Characterizing the Real Field

The real field consists of the set $\mathbb{R}$ of real numbers, together with the operations of addition and multiplication, the additive and multiplicative identities 0 and 1, and the ordering relation $\leq$.

We have come across sixteen properties of the real field. First, on page 2, there are the nine field axioms, P1–P9. (In P7, it should be added that $0 \neq 1$ in order to obtain a correct list of axioms for fields.) Secondly, on page 2, there are the five ordering properties, O1–O5. Thirdly, on page 4, we came to the Archimedian property. And finally, on page 48, we came to the "axiom of completeness," stating that Cauchy sequences converge.

These sixteen properties completely characterize the real field, in a sense to be made precise. That is, any complete Archimedian ordered field is exactly like (the technical term is "isomorphic to") the real field.

To state this more accurately, suppose we have some other set F, together with operations $+$ and $\times$ on F, additive and multiplicative identities 0_F and 1_F, and an ordering relation $\leq_F$. We will say that F is a *complete Archimedian ordered field* if this structure satisfies all sixteen properties: the field axioms, the ordering axioms, the Archimedian property, and the axiom of completeness.

Theorem. Assume that a set F, together with operations $+$ and $\times$, elements 0_F and 1_F, and relation $\leq_F$, is a complete Archimedian ordered field. Then there is a one-to-one function h from $\mathbb{R}$ onto F with the following properties (for all real numbers x and y):

(i) h "preserves addition": $h(x + y) = h(x) + h(y)$ (where the $+$ sign on the right side of the equation denotes addition in F, and the $+$ sign on the left side denotes addition in $\mathbb{R}$).

(ii) h "preserves multiplication": $h(xy) = h(x) \times h(y)$.

(iii) $h(0) = 0_F$ and $h(1) = 1_F$.

(iv) h is "order-preserving": Whenever $x \leq y$ then $h(x) \leq_F h(y)$

Moreover, there is a unique such function h.

The function h is said to be an *isomorphism* from the real ordered field to the F ordered field, and we say that the two fields are *isomorphic*.

A proof will not be given here, but we will see how the function h is built up.

First of all, condition (iii) tells us to define $h(0) = 0_F$ and $h(1) = 1_F$. This gets us started.

Secondly, condition (i) requires that we define, for example, $h(3) = 1_F + 1_F + 1_F$, because $3 = 1 + 1 + 1$. And similarly for any other natural number. So we now have $h(n)$ defined for each non-negative integer n.

Thirdly, because $3 + (-3) = 0$, we need to define $h(-3)$ to be the additive inverse of $h(3)$ in F. And similarly for the other negative integers.

Fourthly, condition (ii) requires that we define, for example, $h(5/3)$ to be the unique d in F for which $d \times h(3) = h(5)$. That is, $h(5/3) = h(5) \times h(3)^{-1}$, where on the right side we take the multiplicative inverse in F. And similarly for other rational numbers. So we now have $h(q)$ defined for each rational q.

Finally, we come to the irrational real numbers. We know that any irrational number r equals the supremum of the set of smaller rational numbers. The field F, being Archimedean and complete, has the least-upper-bound property, by §2.5. Condition (iv) requires that h is order-preserving, so we define

$$h(r) = \sup\{h(q) \mid q < r \text{ and } q \in \mathbb{Q}\}$$

where on the right side we take the supremum in F.

This completes the construction of the isomorphism h. The proof that it is one-to-one, onto, and has properties (i)–(iv) is not given here.

The point is that the sixteen properties give us a full picture of the structure of the real field. We know that non-Archimedean fields exist, which are very different from the reals. We know that incomplete ordered fields exist, for example, the rational field. But there is only one complete Archimedean ordered field, up to isomorphism, and that is the real field.

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