

## LIMIT THEOREMS

**1. Markov's inequality:** "A non-negative random variable is unlikely to be far above its mean."

For any *non-negative* random variable  $X$ ,

$$P(X \geq a) \leq \frac{\mu}{a} \quad \text{for } a > 0.$$

Note that we do not need to know  $\sigma$ , and we do not need to know how  $X$  is distributed. It follows that for a non-negative  $X$  with non-zero mean,  $P(X \geq k\mu) \leq 1/k$ . For example, the probability that a non-negative  $X$  is twice its mean is  $1/2$  or less.

**2. Chebyshev's inequality:** "A random variable is unlikely to be far away from its mean."

$$P(|X - \mu| \geq c) \leq \frac{\sigma^2}{c^2} \quad \text{for } c > 0.$$

Note that we do not need to know how  $X$  is distributed. It follows (if  $\sigma$  is non-zero) that  $P(|X - \mu| \geq k\sigma) \leq \frac{1}{k^2}$ . For example, the probability that  $X$  is within two standard deviations of its mean is  $3/4$  or more, no matter how  $X$  is distributed.

**3. Weak law of large numbers:** "The average of a lot of measurements should be close to the mean."

If  $X_1, \dots, X_n$  are i.i.d. with mean  $\mu$ , and if  $\bar{X}_n$  is their average, then for any  $\varepsilon > 0$ ,

$$P(|\bar{X}_n - \mu| \geq \varepsilon) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Note that we do not need to know how the  $X_i$  are distributed.

**4. Central limit theorem:** "Averages and sums of measurements are always normally distributed, almost."

Assume that  $X_1, \dots, X_n$  are i.i.d. with mean  $\mu$  and standard deviation  $\sigma$ .

(a) Let  $\bar{X}_n$  be their average. Then  $\bar{X}_n$  is approximately normally distributed with mean  $\mu$  and standard deviation  $\sigma/\sqrt{n}$ , and the approximation becomes exact as  $n \rightarrow \infty$ .

(b) Let  $S_n$  be their sum. Then  $S_n$  is approximately normally distributed with mean  $n\mu$  and standard deviation  $\sigma\sqrt{n}$ , and the approximation becomes exact as  $n \rightarrow \infty$ . (A more precise statement of this is given on page 891.)

—H. B. Enderton