PREORDER RELATIONS

In *Analysis of algorithms*, we come to a relation, "f is $O(g)$," that is both reflexive (on the space $\{f \mid f : I\!N^+ \to I\!R\}$) and transitive. This relation gives rise to an equivalence relation, "f is $\Theta(g)$."

The purpose of this supplement is to explain that some of what is said about the O relation and the Θ relation holds for *any* reflexive and transitive relation. *Analysis of algorithms* corresponds to Example 3 below.

Suppose then that R is any binary relation on a set U, i.e., $R \subseteq U \times U$. As usual, we will often write xRy to mean that $(x, y) \in R$. Recall that R is said to be *reflexive* on U if xRx for all x in U. And R is said to be *transitive* if whenever both xRy and yRz then xRz. A relation that is both reflexive on U and transitive is called a *preorder* on U.

The point of this discussion is to show that a preorder relation R on U (a) determines an equivalence relation $\equiv$ on U, and (b) partially orders the set $U/\!\equiv$ of equivalence classes.

Example 1: Suppose that U is the set of nonzero integers (positive or negative), and that

$$(m, n) \in R \iff m \text{ divides } n.$$

Then R is obviously reflexive on U, and transitivity is easy to check.

Example 2: Suppose that U is $\{0, 1\}^*$, the set of all binary strings. For a binary string s, let its *weight* $w(s)$ be the number of 1's in s. Let R be the relation on U defined by

$$(s, t) \in R \iff w(s) \leq w(t).$$

Again R is easily seen to be reflexive on U and transitive.

Example 3: Suppose that U is the collection of all functions from the positive integers into the reals as in *Analysis of algorithms* and that

$$(f, g) \in R \iff f \text{ is } O(g).$$

Then again R is obviously reflexive on U, and transitivity is easy to check.

Assume then that R is any reflexive transitive relation (i.e., a preorder) on a set U. Define the binary relation $\equiv$ on U by the condition:

$$x \equiv y \iff xRy \;\&\; yRx$$

for x and y in U. (In other words, the relation $\equiv$ is $R \cap R^{-1}$.)

Proposition: The relation $\equiv$ is an equivalence relation on U.

Proof: The definition makes it clear that $\equiv$ is symmetric. It inherits reflexivity and transitivity from R: For x in U, we have $x \equiv x$ because xRx. If $x \equiv y$ and $y \equiv z$ then we have four facts: xRy, yRx, yRz, and zRy. Regrouping these and using the transitivity of R, we get xRz and zRx, whence $x \equiv z$. $\dashv$

Hence the relation $\equiv$ partitions U into equivalence classes. Let $[x]$ denote the equivalence class to which x belongs. As you know (right?), $[x] = [y]$ if and only if $x \equiv y$. Now consider the "quotient set" $U/\!\equiv$ of all equivalence classes. We can define a binary relation $\leq$ on $U/\!\equiv$ by the condition:

$$[x] \leq [y] \iff xRy$$

for x and y in U.

Caution: There is something to prove here, namely that $\leq$ is "well defined" or "invariant." Suppose that $\mathbf{a}$ and $\mathbf{b}$ are two equivalence classes. We are attempting to define whether or not $\mathbf{a} \leq \mathbf{b}$ holds by *choosing* a particular x from $\mathbf{a}$ and a y from $\mathbf{b}$, and then testing to see if xRy. We need to verify that the verdict is independent of the particular choices made. Suppose that instead of x and y, we had chosen $x' \in \mathbf{a}$ and $y' \in \mathbf{b}$. What must be shown is that $xRy \Leftrightarrow x'Ry'$.

Once we see what must be shown, actually showing it is easy. Since x and x' are in the same equivalence class, we have $x \equiv x'$. Similarly $y \equiv y'$. It follows from transitivity that

$$x \equiv x' \;\&\; y \equiv y' \;\&\; xRy \implies x'Ry'.$$

Proposition: The relation $\leq$ is reflexive on $U/\equiv$, transitive, and antisymmetric. That is, $\leq$ is a partial order on $U/\equiv$.

Proof: Reflexivity and transitivity are inherited from R. Recall that a relation Q is said to be *antisymmetric* if whenever both aQb and bQa then $a = b$. Suppose, then, that both $[x] \leq [y]$ and $[y] \leq [x]$. Then we have both xRy and yRx, whence $x \equiv y$. Therefore $[x] = [y]$. $\dashv$

For equivalence classes $\mathbf{a}$ and $\mathbf{b}$, we write $\mathbf{a} < \mathbf{b}$ to mean that $\mathbf{a} \leq \mathbf{b}$ and $\mathbf{a} \neq \mathbf{b}$.

Proposition: The relation $<$ is irreflexive (i.e., never $\mathbf{a} < \mathbf{a}$) and transitive.

Proof: Irreflexivity is clear. Suppose that $\mathbf{a} < \mathbf{b} < \mathbf{c}$. Then clearly $\mathbf{a} \leq \mathbf{c}$, but could we have $\mathbf{a} = \mathbf{c}$? No, that would imply $\mathbf{a} \leq \mathbf{b} \leq \mathbf{a}$ whence $\mathbf{a} = \mathbf{b}$ by antisymmetry. $\dashv$

Example 1: In Example 1 above, we have $m \equiv n$ if and only if $|m| = |n|$. Each equivalence class contains exactly two numbers; $[n] = \{n, -n\}$. Under the partial order, [1] is the *least* class, that is, $[1] \leq [n]$ for every n. The partial order is not a total order; for example, [2] and [3] are incomparable.

Example 2: In Example 2, we have $s \equiv t$ if and only if $w(s) = w(t)$, that is, if and only if s and t have exactly the same number of 1's. Here are two of the equivalence classes:

$$\{\lambda, 0, 00, 000, 0000, 0000, \ldots\}$$
$$\{1, 01, 10, 001, 010, 100, 0001, 0010, 0100, 1000, \ldots\}$$

The first of these is the least equivalence class, and the second is the least of the remaining ones. The ordering of $\{0,1\}^*/\equiv$ is exactly like the ordering of the natural numbers. (The more precise way of saying this is that $(\{0,1\}^*/\equiv, \leq)$ is *isomorphic* to the natural numbers together with their usual ordering; the isomorphism is the function $f([s]) = w(s)$. We had better not go into the isomorphism concept right now; it will come up later.)

Example 3: In Example 3 we have seen the equivalence relation $\equiv$ before:

$$f \equiv g \iff f \text{ is } \Theta(g).$$

There is a least equivalence class, containing the functions that are eventually zero. The partial order is not a total order. But when *restricted* to polynomials it becomes a total order; for polynomials p and q we have $[p] \leq [q] \iff \deg p \leq \deg q$.

—H. B. Enderton