Appendix

Countability

Often we want to know the *size* of a set. On the one hand, there are the finite sets. On the other hand, there are the infinite sets. The infinite sets are bigger than the finite sets.

There is more to it, of course. There is the 0-element set, the empty set. There are the 1-element sets, the singletons (like $\{8\}$). There are the 2-element sets, the doubletons (like $\{0, 8\}$). And so forth and so on. Finite sets come in all sizes.

Something similar happens with the infinite sets. All the infinite sets are big, but some are bigger than others. We want to make sense of this idea, by extending some concepts (that are familiar in the finite case) to infinite sets.

For sets A and B, say that A is the *same size* as B (written $A \approx B$) if there is a one-to-one correspondence between them, that is, if there is a one-to-one function f whose domain is A and whose range is B. (In this situation, f^{-1} is a one-to-one function whose domain is B and whose range is A. Hence B is also the same size as A, so the concept is symmetric.)

Applied to finite sets, this concept tells us nothing much that is new. For infinite sets, the situation is more interesting. The possibly surprising fact about infinite sets is that they are *not* all the same size.

One infinite set is the set $\mathbb{N} = \{0, 1, 2, \ldots\}$ of all natural numbers. We can use natural numbers to give an exact characterization of finiteness: A set if finite iff there is a natural number n such that the set is the same size as $\{0, 1, \ldots, n-1\}$. (For the empty set, $n = 0$.)

Definition: A set is said to be *countable* if it the same size as some subset of $\mathbb{N}$. That is, a set S is countable if there is a one-to-one function $f : S \to \mathbb{N}$ mapping S into the natural numbers. Otherwise, the set is said to *uncountable.*

For example, any finite set is countable, because it has the same size as $\{0, 1, \ldots, n-1\}$, for some n. And $\mathbb{N}$ itself is countable, as are all of its subsets.

Theorem: Any infinite countable set has the same size as $\mathbb{N}$.

Thus the countable sets consist of the finite sets, plus the sets that are the same size as $\mathbb{N}$.

Proof. Assume that S is an infinite set that is countable, so that there is a one-to-one function $f : S \to \mathbb{N}$. We want a new function $g : S \to \mathbb{N}$ that is both one-to-one and *onto* $\mathbb{N}$. That is, we know that $\operatorname{ran} f \subseteq \mathbb{N}$ and we want $\operatorname{ran} g = \mathbb{N}$. The idea is push down $\operatorname{ran} f$, to squeeze out all the holes.

First of all, $\operatorname{ran} f$ contains some least member, say $f(s_0)$. (Because f is one-to-one, s_0 is unique.) We define $g(s_0) = 0$. More generally, for each n, there is a unique s_n in S for which $f(s_n)$ is the $(n+1)$st member of $\operatorname{ran} f$. We define $g(s_n) = n$. This gives us the function g we want: $\operatorname{dom} g = S$ and $\operatorname{ran} g = \mathbb{N}$.

Example: The set of all finite sequences of natural numbers is countable. In Chapter 2 we defined the bracket notation:

$$\begin{array}{rcl} [\,] & = & 1 \\ [x] & = & 2^{x+1} \\ [x,y] & = & 2^{x+1}3^{y+1} \\ [x,y,z] & = & 2^{x+1}3^{y+1}5^{z+1} \\ \cdots & & \\ [x_0,x_1,\ldots,x_k] & = & 2^{x_0+1}3^{x_1+1}\cdots p_k^{x_k+1} \end{array}$$

The function

$$\langle x_0,x_1,\ldots,x_k\rangle \mapsto [x_0,x_1,\ldots,x_k]$$

maps the set of sequences of natural numbers into $\mathbb{N}$, and it is one-to-one by the uniqueness of prime factorization.

Theorem: (a) Any subset of a countable set is countable.
(b) The union of two countable sets is countable
(c) The Cartesian product of two countable sets is countable.
(d) If A is a countable set, then the set A^* of all finite sequences of members of A is countable.
(e) The union of countably many countable sets is countable.

For example, the set $\mathbb{Z}$ of all integers (positive, negative, and zero) is a countable set. And the set $\mathbb{Q}$ of all rational numbers is countable. Part (d) tells us that over a countable alphabet A, the set A^* of all words is countable.

But not every set is countable. And by part (a) of the theorem, any set having an uncountable subset must be uncountable.

