Purpose: This document is a compilation of notes generated for discussion in MATH 131A with reference credit due to Kenneth Ross’s text *Elementary Analysis*, 2nd. Ed. If the reader finds any errors/typos, please feel free to email me at heaton@math.ucla.edu and I will address these and post an updated set of notes to my webpage.

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Introduction 1
INTRODUCTION

These notes are provided to compliment the TA discussion sessions on Tuesdays and Thursdays for MATH 131A Sections 2 and 5. Typically, more detail is provided here than on the board during discussions since portions of solutions are given orally in class. The examples provided here are meant to be a constructive reference for students. These illustrate how to use certain logical quantifiers, how to guide the reader through your proofs, and the level of rigor desired from students this quarter. Before reading each solution, I highly encourage students to first seriously attempt the problems on their own. I cannot overstate the value of struggling through these problems before comparing your attempts to the example solutions.

These notes will be updated weekly (if not more often), reflecting the current discussion material.

REMARK 1: In this course, it is of utmost importance to learn how to write complete thoughts and be precise/rigorous. We need to make every effort possible to be clear and concise.
Remark 2: The principle of mathematical induction asserts the statement $P_n$ is true for each $n \in \mathbb{N}$ provided $P_1$ is true and $P_{n+1}$ is true whenever $P_n$ is true. We apply this principle to complete the following examples.

Example 1: Prove $11^n - 4^n$ is divisible by 7 for each $n \in \mathbb{N}$.

Proof:

We proceed by way of induction. For each $n \in \mathbb{N}$, let the $n$-th proposition $P_n$ be the statement

$11^n - 4^n$ is divisible by 7.

(1)

In the base case, we see $11^1 - 4^1 = 11 - 4 = 7 = 7 \cdot 1$, and so $P_1$ holds. For the inductive step, suppose $11^k - 4^k$ is divisible by 7 for some $k \in \mathbb{N}$. This implies there is some $\alpha_k \in \mathbb{Z}$ such that $11^k - 4^k = 7\alpha_k$. Consequently,

$11^{k+1} - 4^{k+1} = (7)11^k + (4)11^k - (4)4^k = (7)11^k + 4 \left(11^k - 4^k\right) = 7 \left(11^k + 4\alpha_k\right),

(2)

where the first equality holds since $7 + 4 = 11$. Because $(11^k + 4\alpha_k) \in \mathbb{Z}$, we see 7 divides $11^{k+1} - 4^{k+1}$, i.e., $P_{k+1}$ holds, and we have closed the induction. The claim then follows by the principle of mathematical induction.
Example 2: Prove $1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2$ for all positive integers $n$.

Proof:
We proceed by using induction. For each $n \in \mathbb{N}$, let $P_n$ be the statement “$1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2$.” First observe $1^3 = 1 = 1^2$, and so the base case holds. For the inductive step, suppose $P_k$ holds for some $k \in \mathbb{N}$. From Example 1 (c.f. page 3) of our text, we know for this $k$

$$1 + 2 + \cdots + k = \frac{k(k+1)}{2}. \tag{3}$$

With the inductive hypothesis, this implies

$$1^3 + 2^3 + \cdots k^3 + (k+1)^3 = (1 + \cdots + k)^2 + (k+1)^3$$

$$= \left( \frac{k(k+1)}{2} \right)^2 + k(k+1)^2 + (k+1)^2$$

$$= \left( \frac{k(k+1)}{2} \right)^2 + 2 \left( \frac{k(k+1)}{2} \right) (k+1) + (k+1)^2$$

$$= \left( \frac{k(k+1)}{2} + (k+1) \right)^2 \tag{4}$$

$$= \left( \frac{k^2 + 3k + 2}{2} \right)^2$$

$$= \left( \frac{(k+1)(k+2)}{2} \right)^2$$

$$= (1 + \cdots + k + (k+1))^2.$$

This shows $P_{k+1}$ holds and, thus, closes the induction. Therefore we conclude, by the principle of mathematical induction, $P_n$ holds for each $n \in \mathbb{N}$. ■
**Remark 3:** Sometimes we may wish to prove a statement, which holds for a certain range of integers not including one. The following example illustrates one approach to tackling these kind of problems. The idea here is roughly to “reindex the statement $P_n$ so it starts with $n = 1$.”

