METHODS FOR CHAPTER 5

•To determine whether a function $T : V \to W$ is linear.

Compare $T(\mathbf{u}+\mathbf{v})$ with $T(\mathbf{u})+T(\mathbf{v})$ and compare $T(\alpha\mathbf{v})$ with $\alpha T(\mathbf{v})$. (For starters, check whether $T(\mathbf{0}_V) = \mathbf{0}_W$.)

•To calculate the matrix A_T representing T.

Apply T to the basis vectors of the domain V, then take coordinate vectors relative to the basis for W. This gives the columns of the matrix.

•To solve the equation $T(\mathbf{v}) = \mathbf{w}$ for $\mathbf{v}$.

Find the representing matrix $A = [\![T]\!]_{\mathcal{C}}^{\mathcal{B}}$ relative to some bases $\mathcal{B}$ and $\mathcal{C}$; solve the equation $A\mathbf{x} = [\![\mathbf{w}]\!]_{\mathcal{C}}$ for $\mathbf{x}$; decoordinatize. (There may be more direct methods, depending on the situation.)

•To test whether a linear transformation T is an isomorphism.

Is it one-to-one (what is its nullity)? Is it onto (what is its rank)? Because

$$\text{nullity} + \text{rank} = \text{dimension of the domain},$$

the two questions are related.

•To find a basis for $\ker T$.

Find a basis for the nullspace of a representing matrix, and decoordinatize it. (There may be more direct methods, depending on the situation.) If all you need is the nullity of T, then all you need is the dimension of the nullspace for the matrix.

•To find a basis for $\operatorname{ran} T$.

Find a basis for the column space of a representing matrix, and decoordinatize it. (There may be more direct methods, depending on the situation.) If all you need is the rank of T, then all you need is the rank of the matrix.

•To test whether two vector spaces (over the same field) are isomorphic.

Do they have the same dimension?

•To find the coordinate vector with respect to a new basis.

Forget the old basis, and calculate the coordinate vector as before. Or else use the equation $[\![\mathbf{v}]\!]_{\mathcal{B}} = Q^{-1}[\![\mathbf{v}]\!]_{\mathcal{S}}$, where $Q = [\![I_V]\!]_{\mathcal{S}}^{\mathcal{B}}$, so that column j of Q is $[\![\mathbf{b}_j]\!]_{\mathcal{S}}$, where $\mathbf{b}_j$ is $\mathcal{B}$'s jth vector. (The latter method gives the connection between the old coordinate vector and the new one.)

•To find the matrix representation of a linear operator with respect to a new basis.

Forget the old basis, and calculate the representing matrix as above. Or else use the equation $[\![T]\!]_{\mathcal{B}}^{\mathcal{B}} = Q^{-1}[\![T]\!]_{\mathcal{S}}^{\mathcal{S}}Q$, where Q is as above. (The latter method gives the connection between the old matrix and the new one.)