Cantor's theorem: (a) The set $\mathbb{R}$ of all real numbers is uncountable.
(b) The set $\mathcal{P}\mathbb{N}$ of all subsets of $\mathbb{N}$ is uncountable.
(c) The set of all infinite binary sequences (i.e., the set of all functions from $\mathbb{N}$ into $\{0,1\}$) is uncountable.
(d) The set $\mathbb{N}^{\mathbb{N}}$ of all function from $\mathbb{N}$ into $\mathbb{N}$ is uncountable.

This theorem is proved by the classical "Cantor diagonal argument."

Decadic notation

There is a simple and natural one-to-one correspondence between the set of all strings over a finite alphabet and the set of natural numbers. The key to the correspondence is to use base-n notation, where n is the size of the alphabet, but without a 0 digit.

Suppose that Σ is a finite set of size n. We refer to Σ as the *alphabet*, we refer to the members of Σ as *letters*, and we refer to finite strings of letters as *words* (over Σ). Let Σ^* be the infinite set of all words (including the empty word λ). For example, if $\Sigma = \{1, 2\}$, then

$$\Sigma^* = \{\lambda, 1, 2, 11, 12, 21, 22, 111, 112, 121, \ldots\}.$$

We assume that the members of the alphabet Σ are ordered in some way (referred to as *alphabetic order*). Thus we have a function $v : \Sigma \to \{1, \ldots, n\}$ where

$$\begin{aligned} v(\text{the first letter}) &= 1 \\ v(\text{the second letter}) &= 2 \\ &\cdots \\ v(\text{the last letter}) &= n \end{aligned}$$

and we refer to $v(a)$ as the *value* of the letter a.

We obtain a one-to-one map from Σ^* onto the set $\mathbb{N} = \{0, 1, 2, \ldots\}$ by mapping the three-letter word *abc* to the number

$$(abc)_{n\text{-adic}} = v(a)n^2 + v(b)n + v(c)$$

and in general,

$$(a_k \cdots a_2 a_1)_{n\text{-adic}} = v(a_k)n^{k-1} + \cdots + v(a_2)n + v(a_1)$$

and $(\lambda)_{n\text{-adic}} = 0$ (the empty sum).

In what follows, the properties of this map will be developed in the special case where $n = 10$ and the ten letters are $\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{6}, \mathbf{7}, \mathbf{8}, \mathbf{9}, \boldsymbol{X}$, in that order. But all of the arguments generalize immediately to alphabets of any size 2 or more.

(An alphabet of size 1 is a somewhat special and somewhat boring case. Where the alphabet is the singleton $\{|\}$, a typical word is the string $|||||||$, and the above equation reduces to

$$(a_k \cdots a_2 a_1)_{1\text{-adic}} = 1 + \cdots + 1 + 1 = k$$

so that $(w)_{1\text{-adic}}$ equals the length of the word w. This obviously produces a one-to-one map onto $\mathbb{N}$.)

In the $n = 10$ case, we will refer to $(w)_{10\text{-adic}}$ as $(w)_{\text{decadic}}$, the number denoted by the word w in *decadic* notation. (The corresponding words for $n = 1, 2, 3, \ldots$ would be monadic, dyadic, triadic, ...) For example,

$$(\mathbf{415})_{\text{decadic}} = 4 \cdot 100 + 1 \cdot 10 + 5 = 415$$

and

$$(\boldsymbol{4X5})_{\text{decadic}} = 4 \cdot 100 + 10 \cdot 10 + 5 = 505.$$

As the first of these equations exemplifies, for any word w not containing the $\boldsymbol{X}$ digit, $(w)_{\text{decadic}}$ is simply the number denoted by the numeral w in the usual base-10 notation. But the alphabet contains no zero digit, and some words contain the $\boldsymbol{X}$ digit (the "ten" digit). For example,

$$(\boldsymbol{XX})_{\text{decadic}} = 10 \cdot 10 + 10 = 110$$

and

$$(\boldsymbol{XXX})_{\text{decadic}} = 10 \cdot 100 + 10 \cdot 10 + 10 = 1,110.$$

First consider the problem of how to add 1 in decadic notation. Here are some examples:

$$\begin{array}{rcl} (\boldsymbol{3X8})_{\text{decadic}} + 1 & = & (\boldsymbol{3X9})_{\text{decadic}} \\ (\boldsymbol{XXX})_{\text{decadic}} + 1 & = & (\boldsymbol{1111})_{\text{decadic}} \\ (\boldsymbol{2XXX})_{\text{decadic}} + 1 & = & (\boldsymbol{3111})_{\text{decadic}} \end{array}$$

In general, a word w consists of a word u (possibly empty) not ending in $\boldsymbol{X}$, followed by a string of k $\boldsymbol{X}$'s (where $k \geq 0$). Thus

$$(w)_{\text{decadic}} = (u)_{\text{decadic}} \cdot 10^k + 10^k + 10^{k-1} + \cdots + 10.$$

