

## POISSON PROCESSES

There is a certain type of situation (a “Poisson process”) to which the Poisson distribution applies. It will be helpful to be able to recognize these situations.

**Example 1.** Cosmic rays hit your detector at random times, with an average rate of  $\lambda$  per day. What is the probability of exactly two hits tomorrow?

**Example 2.** UCLA’s emergency medical technicians get calls now and then. A study of the phone logs reveals no particular pattern to the calls, but in the long run they average  $\lambda$  calls per week. What is the probability that they will get at most three calls on Thursday?

**Example 3.** The cardiac ward has a lot of patients. Now and again, one of them has an acute angina attack. There is no systematic pattern, but the average rate is  $\lambda$  attacks per day. What is the expected number of attacks for next week?

In general, a *Poisson process* is a situation where the following conditions hold:

- Hits occur randomly, but with a long-term average rate  $\lambda$  per unit time.
- One hour is just as likely as another to have a hit. And the likelihood of a hit is completely independent of the past history.
- The hits are rare enough that in a very short time interval, there is a negligible chance of more than one hit.

Whenever we have a Poisson process, with hits occurring at an average rate of  $\lambda$  per unit time, then the probability that the number  $X$  of hits in a period of  $t$  units of time is exactly  $k$  (i.e.,  $P(X = k)$ ) is given by the Poisson distribution:

$$P(X = k) = e^{-\Lambda} \frac{\Lambda^k}{k!} \quad \text{where } \Lambda = \lambda t.$$

Note that  $\Lambda$  must be dimensionless, but typically  $\lambda$  has units of  $\text{time}^{-1}$  or  $\text{space}^{-1}$  and  $t$  has the opposite units.

This formula can be derived by taking the limit of shorter and shorter Bernoulli trials. That is, we can approximate the Poisson process by partitioning the time interval into  $n$  equal subintervals, each of length  $\Delta t = t/n$ . And in each subinterval we imagine a Bernoulli trial: success if we get a hit, failure if we don’t. The probability  $p$  of success equals  $\lambda \Delta t = \Lambda/n$ , so that altogether the expected number of hits is  $np = \Lambda = \lambda t$ .

Then the probability of exactly  $k$  hits is given by the binomial distribution:

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k} = \binom{n}{k} \left(\frac{\Lambda}{n}\right)^k \left(1 - \frac{\Lambda}{n}\right)^{n-k}$$

We take the limit as  $n \rightarrow \infty$  and  $p \rightarrow 0$ , holding  $np = \Lambda$  constant, as on page 845, and find that, in the limit,

$$P(X = k) = e^{-\Lambda} \frac{\Lambda^k}{k!}.$$

### Conclusions:

- The formula for the Poisson distribution is derived from the binomial distribution, by holding  $np = \Lambda$  and letting  $n \rightarrow \infty$ .
- A Poisson process can be thought of as “continuous trials,” where at every instant there is a trial with a tiny probability of success, but with an expected number  $\Lambda$  of successes overall.
- The Poisson distribution has a second application (in addition to its application to Poisson processes): It can be used to approximate the binomial distribution when  $n$  is large and  $p$  is small. As a rule of thumb, this approximation is pretty good when  $n > 100$  and  $np < 5$ . For  $np > 5$ , the normal distribution (§12.5.2) is apt to give a better approximation.