# Uniquely undefinable elements *

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**Abstract**

There exists a model in a countable language having a unique element which is not definable in $\mathcal{L}_{\omega_1,\omega}$.

We answer problem 10.12 from the updated version of Arnie Miller's problem list, [3].

**Definition** For a structure $\mathcal{M}$ and $a \in \mathcal{M}$, we say that $a$ is *definable over* $\mathcal{L}_{\omega_1,\omega}$ if there is a formul $\psi \in \mathcal{L}_{\omega_1,\omega}$ such that for all $b \in \mathcal{M}$

$$a = b \Leftrightarrow \mathcal{M} \models \psi(b).$$

**Theorem 0.1** *There is a countable language $\mathcal{L}$ and a model $\mathcal{M}$ for that language such that there is a unique $a \in \mathcal{M}$ which is not definable over $\mathcal{L}_{\omega_1,\omega}$.*

## 1 Ranks for formulas

For the entirety of this section, $\mathcal{L}$ is a countable language.

**Definition** For $\varphi \in \mathcal{L}$ we define the *rank* of the formula by induction. If $\varphi$ is an atomic formula, then it has rank 0. If $\varphi$ has rank $\alpha$, then

$$\exists x \varphi$$

and

$$\forall x \varphi$$

both have rank $\alpha + 1$ and

$$\neg\varphi$$

has rank $\alpha$. If $(\varphi_n)_{n\in\mathbb{N}}$ is a sequence of formulas, each $\varphi_n$ having rank $\alpha_n$, then

$$\bigwedge_{n\in\mathbb{N}} \varphi_n$$

and

$$\bigvee_{n\in\mathbb{N}} \varphi_n$$

both have rank $\sup_{n\in\mathbb{N}}\alpha_n + 1$.

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**Definition** For $\mathcal{M}$ an $\mathcal{L}$-structure and $\vec{a} \in \mathcal{M}$ a finite sequence, we define

$$\varphi_\alpha^{\vec{a},\mathcal{M}},$$

the $\alpha^{\text{th}}$ *approximation to the Scott sentence of* $\vec{a}$, by induction on the ordinal $\alpha$. $\varphi_0^{\vec{a},\mathcal{M}}$ is the conjunction of all quantifier free, first order sentences $\psi(\vec{x})$ such that

$$\mathcal{M} \models \psi(\vec{a}).$$

$\varphi_{\alpha+1}^{\vec{a},\mathcal{M}}$ is

$$(\bigwedge_{\vec{b}\in\mathcal{M}} \exists \vec{y} \varphi_\alpha^{\vec{a}\frown\vec{b},\mathcal{M}}(\vec{x},\vec{y})) \wedge (\forall \vec{y} \bigvee_{\vec{b}\in\mathcal{M}} \varphi_\alpha^{\vec{a}\frown\vec{b},\mathcal{M}}(\vec{x},\vec{y})).$$

For $\lambda$ a countable limit ordinal,

$$\varphi_\lambda^{\vec{a},\mathcal{M}} = \bigwedge_{\alpha<\lambda} \varphi_\alpha^{\vec{a},\mathcal{M}}.$$

We let

$$\varphi_\alpha^{\mathcal{M}},$$

the $\alpha^{\text{th}}$ *approximation to the Scott sentence of* $\mathcal{M}$, denote

$$\varphi_\alpha^{\emptyset,\mathcal{M}}.$$

**Lemma 1.1** *Let $\mathcal{M}, \mathcal{N}$ be $\mathcal{L}$-structures, $\vec{a} \in \mathcal{M}$, $\vec{b} \in \mathcal{N}$, and let $\psi \in \mathcal{L}_{\omega_1,\omega}$ a formula of rank $\alpha$. If*

$$\varphi_\alpha^{\vec{a},\mathcal{M}} = \varphi_\alpha^{\vec{b},\mathcal{N}},$$

*then*

$$(\mathcal{M} \models \psi(\vec{a})) \Leftrightarrow (\mathcal{N} \models \psi(\vec{b})).$$

**Proof** This well known fact has a routine proof by induction on $\alpha$. □

**Theorem 1.2** *(Scott) For each countable $\mathcal{M}$ there is an ordinal $\alpha(\mathcal{M})$ such that for all $\beta \in \omega_1$, $\vec{a}, \vec{b} \in \mathcal{M}$*

$$\varphi_{\alpha(\mathcal{M})}^{\vec{a},\mathcal{M}} = \varphi_{\alpha(\mathcal{M})}^{\vec{b},\mathcal{M}} \Rightarrow \varphi_\beta^{\vec{a},\mathcal{M}} = \varphi_\beta^{\vec{b},\mathcal{M}}.$$

