Purpose: This document is a compilation of notes generated for discussion in MATH 131B with reference credit due to Terrence Tao’s text *Analysis II*. If the reader finds any errors/typos, please feel free to email me at heaton@math.ucla.edu and I will address these and post an updated set of notes to my webpage.

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INTRODUCTION

These notes are provided to compliment the TA discussion sessions on Tuesdays for MATH 131B. Typically, more detail is provided here than on the board during discussions since portions of solutions are given orally in class. The examples provided here are meant to be a constructive reference for students. These illustrate how to use certain logical quantifiers, how to guide the reader through your proofs, and the level of rigor expected from students this quarter. Before reading each solution, I highly encourage students to first seriously attempt the problems on their own. I cannot overstate the value of struggling through these problems before comparing your attempts to the example solutions.

These notes will be updated weekly (if not more often) on Mondays or Tuesdays, reflecting the current discussion material.
**Metric Spaces**

**Remark 1:** For \( p \in [1, \infty) \), we can define the \( p \)-norm (also called the \( \ell^p \) norm) of vectors in \( \mathbb{R}^n \) by

\[
\|x\|_p := \left( \sum_{i=1}^{n} |x_i|^p \right)^{1/p}.
\] (1)

This can in turn be used to define a metric \( d_{\ell^p} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \) by

\[
d_{\ell^p}(x, y) := \|x - y\|_p = \left( \sum_{i=1}^{n} |x_i - y_i|^p \right)^{1/p}.
\] (2)

We also define the sup norm (or called “max” norm) on vectors in \( \mathbb{R}^n \) by

\[
\|x\|_\infty := \max_{1 \leq i \leq n} |x_i|,
\] (3)

which leads to the sup metric \( d_{\ell^\infty} \) defined by

\[
d_{\ell^\infty}(x, y) := \|x - y\|_\infty = \max_{1 \leq i \leq n} |x_i - y_i|.
\] (4)

Consequently, \( (\mathbb{R}^n, d_{\ell^p}) \) is a metric space for \( p \in [1, \infty] \). If we simply say the space \( \mathbb{R}^n \), then we implicitly mean the space \( (\mathbb{R}^n, d_{\ell^2}) \), with which we are well-familiar.

**Example 1:** Let \( (x^{(n)})_{n=1}^{\infty} \) be a sequence in \( \mathbb{R}^m \) converging to \( z \) with respect to the \( d_{\ell^\infty} \) metric. Prove \( x^{(n)} \to z \) with respect to the \( d_{\ell^1} \) metric.

**Proof:**

Let \( \varepsilon > 0 \) be given. We must show there is \( N > 0 \) such that

\[
d_{\ell^1}(x^{(n)}, z) < \varepsilon \quad \forall \ n > N.
\] (5)

First observe

\[
d_{\ell^1}(x^{(n)}, z) = \sum_{i=1}^{m} |x_i^{(n)} - z| \leq \sum_{i=1}^{m} \max_{1 \leq i \leq m} |x_i^{(n)} - z| = m \cdot \max_{1 \leq i \leq m} |x_i^{(n)} - z| = m \cdot d_{\ell^\infty}(x^{(n)}, z).
\] (6)

By the convergence given in our hypothesis, there is \( N > 0 \) such that

\[
d_{\ell^\infty}(x^{(n)}, z) < \frac{\varepsilon}{m} \quad \forall \ n > N,
\] (7)

and so

\[
d_{\ell^1}(x^{(n)}, z) \leq m \cdot d_{\ell^\infty}(x^{(n)}, z) < m \cdot \frac{\varepsilon}{m} = \varepsilon \quad \text{for all } n > N.
\] (8)

