Schröder–Bernstein Theorem

The following theorem is useful in showing that two sets are the same size. It is somewhat analogous to showing that two numbers are equal by doing separately the "$\leq$" and the "$\geq$" parts, or to showing that two sets are equal by doing separately the two inclusions.

Schroöder–Bernstein Theorem. Assume that f is a one-to-one function from A into B, and g is a one-to-one function from B into A. Then A and B have the same cardinality; there is a one-to-one function h from A *onto* B.

Proof: Let $C_0 = A \setminus \operatorname{ran} g$. C_0 is the troublesome part that keeps g from being a one-to-one correspondence between B and A. If $C_0 = \emptyset$ then there is nothing to prove; we can simply take $h = g^{-1}$ and stop.

The strategy involves bouncing C_0 back and forth between A and B; that is, let $C_1 = \{g(f(x)) \mid x \in C_0\}$, and more generally, define C_n by recursion:

$$C_{n+1} = \{g(f(x)) \mid x \in C_n\}.$$

Here is the function h that we need:

$$h(x) = \begin{cases} f(x) & \text{if } x \in C_n \text{ for some } n, \\ g^{-1}(x) & \text{otherwise.} \end{cases}$$

Note that in the second case (where $x \in A$ but $x \notin C_n$ for any n), it follows that $x \notin C_0$ and hence $x \in \operatorname{ran} g$. So $g^{-1}(x)$ makes sense in this case.

Thus h is "usually" g^{-1}, but on the C_n's we use f. Does it work? We must verify that h is one-to-one and has range B. Define $D_n = \{f(x) \mid x \in C_n\}$, so that $C_{n+1} = \{g(y) \mid y \in D_n\}$. To show that h is one-to-one, consider distinct x and x' in A. Since both f and g^{-1} are one-to-one, the only possible problem arises when, say, $x \in C_m$ (so that $h(x) = f(x) \in D_m$) and $x' \notin C_n$ for any n (so that $h(x') = g^{-1}(x')$). In the latter situation, $h(x') = g^{-1}(x') \notin D_m$, lest x' be in C_{m+1}. So $h(x) \neq h(x')$.

Finally, we must check that $\operatorname{ran} h$ exhausts B. Certainly each $D_n \subseteq \operatorname{ran} h$, because $D_n = \{h(x) \mid x \in C_n\}$. Consider then a point y in B that is not in any D_n. Where is $g(y)$? Certainly not in C_0. Also $g(y) \notin C_{n+1}$, because $C_{n+1} = \{g(t) \mid t \in D_n\}$ and $y \notin D_n$ and g is one-to-one. So $g(y) \notin C_n$ for any n. Therefore $h(g(y)) = g^{-1}(g(y)) = y$. This shows that $y \in \operatorname{ran} h$. $\dashv$

The Schröder–Bernstein theorem is sometimes called the "Cantor–Bernstein theorem." Georg Cantor proved the theorem in his 1897 paper, but his proof utilized a principle that is equivalent to the axiom of choice. Ernst Schröder announced the theorem in an 1896 abstract. His proof, published in 1898, was imperfect, and he published a correction in 1911. The first fully satisfactory proof was given by Felix Bernstein and was published in an 1898 book by Borel.

Example: The powerset of $\mathbb{N}$ has the same cardinality as $\mathbb{R}$ and the open unit interval, $(0,1)$.

We already know that $\mathbb{R}$ and $(0,1)$ have the same cardinality. We can construct a one-to-one function f from $\mathcal{P}(\mathbb{N})$, the powerset of $\mathbb{N}$, to $(0,1)$ as follows: For each subset S of $\mathbb{N}$, let $f(S) = 0.\_\,\_\ldots$ where

$$\text{the } n\text{th decimal place} = \begin{cases} 3 & \text{if } n \in S \\ 7 & \text{if } n \notin S. \end{cases}$$

Then f is one-to-one, but it is not *onto* $(0,1)$. In fact we always have $1/3 \leq f(S) \leq 7/9$.

In the other direction, we can construct a one-to-one function g from $(0,1)$ to $\mathcal{P}(\mathbb{N})$ by representing each number in $(0,1)$ by its non-terminating binary expansion, and defining

$$g(x) = \{n \in \mathbb{N} \mid \text{the } n\text{th bit in the binary expansion of } x \text{ is } 1\}.$$

Then g is one-to-one, but is not *onto* $\mathcal{P}\mathbb{N}$; the set $g(x)$ is never finite.

The Schröder–Bernstein theorem can now be applied to give us the desired result.

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