*Moreover, for each $\vec{a}, \vec{b} \in \mathcal{M}$,*

$$\varphi_{\alpha(\mathcal{M})}^{\vec{a},\mathcal{M}} = \varphi_{\alpha(\mathcal{M})}^{\vec{b},\mathcal{M}}$$

*implies there is an automorphism*

$$\pi : \mathcal{M} \cong \mathcal{M}$$

*with $\pi(\vec{a}) = \vec{b}$. Finally, given any two countable $\mathcal{L}$-structures $\mathcal{M}, \mathcal{N}$, $\vec{a} \in \mathcal{M}$, $\vec{b} \in \mathcal{N}$, if*

$$\varphi_{\alpha(\mathcal{M})+2}^{\vec{a},\mathcal{M}} = \varphi_{\alpha(\mathcal{M})+2}^{\vec{b},\mathcal{N}},$$

*then there is an isomorphism $\pi : \mathcal{M} \cong \mathcal{N}$ with $\pi(\vec{a}) = \vec{b}$.*

**Proof** See [2]. □

**Definition** For $\mathcal{M}$ an $\mathcal{L}$-structure and $a \in \mathcal{M}$, we say that $a$ is *definable over* $\mathcal{L}_{\omega_1,\omega}$ if there is some $\psi \in \mathcal{L}_{\omega_1,\omega}$ such for all $b \in \mathcal{M}$

$$(\mathcal{M} \models \psi(b)) \Leftrightarrow b = a.$$

Note then by Scott's theorem we obtain that a countable model with no automorphisms has *every* element definable over $\mathcal{L}_{\omega_1,\omega}$.

## 2 Indiscernibility

The next lemma should be considered folklore. For instance one might compare [1] for similar constructions.

**Lemma 2.1** *There is an uncountable sequence of non-isomorphic countable models of a linear ordering, $(\mathcal{M}_\alpha)_{\alpha\in\omega_1}$, such that for all $\gamma < \alpha < \beta < \omega_1$*

$$\varphi_\gamma^{\mathcal{M}_\alpha} = \varphi_\gamma^{\mathcal{M}_\beta}$$

*and every element in every $\mathcal{M}_\alpha$ is definable over $\mathcal{L}_{\omega_1,\omega}$.*

**Proof** Consider the structure $(V_{\omega_2}, \in)$. Take some chain of elementary submodels, $X_\alpha \prec V_{\omega_2}$ such that:

(i) each $X_\alpha$ is countable;

(ii) $\alpha < \beta$ implies $X_\alpha \subset X_\beta$;

(iii) $\alpha < \beta$ implies $X_\alpha \cap \omega_1 < X_\beta \cap \omega_1$.

Note that (iii) gives $\alpha \subset X_\alpha$

Let $\pi_\alpha : X_\alpha \cong \mathcal{N}_\alpha$ be the Mostowski collapse. Let

$$\rho_\alpha : \mathcal{N}_\alpha \to V_{\omega_2}$$

be the inverse of the collapsing map.

Let $\mathcal{M}$ be the structure $(\omega_1, <)$. At $\alpha \in \omega_1$, let

$$\mathcal{M}_\alpha = \pi_\alpha(\mathcal{M}).$$

Note that at each $\alpha \in \omega$ and $x$ hereditarily countable in $\mathcal{N}_\alpha$,

$$\rho_\alpha(x) = x.$$

In particular for $\psi \in \mathcal{L}_{\omega_1,\omega} \cap \mathcal{N}_\alpha$,

$$\mathcal{N}_\alpha \models (\mathcal{M}_\alpha \models \psi) \Leftrightarrow V_{\omega_2} \models (\mathcal{M} \models \psi)$$

$$\therefore \mathcal{M}_\alpha \models \psi \Leftrightarrow \mathcal{M} \models \psi.$$

Thus we obtain that for any $\gamma < \beta < \alpha < \omega_1$

$$\varphi_\gamma^{\mathcal{M}_\alpha} = \varphi_\gamma^{\mathcal{M}_\beta}.$$

Finally to see that every element in every $\mathcal{M}_\alpha$ is definable, we observe that each of these structures are rigid and countable. From this it follows by Scott's theorem. □

