WHAT IS THE GEOMETRY

OF PHYSICAL SPACE?

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RIEMANN'S INAUGURAL TALK

June 10, 1854, Göttingen

Über die Hypothesen, welche der geometrie zu Grunde liegen (*On the Hypotheses which lie at the Foundations of Geometry*)

- **Concept of an n–dimensional manifold**
- **Riemannian geometry**
- **Curvature tensor and Flat manifolds**
- **Metric in space of constant curvature**

$$\frac{1}{1+\frac{\alpha}{4}\sum x^2}\sqrt{\sum dx^2}$$

- **Problem of Space**

PROBLEM OF SPACE

- **Different geometries on same manifold**
- **Physics $\Longrightarrow$ Geometry of Physical Space**
- **Geometry of Physical Space at small distances**

SPACE OF METRICS

In his work on Riemann surfaces which are complex manifolds of dimension 1 Riemann had already discovered that on a 2–dimensional compact topological (or smooth) manifold of genus g one can have many inequivalent complex structures.

Riemann was thus aware that on a given manifold there are many possible metric structures and so the problem of which structure is the one on physical space requires empirical methods for its solution.

He introduced the idea that to define a metric geometry it is sufficient to give the form of the distance function between infinitesimally near points, and then to define finite distances by computing the lengths of paths and taking the shortest paths.

RIEMANNIAN GEOMETRY

$$(ds)^2 = (dx^1)^2 + (dx^2)^2 + \ldots + (dx^n)^2 \qquad (\text{ euclidean})$$

$$(ds)^2 = \sum_{ij} g_{ij} dx^i dx^j \qquad (\text{ general Riemannian})$$

$$(ds)^2 = \frac{1}{y^2}(dx^2 + dy^2) \qquad (y > 0, \text{ Poincaré noneuclidean})$$

$$ds = F(x^1, \ldots, x^n, dx^1, \ldots, dx^n) \qquad (\text{ Finsler })$$

where F is homogeneous of degree 1 in the dx^i. This case, especially when F is the 4^{th} root of a homogeneous polynomial of degree 4 was already remarked on by Riemann and has come up surprisingly in recent work of Connes and Moscovici.

TWO THEMES OF RIEMANN

- Space does not exist independently of phenomena and its structure depended on the extent to which we can observe and predict what happens in the physical world.

  Riemann's vision became a reality when Einstein showed that it is spacetime and not space by itself that has an intrinsic significance, and that far from being a background to events, spacetime is a dynamic structure. Moreover the geometry of spacetime becomes noneuclidean in the presence of matter and is in fact pseudo Riemannian.

•• In the infinitely small the manifold structure of space may not be valid.

  This idea lay dormant till the search for a unified field theory at the quantum level forced the physicists to reconsider the structure of spacetime at extremely small distances.

"Now it seems that the empirical notions on which the metric determinations of Space are based, the concept of a solid body and a light ray, lose their validity in the infinitely small; it is therefore quite definitely conceivable that the metric relations of Space in the infinitely small do not conform to the hypotheses of geometry; and in fact, one ought to assume this as soon as it permits a simpler way of explaining phenomena ..."

An answer to these questions can be found only by starting from that conception of phenomena which has hitherto been approved by experience, for which Newton laid the foundation, and gradually modifying it under the compulsion of facts which cannot be explained by it. Investigations like the one just made, which begin from general concepts, can serve only to ensure that this work is not hindered by too restricted concepts, and that the progress in comprehending the connection of things is not obstructed by traditional prejudices. (Riemann, *Inaugural Talk*)

EINSTEIN

- **Spacetime as a pseudo Riemannian manifold**
- **Gravitation as curvature of spacetime**
- **Bending of light in a gravitational field**

MINKOWSKI

Space and Time

"*The views of space and time which I wish to lay before you have sprung from the soil of experimental physics, and therein lies their strength. They are radical. Henceforth space by itself, and time by itself, are doomed to fade away into mere shadows, and only a kind of union of the two will preserve an independent reality.*"

(From an Address delivered at the 80^{th} Assembly of German Natural Scientists and Physicians, at Cologne, 21 September, 1908.)

Minkowski's work ranges from the most abstract aspects of number theory to the most concrete studies of physics. He was a close friend of Hilbert and exerted a profound influence on him.

