RECURSIVE PARTIAL FUNCTIONS

Recursive total functions. Before discussing partial functions, let's review some facts about the total ones.

A total function $f : \mathbb{N}^m \to \mathbb{N}$ is said to be recursive if its graph is an $(m+1)$-ary recursive relation (page 210, page 247).

By *Theorem 33H (page 209), f is an effectively computable function iff its graph is a decidable relation. Therefore Church's thesis tells us that for total functions, the concept of a recursive function is the correct formalization of the concept of an effectively computable function.

Sections 3.3 and 3.4 produce a substantial supply of total recursive functions, including the characteristic functions of various recursive relations. Moreover, the class of total recursive functions is closed under composition (Theorem 33L, page 215), closed under the μ-operator (Theorem 33M, page 216), and closed under primitive recursion (Theorem 33P, page 222 and Exercise 8, page 224).

Partial functions. The concept of a partial function is defined on page 250. (As noted there, "partial" does not mean "non-total"; the class of partial functions includes both total and non-total functions.) The study of effectively computable functions is much more natural in the context of partial functions. We will write $f(\vec{a})\downarrow$ to mean that $f(\vec{a})$ is defined (i.e., $\vec{a} \in \operatorname{dom} f$). And we write $f(\vec{a})\uparrow$ to mean that $f(\vec{a})$ is undefined (i.e., $\vec{a} \notin \operatorname{dom} f$).

An m-place partial function is defined (page 251) to be a recursive partial function if its graph is an $(m+1)$-ary recursively enumerable (r.e.) relation (or equivalently, is a Σ_1 relation).

From *Theorem 36C (page 250) we see that a partial function is effectively computable iff its graph is an effectively enumerable relation. Combining this with Church's thesis (the second form, page 240), we can conclude that the concept of a recursive partial function is the correct formalization of the concept of an effectively computable partial function.

Any total recursive function is automatically a recursive partial function, so from Sections 3.3 and 3.4 we inherit some initial examples of recursive partial functions.

To show that certain partial functions are in fact recursive partial functions, it will be useful to have techniques for showing that relations are Σ_1. From Exercise 7 on page 246 we see that the union and intersection of Σ_1 relations will be Σ_1. And for total recursive functions $f_1, \ldots, f_m$, the relation

$$\{\vec{a} \mid \langle f_1(\vec{a}), \ldots, f_m(\vec{a})\rangle \in A\}$$

will be Σ_1 if A is.

In addition, the following two quantifier manipulation rules will be valuable.

Theorem 35E (Collapsing quantifiers). If Q is recursive and

$$\vec{a} \in R \iff \exists b_1 \cdots \exists b_n \langle \vec{a}, b_1, \ldots, b_n\rangle \in Q$$

then R is Σ_1, and hence r.e. (see page 238).

In fact it would be enough here to assume that Q is r.e.

The second quantifier manipulation rule (for bounded universal quantifiers) appears near the bottom of page 244:

$$(\forall c < b)(\exists d)\langle c, d\rangle \in P \iff (\exists d)(\forall c < b)\langle c, (d)_c\rangle \in P.$$

From this it follows that if Q is Σ_1 and

$$\langle \vec{a}, b\rangle \in R \iff (\forall c < b)\langle \vec{a}, b, c\rangle \in Q$$

then R is also Σ_1.

Theorem. The class of recursive partial functions is closed under composition. That is, if g is an n-place recursive partial function and $h_1, \ldots, h_n$ are m-place recursive partial functions, then the m-place partial function f defined by the equation

$$f(\vec{a}) = g(h_1(\vec{a}), \ldots, h_m(\vec{a}))$$

is a recursive partial function. (Here it is to be understood that $f(\vec{a})\downarrow$ iff $h_1(\vec{a})\downarrow$, $\ldots$, $h_n(\vec{a})\downarrow$, and moreover $g(h_1(\vec{a}), \ldots, h_n(\vec{a}))\downarrow$.)

Proof: Look at the graph of f:

$$f(\vec{a}) = b \iff \exists c_1 \cdots \exists c_n [h_1(\vec{a}) = c_1 \,\&\cdots\&\, h_n(\vec{a}) = c_n \,\&\, g(c_1, \ldots, c_n) = b].$$

On the right side, each equation expresses a Σ_1 condition on its variables. Pulling all the existential quantifiers into prenex position and then collapsing the quantifiers, we obtain a Σ_1 expression for the graph of f. ⊣

Theorem. The class of recursive partial functions is closed under the μ-operator. That is, if g is an $(m+1)$-place recursive partial function then the m-place partial function f defined by the equation

$$f(\vec{a}) = \mu b\,[g(\vec{a}, b) = 0]$$

is a recursive partial function. (Here it is to be understood that $f(\vec{a})\downarrow$ iff there exists some number b for which $g(\vec{a}, b)\downarrow$ and is 0, and moreover for all c less than b, $g(\vec{a}, c)\downarrow$ and is non-zero.)

Proof: Look at the graph of f:

$$f(\vec{a}) = b \iff g(\vec{a}, b) = 0 \,\&\, (\forall c < b)(\exists d)[g(\vec{a}, c) = d \,\&\, d > 0].$$

Again, on the right side, each equation expresses a Σ_1 condition on its variables. Pulling all the existential quantifiers to the left of the bounded universal quantifier and into prenex position, we again obtain a Σ_1 expression for the graph of f. ⊣