This verifies (5) and completes the proof.
**Remark 2:** We can also talk about metric spaces in similar fashion to \( (\mathbb{R}^n, d_{\ell^p}) \), but that are of infinite dimension. Indeed, let \( p \in [1, \infty) \) Then for each sequence \( x = (x_i)_{i=1}^{\infty} \) of numbers in \( \mathbb{R} \) we can take\(^1\)

\[
\|x\|_p := \left( \sum_{i=1}^{\infty} |x_i|^p \right)^{1/p} \quad \text{and} \quad d_{\ell^p}(x, y) := \|x - y\|_p = \left( \sum_{i=1}^{\infty} |x_i - y_i|^p \right)^{1/p}.
\]  

(9)

Note the only difference here is that our sums from before now turn into infinite series. Then let \( X \) be the space consisting of all sequences with finite \( p \)-norm, i.e.,

\[
X := \left\{ x = (x_i)_{i=1}^{\infty} : \|x\|_{\ell^p} := \left( \sum_{i=1}^{\infty} |x_i|^p \right)^{1/p} < \infty \right\}.
\]

(10)

Then \( (X, d_{\ell^p}) \) forms a metric space. This metric space is commonly written simply as \( \ell^p \).

In similar fashion to above, now consider \( p = \infty \). For a sequence \( x = (x_i)_{i=1}^{\infty} \) we set

\[
\|x\|_\infty := \sup_{i \in \mathbb{N}} |x_i| \quad \text{and} \quad d_{\ell^\infty}(x, y) := \sup_{i \in \mathbb{N}} |x_i - y_i|.
\]

(11)

When

\[
X := \left\{ x = (x_i)_{i=1}^{\infty} : \|x\|_{\ell^\infty} = \sup_{i \in \mathbb{N}} |x_i| < \infty \right\}.
\]

(12)

the combination \( (X, d_{\ell^\infty}) \) forms a metric space, denoted \( \ell^\infty \).

**Remark 3:** The sequence space \( \ell^2 \) is complete. That is, if a sequence \( (x^{(n)})_{n=1}^{\infty} \) of sequences is in \( \ell^2 \) and is Cauchy, then it converges to some limit in \( \ell^2 \).

\^1\Note here we use different notation to denote the sequences in order to avoid confusion in the example below.
Example 2: For each \( n \in \mathbb{N} \), let \( e^{(n)} = (e_i^{(n)})_{i=1}^{\infty} \) be the sequence of real numbers satisfying

\[
e_i^{(n)} = \begin{cases} 
1 & \text{if } n = i, \\
0 & \text{otherwise.}
\end{cases}
\]  

Note \( e^{(n)} \) is itself a sequence for fixed \( n \in \mathbb{N} \).

a) Show \( e^{(n)} \in \ell^2 \) for each \( n \in \mathbb{N} \).

b) Show the sequence \((e^{(n)})_{n=1}^{\infty} \) is not Cauchy in \( \ell^2 \).

c) Define the sequence \((x^{(n)})_{n=1}^{\infty} \) by \( x^{(n)} := n \cdot e^{(n)} \) for each \( n \in \mathbb{N} \). Is this sequence \((x^{(n)})_{n=1}^{\infty} \) in \( \ell^2 \)?

d) Is \((x^{(n)})_{n=1}^{\infty} \) bounded?

Solution:

a) Observe for each \( n \in \mathbb{N} \) we have

\[
\|e^{(n)}\|_{\ell^2} = \left( \sum_{i=1}^{\infty} |e_i^{(n)}|^2 \right)^{1/2} = \left( |e_n^{(n)}|^2 \right)^{1/2} = (1^2)^{1/2} = 1 < \infty.
\]  

Consequently, \( e^{(n)} \in \ell^2 \) for each \( n \in \mathbb{N} \).

b) It suffices to provide a single counter example. Let \( \varepsilon = 1 \). Then for \( m \neq n \) we have

\[
d_{\ell^2}(e^{(n)}, e^{(m)}) = \|e^{(n)} - e^{(m)}\|_{\ell^2} \\
= \left( \sum_{i=1}^{\infty} |e_i^{(n)} - e_i^{(m)}|^2 \right)^{1/2} \\
= \left( |e_m^{(m)} - e_m^{(m)}|^2 + |e_n^{(n)} - e_n^{(m)}|^2 \right)^{1/2} \\
= \left( |0|^2 + |1|^2 \right)^{1/2} \\
= \sqrt{2}.
\]  

