}\). Consequently, letting \(N = \max F\) yields
\[
[0, 1] \subseteq \bigcup_{i \in F} g_i^{-1}((\infty, \varepsilon)) = g_N^{-1}((\infty, \varepsilon)) \subseteq g_n^{-1}((\infty, \varepsilon)) \quad \forall \ n > N. \tag{115}
\]
Together (115) and the fact \(g_n \geq 0\) show \(n > N\) implies
\[
-\varepsilon < 0 \leq g_n(x) = f_n(x) - f(x) < \varepsilon \quad \forall \ x \in [0, 1], \tag{116}
\]
and so (113) holds. This completes the proof. 

\[\square\]