GROWTH RATES

In the section *Analysis of algorithms*, we compare the growth rates of various functions. Typically we are looking at functions f for which $\lim_{n\to\infty} f(n) = \infty$, and we want to say something quantitative about how *fast* $f(n)$ approaches infinity.

Although the intent of *Analysis of algorithms* is to apply the concepts to functions f where $f(n)$ is the worst-case running time of an algorithm on an input of size n, initially we will not focus on this application, but instead focus on the techniques themselves for measuring the growth rates of functions.

The "space" we are concerned with is the set S of all functions from the set $\mathbb{N}^+$ of positive integers into the set $\mathbb{R}$ of real numbers:

$$S = \{f \mid f : \mathbb{N}^+ \to \mathbb{R}\}$$

In other words, S is the set of infinite real-valued sequences.

On the space S we define two relations, O and Θ as in *Analysis of algorithms*. Instead of writing $(f, g) \in O$ or fOg, we write instead "f is $O(g)$." Similarly we write "f is $\Theta(g)$" to mean that f bears the Θ relation to g. (In fact some people even write "$f(n) = O(g(n))$" and "$f(n) = \Theta(g(n))$," but I am not one of them.)

Definition: (a) The function f is $O(g)$ if there is some constant C such that

$$|f(n)| \le C|g(n)| \quad \text{for almost all } n$$

(that is, the inequality holds for all but finitely many values of n).

(b) The function f is $\Theta(g)$ if both f is $O(g)$ and also g is $O(f)$.

Proposition: The O relation is reflexive on S and is transitive. The Θ relation is an equivalence relation on S.

Some of the important Θ-equivalence classes are listed in Table 4.3.3, such as

$$[1], \quad [n \log n], \quad [n^2], \quad [2^n], \quad [n!],$$

and so forth.

In many cases, we are able to calculate $\lim\limits_{n\to\infty} \left|\dfrac{f(n)}{g(n)}\right|$. In such cases, the following result compares the growth rates of f and g.

Theorem:

(a) If $\lim\limits_{n\to\infty} \left|\dfrac{f(n)}{g(n)}\right| = L$, with $0 < L < \infty$, then the functions f and g are Θ of each other. Hence their Θ-equivalence classes are the same: $[f] = [g]$.

(b) If $\lim\limits_{n\to\infty} \left|\dfrac{f(n)}{g(n)}\right| = 0$, then f is $O(g)$, but not the other way around. Hence $[f] < [g]$ in the partial order described on the "Preorder Relations" supplement.

(c) If $\lim\limits_{n\to\infty} \left|\dfrac{f(n)}{g(n)}\right| = \infty$, then g is $O(f)$, but not the other way around. Hence $[g] < [f]$ in the partial order.

If $\lim\limits_{n\to\infty} \left|\dfrac{f(n)}{g(n)}\right|$ does not exist, then we get no information from that fact.

In calculating $\lim\limits_{n\to\infty} \left|\dfrac{f(n)}{g(n)}\right|$, it will be useful to recall the following fact from calculus.

L'Hôpital's rule (1696): Assume that both $f(x) \to \infty$ and $g(x) \to \infty$ as $x \to \infty$. Then

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}$$

provided this limit exists.

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