AGING

First, recall the following for a Poisson process with rate λ: One time period is just as likely as any other for a hit. And the probability of *no* hits in a time period of length t is

$$P(S = 0) = e^{-\lambda t} \cdot \frac{(\lambda t)^0}{0!} = e^{-\lambda t}.$$

Secondly, the exponential distribution was introduced for the waiting time until the first hit in a Poisson process:

$$P(L > t) = P(\text{no hits in time } 0 \text{ to } t) = e^{-\lambda t}$$

One application of this distribution is to lifespan (where a "hit" means dying) in situations where one time period is just as likely as any other for dying. (This can occur when the main causes of death are infections or injuries, and there is no significant chance of simply dying of old age.) Here is one way to look at what λ signifies here:

$$\lim_{\Delta t \to 0} \frac{P(\text{dying in the first } \Delta t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{1 - e^{-\lambda \Delta t}}{\Delta t}$$

This is a "0/0" limit. We apply L'Hôpital's rule, differentiating numerator and denominator with respect to Δt:

$$\lim_{\Delta t \to 0} \frac{0 + \lambda e^{-\lambda \Delta t}}{1} = \lambda e^0 = \lambda$$

And since one time interval is just like another, we can conclude that whenever the organism is alive,

$$\lim_{\Delta t \to 0} \frac{P(\text{dying in the next } \Delta t)}{\Delta t} = \lambda.$$

This justifies calling the constant λ the "hazard rate." It is a measure of how great the danger of dying is.

Thirdly, suppose that instead of a constant hazard rate λ, we allow a hazard rate function $\lambda(t)$, depending on the time t (where $t = 0$ at birth). Some time periods may be safer than others. And some time periods may be more dangerous. For a very old organism—when t is large—the hazard rate may increase to reflect the likelihood of dying of old age.

What now is the probability of surving to age t^*? We can partition the interval $[0, t^*]$ into n equal pieces of length Δt:

$$0 = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = t^*$$

where $t_k = k\Delta t$. To survive to age t^*, the organism must survive all n subintervals. In any one subinterval, the hazard rate does not change very much; throughout the subinterval $[t_{k-1}, t_k]$ the hazard rate is about $\lambda(t_k)$. So the probability of surviving all n subintervals is approximately given by the product

$$e^{-\lambda(t_1)\Delta t} \cdot e^{-\lambda(t_2)\Delta t} \cdot \ldots \cdot e^{-\lambda(t_n)\Delta t}$$

because $e^{-\lambda(t_k)\Delta t}$ is the probability, if the organism is still alive at the beginning of the kth subinterval, of making it through that subinterval. By the usual exponent laws, this product can be written

$$e^{-\sum_{k=1}^{n} \lambda(t_k)\Delta t}$$

and in the limit as $n \to \infty$ and $\Delta t \to 0$, the Riemann sum becomes the integral

$$e^{-\int_0^{t^*} \lambda(u)\, du}.$$

This gives us the following conclusion: For a hazard rate function $\lambda(t)$, the probability of surviving to age t^* is given by the equation

$$P(L > t^*) = e^{-\int_0^{t^*} \lambda(u)\, du}.$$

This is the equation on page 882 (where the derivation is presented somewhat differently). See Examples 20 and 21 there.

—H. B. Enderton