The Coming Infinite-Dimensional Calculus

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Abstract.

The calculational techniques developed by physicists over the last century in their search for the ultimate quantum description of the universe—shunned by mathematicians during most of their gestation for lack of rigor—have finally in the last decade begun to exert profound and startling effects on many areas of mathematics. Just as it took centuries for mathematicians to put the original, finite-dimensional calculus of classical physics on a rigorous footing, it will probably be a major, long-term project to make complete sense of the rules and heuristics of the new, infinite-dimensional calculus of quantum physics. But in the meantime, as the ideas reverberate through mathematics, no graduate student should remain unaware.

A Brief History of Calculus.

−300	
	Archimedes (exhaustion, geometric series)
−200	
⋮	
1600	
	Kepler, Cavalieri, Wallis (indivisibles)
	Newton, Leibniz (fundamental theorem)
1700	
	Euler, Maupertuis (calculus of variations)
1800	Abel, Cauchy (convergence)
	Weierstrass, Dedekind, Cantor, Poincaré (foundations of analysis, asymptotics)
1900	Lebesgue (measure theory)
	Von Neumann (computers)
	Connes (noncommutative differentials)
2000	

Notes on History of Calculus.

Centuries of development not surprising; calculus really is complicated. Consider that definition of limit involves 3 levels of quantification: $\forall \varepsilon > 0\ \exists \delta > 0 :\ \forall x, \ldots$

As calculus was rigorized and neatly packaged, some important things were lost:

- Important subjects fell out of favor: asymptotic (divergent) series, infinitesimals.
- The deep links to nonrigorous physics were anomalously abandoned for half a century.

Connes: infinitesimals are compact operators with certain eigenvalue decay, e.g.,

$$dx := \left(\tfrac{d^2}{dx^2}\right)^{-\frac{1}{2}} \text{ (note } x\,dx \neq dx\,x\text{).}$$

$$\int f(x)\,dx := \mathrm{tr}(f(x)\,dx).$$

History of Quantum Field Theory.

1900 Planck (quanta)

Einstein (special relativity)

Einstein (general relativity)

1920

Schrödinger, Heisenberg, Dirac (QM)
Dirac (unified QM and special relativity)
Kaluza, Klein (internal geometry)

1940 Feynman (path integrals, unified QM/EM)

1960

Nambu (string theory)
Golfand, Likhtman (supersymmetry)
(nonabelian gauge theory)
Yau (Calabi-Yau manifolds)
1980 Schwarz (superstring theory)
Donaldson (4D differential topology)
Witten, Seiberg (topological QFT)
Vafa, Witten (mirror symmetry)
Hull, Townsend, Witten (dualities, M-theory)
2000

Notes on History of QFT.

Drama of the century: QM and General Relativity—and the profound difficulty (theoretical and experimental) of their unification.

Uncertainty Principle: $dx\,dp \geqslant \hbar$.
Quantum Gravity: $dx\,dx \geqslant \ell_P^2$, where $\ell_P = 10^{-35}$m!

Feynman's infinite-dimensional integrals set off a long (and uncharacteristic) rift between physics and mathematics.

String theory was originally a failed theory for strong nuclear interactions, in which the (unrecognized) graviton produced by (unrecognized) strings was an annoyance.

High noon: physics and math conflict over number of cubic rational curves in a generic quintic Calabi-Yau 3-fold (mathematicians couldn't count beyond cubics).

Nonrigorous physics wins! (Answer: 317,206,375.)
Lesson: Rigor without good heuristics is brittle!

This subject is not yet neatly packaged; lots of math and physics background is needed. And the time to learn it is now—in grad school.

“Dualities” are exact, non-perturbative isomorphisms and automorphisms of QFTs.

Some Mathematical Areas of Impact of QFT.

Differential geometry: Morse theory, instantons, Calabi-Yau spaces.

Differential topology: Donaldson theory.

Algebraic geometry: Moduli spaces, Calabi-Yau spaces.

Representation theory: Integrable systems.

Topology: Knot invariants.

Why Infinite-Dimensional Calculus?

CM = infinite-dimensional differential calculus:

Maupertuis' principle of least action (1744): path taken by system is determined by

$$\delta S = 0,$$

where the *action* $S = \int (\mathsf{KE} - \mathsf{PE})\, dt$.

QM = infinite-dimensional integral calculus:

Feynman's principle of action diffraction (1942):

$$Z = \int_{\mathsf{paths}} e^{iS/\hbar}.$$

As $\hbar \to 0$, constructive interference near critical paths ($\delta S = 0$) dominates the integral.

Feynman Diagrams.