Consequently,

\[
d_{\ell^2}(e^{(n)}, e^{(m)}) = \sqrt{2} > \varepsilon \text{ whenever } n \neq m.
\]  

This shows there does not exist \( N > 0 \) such that

\[
d_{\ell^2}(e^{(n)}, e^{(m)}) < \varepsilon \text{ for all } m, n > N,
\]  

and so the sequence \((e^{(n)})_{n=1}^{\infty} \) is not Cauchy.
c) We claim the sequence \( (x^{(n)})_{n=1}^\infty \) is in \( \ell^2 \). Indeed, for \( n \in \mathbb{N} \) we discover
\[
\|x^{(n)}\|_{\ell^2} = \left( \sum_{i=1}^\infty |x_i^{(n)}|^2 \right)^{1/2} = \left( |x_n^{(n)}|^2 \right)^{1/2} = (n^2)^{1/2} = n < \infty. \tag{18}
\]

\[\text{d) We claim the sequence } (x^{(n)})_{n=1}^\infty \text{ is unbounded. Indeed, by way of contradiction, suppose this sequence is bounded by some } M > 0. \text{ Then by the Archimedean property of } \mathbb{R} \text{ there is } k \in \mathbb{N} \text{ such that } k > M. \text{ Consequently, (18) shows}
\]
\[
\|x^{(k)}\|_{\ell^2} = k > M, \tag{19}
\]
contradicting the fact we assumed \( M \) was an upper bound. Thus the initial assumption was false and we conclude \( (x^{(n)})_{n=1}^\infty \) is unbounded.
\]

\[\square\]

**Remark 4:** The above example contains a sequence in a sequence space. In other words, \( (e^{(n)})_{i=1}^\infty \) is a sequence of sequences.

\[\diamond\]

**Remark 5:** The above example shows results we know in \( \mathbb{R}^n \) do not necessarily hold in infinite dimensional spaces. Indeed, recall the Bolzano-Weierstrass theorem discussed in MATH 131A last quarter, which tells us every bounded sequence in \( \mathbb{R}^n \) has a convergent subsequence, which in turn is Cauchy. However, in the above example, the sequence \( (e^{(n)})_{n=1}^\infty \) is bounded, but it does not have a Cauchy subsequence.

\[\diamond\]
**Lemma:** Let $\alpha \in (0, 1)$. Then $\alpha^n \rightarrow 0$ as $n \rightarrow \infty$. \(\triangle\)

**Proof:**
We claim this sequence is decreasing. Indeed, for each $n \in \mathbb{N}$ we see $\alpha^{n+1} = \alpha \alpha^n < \alpha^n$. And the sequence is nonnegative. Consequently, the monotone convergence theorem implies this sequence converges to a limit $z$. Moreover,

$$z = \lim_{n \to \infty} \alpha^n = \lim_{n \to \infty} \alpha^{n+1} = \alpha \lim_{n \to \infty} \alpha^n = \alpha z,$$

which holds precisely when $z = 0$. This completes the proof. \(\blacksquare\)

**Example 3:** Let $(X, d)$ be a metric space. Suppose the sequence $(x^{(n)})_{n=1}^{\infty}$ satisfies the inequality

$$d(x^{(n+1)}, x^{(n)}) \leq \alpha^{n+1} \text{ for } n \in \mathbb{N},$$

where $\alpha \in (0, 1)$. Is this sequence Cauchy? Prove your answer.