REMARKS

Even in Newtonian mechanics one admits coordinate frames which are in uniform motion with respect to each other and the transformation of coordinates between these frames couple the space and time coordinates. The splitting of space and time is thus not an invariant process and one can speak of only *spacetime*, the set of *world points,* in an invariant manner. In special relativity this is an *affine space* with a distinguished nondegenerate quadratic form of signature $(+++-)$, called *Minkowski spacetime.* From our point of view we view the change as the replacing of the euclidean geometry of space by the *Minkowskian geometry* governed by an *indefinite quadratic form.* In general relativity spacetime is allowed to be Minkowskian only infinitesimally. Thus spacetime becomes a manifold with an indefinite metric, a *pseudo Riemannian* manifold. Gravitation is then a manifestation of the curvature of spacetime. Einstein's calculation of the deflection of light in the gravitational field of the sun and its subsequent verification in 1919 must therefore be regarded as the climax of a long series of ideas that began with the measurements of Gauss of triangles in the Hannover region.

GEOMETRY FROM ALGEBRA

- **Gel'fand Principle**

  The geometric structure of space can be recovered from the commutative ring of functions on it.

  $C(X)$, the ring of continuous functions on a compact Hausdorff spacxe X, determines X up to a homeomorphism

  $\mathbf{C}(X)$, the field of meromorphic functions on a compact Riemann surface X, determines X up to a complex analytic isomorphism

- **Grothendieck Principle**

  Any commutative ring is essentially the ring of functions on some space. The ring is allowed to have nilpotents whose numerical values are 0 but which play an essential role in determining the geometric structure.

  $\mathbf{C}[X, Y]/(X)$ is the ring of functions on the line $X = 0$ in the XY–plane

  $\mathbf{C}[X, Y]/(X^2)$ is the ring of functions on the double line $X^2 = 0$ in the XY–plane

DESCRIPTION OF GROUPS BY THEIR COORDINATE RINGS

- **Coordinate rings of groups**

  If G is a group, its coordinate ring $A(G)$ admits a comultiplication, antipode, and counit, all naturally coming from the multiplication, inverse, and unit of G. For instance, if f is in $A(G)$ and $f(xy)$ can be expressed as $\sum_i f_i(x)g_i(y)$ for suitable $f_i, g_i \in A(G)$ for all $x, y \in G$, the multiplication takes f to $\sum f_i \otimes g_i$.

  comult : $A(G) \longrightarrow A(G) \otimes A(G)$

  antipode : comult : $A(G) \longrightarrow A(G)$

  counit : comult : $A(G) \longrightarrow k$ (k is the ground field)

  For $G = GL(n)$ these are respectively

$$a_{ij} \longrightarrow \sum_r a_{ir} \otimes a_{rj}$$
$$a_{ij} \longrightarrow a^{ij}$$
$$a_{ij} \longrightarrow \delta_{ij}$$

- **Hopf Algebras**

  An abstract algebra with the structure of comultiplication, antipode, and counit, is a *Hopf Algebra.*

- **Groups=Commutative Hopf Algebras**

THE ORIGIN OF SUPERGEOMETRY

- **Quantum Fermions**

  Fermions are elementary particles with highly non-classical properties (spin, exclusion principle). Protons, neutrons, electrons which make up matter are fermions. Radiation consists of photons which are *Bosons.* In quantum electrodynamics transformations between Ferminons and Bosons are possible and occur all the time.

- **Classical Fermions**

  Quantum systems are usually obtained from classical systems by *quantization.* But no fermionic system is ever obtained this way. In the 1970's, Physicists (**Zumino, Wess, Ferrara, Salam, Strathdee,** and others) asked the question whether it is possible to invent classical systems whose quantizations lead to both fermions and bosons. The exclusion principle and spin properties of fermions imply that the coordinates describing a classical fermion must be *anticommuting.* A *superspace* or a *supermanifold* is a space which requires both commuting and anticommuting coordinates for its local description. Transformations that excahnge commuting and anticommuting coordinates are called *supersymmetries.*

LINEAR SUPERALGEBRA

- **Supervector spaces**

$$V = V_0 \oplus V_1 \qquad (\mathbf{Z}_2 - \text{grading})$$

Morphisms preserve grading (even)

- **Superalgebra**

The underlying space of A is super and the multiplication $A \otimes A \longrightarrow A$ is even.

- **Rule of signs**

Whenever we permute two elements a, b in a classical formula one has to have a sign factor $(-1)^{p(a)p(b)}$ where p is the parity function of homogeneous elements (0 for even and 1 for odd). For example, by a *supercommutative algebra* is meant an algebra such that for all homogeneous elements a, b,

$$ab = (-1)^{p(a)p(b)} ba$$

- **Superdeterminant (Berezinian)**

$$\text{Ber} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(A) \det(I - BD^{-1}C) \det(D)^{-1}$$

(A, D are even and B, C are odd)

SUPERMANIFOLDS

- **Exterior Algebra**

  An *Exterior Algebra over a commutative algebra* A is the algebra $A[\theta_1, \ldots, \theta_q]$ generated over A by elements

  $$\theta_1, \theta_2, \ldots, \theta_q$$

  with the relations

  $$\theta_i^2 = 0, \qquad \theta_i \theta_j = -\theta_j \theta_i \ (i \neq j)$$

  Exterior algebras are supercommutative.