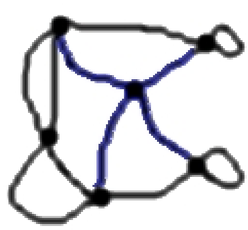
Prototype (0+0)-dimensional Euclidean QFT:

$$Z(\lambda) = \int_{-\infty}^{\infty} \exp\Bigg(\overbrace{- \underbrace{\frac{y^2}{2}}_{\text{kinetic energy}} + \underbrace{\lambda}_{\text{coupling constant}} \underbrace{\frac{y^4}{4!}}_{\text{interaction potential}}}^{\substack{\text{(unnormalized) probability density} \\ \text{makes higher energies less probable}}} \Bigg) \underbrace{dy}_{\text{integrate over all fields } y} .$$

As $\lambda \to 0$ this has asymptotic expansion

$$Z(\lambda) \sim \sqrt{2\pi} \sum_{k=0}^{\infty} \frac{(4k-1)(4k-3)\cdots 3 \cdot 1}{k!\, 4!^k} \lambda^k.$$

Coefficient of λ^k is the "symmetry-corrected" number of 4-valent graphs with exactly k vertices (loops and multiple edges allowed).



Feynman's intepretation: each edge represents a propagating particle, each vertex represents a pair-pair interaction.

Bosonic Integration.

Recall that

$$\int \frac{dx}{\sqrt{\pi}} e^{-bx^2} = \frac{1}{\sqrt{b}}.$$

Taking a product yields

$$\int \cdots \int \frac{dx_1}{\sqrt{\pi}} \cdots \frac{dx_n}{\sqrt{\pi}} \exp\left(- \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix} \begin{bmatrix} b_1 & & \\ & \ddots & \\ & & b_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \right) = \frac{1}{\sqrt{b_1 \cdots b_n}}.$$

More generally, for B a symmetric positive definite matrix,

$$\int \frac{d^n x}{\pi^{n/2}} \exp\left(-x^* B x\right) = \frac{1}{\sqrt{\det B}}.$$

Can be extrapolated to infinite dimensions by techniques for evaluating divergent infinite products.

Fermionic Integration.

Now consider *anti-commuting* coordinates θ obeying $\theta_i\theta_j = -\theta_j\theta_i$ (in particular $\theta_i^2 = 0$). The definition of integration is

$$\int d\theta_i = 0, \qquad \int \theta_i d\theta_i = 1.$$

For example,

$$\int d\theta_2\, d\theta_1 \exp\left(\frac{1}{2}\begin{bmatrix} \theta_1 & \theta_2 \end{bmatrix}\begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix}\begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}\right) = a.$$

More generally, for A an anti-symmetric matrix,

$$\int d^n\theta \exp\left(\frac{1}{2}\theta^* A\theta\right) = \mathrm{Pf}(A) = \pm\sqrt{\det A}.$$

Again, can be extrapolated to infinite dimensions by tricks.

Supersymmetric Integration.

Witten: Let D be a real matrix, and ϕ, ψ be two sets of anti-commuting coordinates. Then using the (respectively) symmetric and anti-symmetric forms

$$D^* D \text{ and } \begin{bmatrix} 0 & D \\ -D & 0 \end{bmatrix},$$

we obtain

$$\int \frac{d^n x}{\pi^{n/2}} d^n\phi \, d^n\psi \;\; \exp\Big(-(Dx)^*(Dx) + \psi^* D\phi\Big) = \mathrm{sgn}(\det(D)),$$

again extendable to infinite dimensions.

This is a topological invariant of D, counting the zero of the linear function Dx as $+1$ or -1 depending on its orientation.

Physically, this type of integral describes a free, supersymmetric field or particle.

A Topological QFT.

Let Σ be a Riemann surface, and X a Kähler manifold. Tangent, cotangent bundles split as

$$\begin{aligned} TX &= T_{1,0}X \oplus T_{0,1}X \\ T^*X &= T^{1,0}X \oplus T^{0,1}X. \end{aligned}$$

"Bosons" are maps $\Sigma \xrightarrow{f} X$. "Fermions" are

$$\begin{aligned} \phi_+ &\in f^*(T_{1,0}X) \\ \phi_- &\in T^{0,1}\Sigma \otimes f^*(T_{1,0}X) \\ \psi_+ &\in T^{1,0}\Sigma \otimes f^*(T_{0,1}X) \\ \psi_- &\in f^*(T_{0,1}X) \end{aligned}$$

Together, the bosons and fermions form an infinite-dimensional supermanifold.

Then Witten's $N = 2$ supersymmetric sigma A-model for counting rational curves in X is

$$Z = \int \mathcal{D}f\, \mathcal{D}\psi\, \mathcal{D}\phi\, \exp\left\{-2\tau \int_\Sigma d^2z\, \left(\tfrac{1}{2}\partial_z f \cdot \partial_{\overline{z}} f + \psi_+ \cdot i\partial_{\overline{z}}\phi_+ + \psi_- \cdot i\partial_z\phi_- + R_X(\phi_+, \psi_+, \phi_-, \psi_-)\right)\right\}$$

Resources.

Professors:

- Kreimer (algebra of renormalization)
- Kefeng Liu (symplectic geometry)
- Gieseker (integrable systems)
- Varadarajan (arithmetic physics)
- Takesaki (operator algebras)
- L. Chayes, Schonmann (statistical mechanics)

Other Researchers:

- Y.-P. Lee (algebraic geometry)
- Burchard (everything)

Workshops.

- IPAM Conformal Field Theory program (Fall 2001)

Books.

- *Quantum Fields and Strings,* IAS and AMS (1999)
- *The Elegant Universe,* Brian Greene (1999)