The Branch Problem

Is it possible to characterize the branchless trees of height and cardinality ω_1 that have a branch in some outer model with the same reals?

If V is a countable standard transitive model of ZFC, say that W is an *outer model* of V if $V \subseteq W$, W has the same ordinals as V, and $(W; V)$ satisfies ZFC in a language with a predicate symbol for V. For short, let us use "tree" to mean "branchless (in the inner model V) normal tree of height and cardinality ω_1"; say (painfully) that a tree is *branchable* if it has a cofinal branch in an outer model of V with the same reals.

The *branch problem* is to give a parameter-free first-order definition in V of the set of branchable trees. As stated, this problem is only interesting if we assume the CH in V.

In general, the set of branchable trees is not an element of V. If V is "sufficiently non-minimal", then it is. But there is no uniform definition of the set of branchable trees that works in all outer models of a given model.

The question, then, is whether the branch problem might be solvable if we work only with models of some recursive extension ZFC^+ of ZFC. To rule out trivial examples like ZFC + $V{=}L$, let us require also that ZFC^+ be compatible with large cardinals.

Obviously, the branch problem for set-generic outer models is solvable. Trees associated with coding proper classes are problematic, so ZFC^+ must have some "anti-coding" force. Generic absoluteness axioms cannot suffice because under CH the set of trees of height and cardinality ω_1 with branches is $\mathbf{\Sigma}^2_1(\mathcal{P}\mathbb{R})$ complete. If we require that certain large cardinals are preserved in the outer model in special ways, it is possible to give a Σ^2_2 (in the langauge of number theory) parameter-free characterization of branchable trees.