In order to add 1, we use the word w^+ consisting of u^+ followed by a string of k $\mathbf{1}$'s, where u^+ is obtained by incrementing u's rightmost digit (and $\lambda^+ = \mathbf{1}$). This works because

$$\begin{array}{rcl} (w^+)_{\text{decadic}} & = & ((u)_{\text{decadic}} + 1) \cdot 10^k + 10^{k-1} + \cdots + 10 + 1 \\ & = & (u)_{\text{decadic}} \cdot 10^k + 10^k + 10^{k-1} + \cdots + 10 + 1 \\ & = & (w)_{\text{decadic}} + 1. \end{array}$$

Now that we know how to add 1, we obtain a theorem:

Theorem. For every natural number n, we can find a word w for which $(w)_{\text{decadic}} = n$.

Proof. We prove the existence of w by induction on n. The basis, $n = 0$, holds because $(\lambda)_{\text{decadic}} = 0$. For the inductive step, we add 1 as above.

And not only does w exist, but we know how to calculate it by adding 1 many times (n times). (Of course, there are much faster ways to find w, if we are in a hurry.) $\dashv$

Next consider the problem of comparing decadic numerals to see which one denotes a larger number. Among 4-digit numbers (i.e., among the numbers denoted by 4-digit words), the smallest is obviously $(\mathbf{1111})_{\text{decadic}}$. Any other 4-digit word will give us a larger number. Similarly, the largest number denoted by a 3-digit word is $(\boldsymbol{XXX})_{\text{decadic}}$. Any other 3-digit word will give us a smaller number. Moreover,

$$(\boldsymbol{XXX})_{\text{decadic}} < (\mathbf{1111})_{\text{decadic}}$$

because adding 1 to the left side gives the right side.

We can conclude from this that any number m denoted by a 3-digit word is less than any number denoted n by a 4-digit word:

$$m \leq (\boldsymbol{XXX})_{\text{decadic}} < (\mathbf{1111})_{\text{decadic}} \leq n$$

And the argument generalizes: Any number denoted by a k-digit word is less than any number denoted by a $(k+1)$-digit word. And then by iterating this argument, we see that any number denoted by a k-digit word is less than any number denoted by word of more than k digits. That is, shorter words always give smaller numbers.

What then about two words of the same length? For example, take the four-letter words $\mathbf{2}{*}{*}{*}$ and $\mathbf{3}{\star}{\star}{\star}$, where in both cases we don't know the last three digits. Even so, we know that

$$\begin{array}{rcll} \mathbf{2}{*}{*}{*} & \leq & \mathbf{2}\boldsymbol{XXX} & \\ & < & \mathbf{3111} & \text{by 1} \\ & \leq & \mathbf{3}{\star}{\star}{\star}. & \end{array}$$

Similary, for the six-letter words $\mathbf{472}{*}{*}{*}$ and $\mathbf{473}{\star}{\star}{\star}$, we have

$$\begin{array}{rcll} \mathbf{472}{*}{*}{*} & \leq & \mathbf{472}\boldsymbol{XXX} & \\ & < & \mathbf{473111} & \text{by 1} \\ & \leq & \mathbf{473}{\star}{\star}{\star}. & \end{array}$$

In general, we need to look only at the first (i.e., leftmost) digit where two words of the same length disagree. The larger digit produces a larger number, no matter what the later digits are. That is, for words of the same length, we simply use lexicographic order. We can summarize these ideas as follows:

Theorem. (a) For two words of different lengths, the shorter word denotes a smaller number than does the longer word.

(b) For two words u and w of the same length, $(u)_{\text{decadic}} < (w)_{\text{decadic}}$ iff u lexicographically precedes w.

In particular, two *different* words must denote different numbers, because either they will have different lengths (and clause (a) of the theorem will apply) or else there will be a first digit where they differ (and clause (b) will apply). That is, our map from words to numbers is one-to-one. Together with the previous theorem, we now have the following:

Theorem. Decadic notation yields a one-to-one correspondence between the set of all words over our 10-letter alphabet and the set $\mathbb{N}$ of all natural numbers.

This theorem illustrates a property of decadic notation which standard decimal notation lacks. In decimal notation, there is the problem of "leading zeros"; the words **3** and **03** denote the same number (or else **03** needs to be declared an illegal word).

Roman numerals are sometimes criticized for lacking a numeral for zero. But the real difficulty with Roman numerals is the lack of place-value notation. Zero itself is nothing.