- **Supermanifold**

  On a supermanifold at each point of one can establish local coordinates

  $$x_1, \ldots, x_m, \theta_1, \ldots, \theta_q$$

  where the x_i are usual commutative coordinates and the θ_j are anticommuting coordinates, the x's and θ's commuting with each other.

  $$\mathbf{R}^{p|q} \qquad \text{(coordinate ring } C^\infty(\mathbf{R}^p)[\theta_1, \ldots, \theta_q])$$

- **Berezin** was one of the first to treat supergeometry from a serious mathematical perspective.

MANIFOLDS AND SUPERMANIFOLDS

- **Commuting coordinates $\longrightarrow$ supercommuting coordinates**

  The coordinate algebras of supermanifolds are supercommutative

- **Nilpotent elements in coordinate rings**

  Coordinate algebras contain elements whose numerical values are 0

- **Schemes and supermanifolds**

  Supermanifolds are more general because of noncommutativity, but less general because the coordinate rings are more structured. Further they are smooth.

- **Superschemes**

  These encompass both supermanifolds and schemes and were first systematically studied by **Manin.**

SUPER LIE GROUPS

- **Super Lie group** is a group object in the category of supermanifolds.

- **Supergroups=Hopf superalgebras**

  Both definitions are essentially equivalent. Supergroups are *not* groups but their points over supercommuting rings *are* groups.

  $R^{1|1}$: The multiplication is

  $$(t_1, \theta_1), (t_2, \theta_2) \longmapsto (t_1 + t_2 + \theta_1\theta_2, \theta_1 + \theta_2)$$

  $GL(p|q)$: $p|q$–block matrices

  $$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

  where A, D are invertible matrices of even coordinates and B, C are matrices of odd coordinates, the multiplication law being the usual one.

  For $GL(n)$ the points over a commuting ring form the group $GL(n, A)$. For $GL(p|q)$ the points over a supercommuting ring A still form a group $GL(p|q, A)$. This is the reason why in the above examples we can manipulate the symbols as if we are dealing with groups.

SUPERSPACETIME

- **Minkowski superspacetime**

  There are many possibilities. The commutative part is of course M, the usual Minkowski spacetime. The odd part has several possibilities depending on the structure of the representation of the Lorentz group on the odd variables. One of the most interesting is $M^{4|4}$ where there are 4 anticommuting coordinates on usual Minkowski space M arising from the two spin representations of the Lorentz group. The super Poincaré group is the group of automorphisms of $M^{4|4}$.

  Physicists call this *rigid supersymmetry* because the affine character of spacetime is preserved. For constructing *supergravity* one has to construct *local supersymmetries.* This is a much deeper affair.

SUPERFIELDS AND WAVE EQUATIONS

- The superfields are elements of the coordinate ring of M^4 and $M^{4|4}$ is a super Lie group so that it acts by left translation on the space of superfields. It turns out that one can introduce complex odd coordinates and obtain wave equations by using supersymmetric Lagrangians.

- **Wave equations**

  The left invariant vector fields on $M^{4|4}$ are D_a. The simplest wave equation is

  $$D_a D^a \Phi = 0$$

  where Φ is a superfield:

  $$\Phi = \varphi_0 + \psi_a \theta^a + F\theta^1\theta^2$$

  This single wave equation on $M^{4|4}$ is equivalent to the equations on M given by

  $$\partial_a \partial^a \varphi = 0, \qquad \sigma_a \psi_a = 0, \qquad F = 0$$

  where φ is a scalar massless boson, ψ a Weyl fermion.

- **SUSY partners** The massless boson and the Weyl fermion (*photon* and *photino*) are *susy partners.*

PREDICTIONS

- **SUSY Partners**

  The known elementary particles have susy partners or more generally part of *supermultiplets.*

- **Prediction of mass**

  Susy quantum field theory predicts bounds for the mass of the heavy quark.

  The new super collider LHC being built at CERN has probably high enough energy to see supersymmetric partners or at least evidence of supersymmetry. There are recent measuremnts that are troblesome for the standard model and they are closer to the numbers predicted by the susy standard model. Whatever the future, supersymmetric mathematics is a beautiful generalization of classical differential geometry, and unifies a great number of mathematical disciplines.