**Proof:**
We claim this sequence is Cauchy. Let $\varepsilon > 0$ be given. We must show there is $N > 0$ such that

$$d(x^{(m)}, x^{(n)}) < \varepsilon \quad \forall \ m, n > N. \tag{22}$$

Observe for $m, n \in \mathbb{N}$ with $n \geq m$ the triangle inequality yields

$$d(x^{(m)}, x^{(n)}) \leq \sum_{j=1}^{n-m} d(x^{(m-1+j)}, x^{(m+j)}) \leq \sum_{j=1}^{n-m} \alpha^{m+j}. \tag{23}$$

Evaluating this geometric sum gives

$$d(x^{(m)}, x^{(n)}) \leq \alpha^m \sum_{j=1}^{n-m} \alpha^j = \alpha^m \left( \frac{1 - \alpha^{n-m+1}}{1 - \alpha} \right) \leq \frac{\alpha^m}{1 - \alpha}. \tag{24}$$

From the above lemma, we know $\alpha^m / (1 - \alpha) \rightarrow 0$ as $m \rightarrow \infty$. This implies there exists $N > 0$ such that

$$\left| \frac{\alpha^m}{1 - \alpha} \right| < \varepsilon \quad \forall \ m > N. \tag{25}$$

Equations (24) and (25) together imply (22) holds, and we are done. \(\blacksquare\)
Example 4: Let \((X, d)\) be a metric space and fix \(z \in X\). The define the function \(f : X \to \mathbb{R}\) by \(f(x) := d(x, z)\). Prove \(f\) is continuous.

Proof:
This is currently left to the reader. A solution will be posted at a later date.

Example 5: Let \((X, d)\) be a metric space and \(x \in X\). Prove the set \(S := \{x\} \subset X\) is closed.

Proof:
We use the fact a set is closed if and only if its complement is open. It thus suffices to show \(S^c = X - \{x\}\) is open. Pick \(y \in S^c\). Then \(y \neq x\), and so \(d(x, y) > 0\). Let \(r = d(x, z)/2\) and \(z \in B(y, r)\). Then the triangle inequality yields

\[
d(x, y) \leq d(x, z) + d(z, y) \implies d(x, z) \geq d(x, y) - d(y, z) = s - d(y, z) > s - \frac{s}{2} = \frac{s}{2} > 0, \tag{26}
\]

and so \(z \neq x\), which implies \(z \in S^c\). This reveals \(B(y, r) \subset S^c\), and so \(y \in \text{int}(S^c)\). Because \(y\) was chosen arbitrarily in \(S^c\), it follows that \(S^c\) is open, from which we conclude \(S = \{x\}\) is closed.
Remark 6: For the following example, it may be helpful to draw several pictures, visualizing $E$ and the multiple open balls inside it, each centered at $x \in E$.

Example 6: Is every point of every open set $E \subset \mathbb{R}^n$ an adherent point of $E$? (Use the Euclidean norm.)

Proof: We claim the answer is yes. Let $x = (x_1, x_2, \ldots, x_n) \in E$. To verify $x$ is a limit point of $E$, we must show for each neighborhood of $x$ there exists $y \in E$ distinct from $x$ (i.e., $y \neq x$) contained in the neighborhood. Let $r > 0$ be given. Then it suffices to find $y \in E \cap B(x, r)$ with $y \neq x$. We do this as follows. Since $E$ is open and $x \in E$, there is $s > 0$ such that $B(x, s) \subset E$. Let $d = \frac{1}{2} \min \{s, r\}$ so that $B(x, d) \subset B(x, s) \cap B(x, r) \subset E \cap B(x, r)$. Then choose

$$y = (y_1, y_2, \ldots, y_n) = (x_1 + d/2, x_2, \ldots, x_n),$$

(27)

noting $y \neq x$, and observe

$$
\|x - y\|_2 = \left( \sum_{i=1}^{n} (y_i - x_i)^2 \right)^{1/2} = (\frac{d}{2})^{1/2} = d/2 < d.
$$

(28)

Consequently, $y \in B(x, d) \subset E \cap B(x, r)$, and we are done.

Example 7: Is every point of every closed set $E \subset \mathbb{R}^n$ an adherent point of $E$?

Proof: We claim the answer is no. Let $x \in \mathbb{R}$. Then the singleton set $E = \{x\}$ is closed. And, there are no elements in $E$ distinct from $x$ contained in the open ball $B(x, r)$ for any $r > 0$, and we are done.