**Example 3:** Prove $2^n > 2n$ for each $n \in \mathbb{N}$ satisfying $n > 2$.

Proof: For each $n \in \mathbb{N}$, let $P_n$ be the statement “$2^{n+2} > 2(n + 2)$.” It suffices to show $P_n$ holds for each $n \in \mathbb{N}$. Observe the base case $P_1$ holds since

$$2^{1+2} = 2^3 = 8 > 6 = 2(3) = 2(1 + 2).$$

Inductively, suppose $P_k$ holds for some $k$. Then observe

$$2^{(k+1)+2} = 2 \cdot 2^{k+2} > 2 \cdot 2(k + 2) = 2(k + 2) + 2(k + 2) > 2(k + 2) + 2 = 2((k + 1) + 2),$$

noting $k + 2 > 1$ for each $k \in \mathbb{N}$. This shows $2^{(k+1)+2} > 2((k + 1) + 2)$, i.e., $P_{k+1}$ holds, which closes the induction. Whence $P_n$ holds for each $n \in \mathbb{N}$ by the principle of mathematical induction. ■
Rational Zeros Theorem: Suppose $c_0, c_1, \ldots, c_n$ are integers and $r \in \mathbb{Q}$ is a solution to the equation

$$c_nx^n + c_{n-1}x^{n-1} + \cdots + c_1x + c_0 = 0,$$

where $n \geq 1$, $c_n \neq 0$ and $c_0 \neq 0$. Since $r$ is rational, there exists $c,d \in \mathbb{Z}$ with $d \neq 0$ and no common factors among $c$ and $d$ and such that $r = c/d$. Then $c$ divides $c_0$ and $d$ divides $c_n$. △

Example 4: Show $\sqrt[3]{10}$ is irrational.

Proof:
By way of contradiction, suppose $r := \sqrt[3]{10} \in \mathbb{Q}$. Then observe $r$ is a solution to the equation $x^3 - 10 = 0$ since $(\sqrt[3]{10})^3 = 10$. Because $r$ is rational, there exist $c,d \in \mathbb{Z}$ be such that $c$ and $d$ have no common roots, $d \neq 0$, and $r = c/d$. The Rational Zeros Theorem then asserts $d$ divides 1 and $c$ divides 8. This implies $r \in \{\pm 1, \pm 2, \pm 4\}$. However, this contradicts the fact $\sqrt[3]{8} \notin \{\pm 1, \pm 2, \pm 4\}$ since $1^3 = 1 \neq 10$, $2^3 = 8 \neq 10$, $5^3 = 125 \neq 10$ and $10^3 = 1000 \neq 10$. Thus the initial assumption was false and we conclude $\sqrt[3]{10}$ is irrational. ■

Example 5: Find all rational solutions to $x^3 - 3x^2 - 2x + 6 = 0$.

Proof:
We shall proceed by applying the Rational Zeros Theorem. Suppose $r \in \mathbb{Q}$ is a solution to $x^3 - 3x^2 - 2x + 6 = 0$. Because $r$ is rational, there exist $c,d \in \mathbb{Z}$ be such that $c$ and $d$ have no common roots, $d \neq 0$, and $r = c/d$. The Rational Zeros Theorem then asserts $d$ divides 1 and $c$ divides 6. This implies $r \in \{\pm 1/1, \pm 2/1, \pm 3/1, \pm 6/1\}$. For notational compactness, define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^3 - 3x^2 - 2x + 6$. Then observe

$$f(1) = 1^3 - 3 \cdot 1^2 - 2 \cdot 1 + 6 = 1 - 3 - 2 + 6 = 2 \neq 0.$$

Similarly,

$$f(-1) = 4, \quad f(2) = -2, \quad f(-2) = -10, \quad f(3) = 0, \quad f(-3) = -42, \quad f(6) = 102, \quad f(-6) = -306.$$

Then (8) and (9) together show the only element in $\{\pm 1, \pm 2, \pm 3, \pm 6\}$ that is a solution to $f(x) = 0$ is $r = 3$. Thus we conclude $r = 3$ is the unique rational solution to $f(x) = 0$. ■