## 3 Proof

**Theorem 3.1** *There is a finite language $\mathcal{L}$ and a structure $\mathcal{M}$ of size $\omega_1$ with a unique $a \in \mathcal{M}$ which is not definable over $\mathcal{L}_{\omega_1,\omega}$.*

**Proof** We begin with an informal description of the construction.

First we take $\omega_1 \times \mathbb{Q}$ in the lexicographic linear ordering. Thus this is a dense linear ordering which has a cofinal, strictly increasing sequence of order type $\omega_1$. Then we let $(\omega_1 \times \mathbb{Q})^*$ be the linear ordering obtained by reversing the ordering. Then we take

$$(\omega_1 \times \mathbb{Q})^\frown\{1\}^\frown(\omega_1 \times \mathbb{Q})^*.$$

Thus we have a linear ordering with an element sandwiched between two dense lnear orderings, one with an increasing sequence of cofinality $\omega_1$, the other with a decreasing sequence of cofinality $\omega_1$. The structure currently has size $\omega_1$. Thus, referring back to 2.1, we can attach a different $\mathcal{M}_\alpha$ to a dense, co-dense subset of the elements of that linear ordering. This two part structure, consisting of one linear ordering of size $\aleph_1$, with a sequence of distinct rigid linear orderings attached to some of these elements, gives the required model.

So much for the picture. Now for formalities.

Technically our language will consist of a binary relation $<$, which will linearly order certain parts of the structure, a binary relation $R$, which will attach densely, codensely many elements of the main linear ordering of size $\aleph_1$ to the various structures to the side, the unary predicate $Q$, which will distinguish the main linearly ordered part from the branches to the side, and the unary predicate $P$ which will indicate the elements of the main linear ordering which are associated with the branches on the side.

The structure will consist of the following objects:

(a) $(0, \alpha, q)$ for $\alpha \in \omega_1, q \in \mathbb{Q}$;
(b) $(1, \alpha, q)$ for $\alpha \in \omega_1, q \in \mathbb{Q}$;
(c) an object $\infty$;
(d) $(\alpha, \beta)$ where $\alpha \in \mathcal{M}_\beta$.

We let the predicate $Q$ apply exactly to the objects in (a), (b), and (c). We let $A \subset \mathbb{Q}$ be a set which is dense and codense in the rationals, and we then let $P(i, \alpha, q)$ hold exactly when $q \in A$.

We then define $<$ on the objects in $Q$ by

$$(i, \alpha, q) < (j, \beta, r)$$

if either (i) $i < j$, or (ii) $i = i = 0$ and $\alpha < \beta$, or (iii) $i = j = 0$, $\alpha = \beta$, $q < r$, or (iv) $i = j = 1$, and $\alpha > \beta$, or (iii) $i = j = 1$, $\alpha = \beta$, $q > r$. We then set

$$(0, \alpha, q) < \infty < (1, \alpha, q)$$

all $\alpha \in \omega_1, q \in \mathbb{Q}$.

For the objects outside $Q$, we set

$$(\alpha, \beta) < (\alpha', \beta')$$

if $\beta = \beta'$ and $\alpha < \alpha'$.

We then attach using $R$. We let $(p_\gamma)_{\gamma \in \omega_1}$ enumerate the objects in $P$, and set

$$R(p_\gamma, (\alpha, \beta))$$

exactly when $\gamma = \alpha$.

**Claim:** Each element in $P$ is definable in $\mathcal{L}_{\omega_1,\omega}$.
**Proof of Claim:**If $p_\beta \in P$, then, by Scott's theorem, it is the unique element $p \in P$ such that

$$(\{x : R(p, x)\}, <) \models \varphi^{\mathcal{M}_\beta}_{\alpha(\mathcal{M}_\beta)+2}.$$

(□Claim)

**Claim:** If $a \notin Q$, then it is definable in $\mathcal{L}_{\omega_1,\omega}$.
**Proof of Claim:**Similar to the last argument, and using that each $\mathcal{M}_\alpha$ has all its elements definable in $\mathcal{L}_{\omega_1,\omega}$. (□Claim)

**Claim:** If $a = (i, \alpha, q)$ is in $Q$, then it definable in $\mathcal{L}_{\omega_1,\omega}$.
**Proof of Claim:**We can choose a sequence $(r_n)_{n \in \mathbb{N}}$ inside $P$ which converge to $a$ from below in $Q$ under the linear ordering $<$. If at each $n$ we have $\psi_n \in \mathcal{L}_{\omega_1,\omega}$ defining $r_n$, then $x = a$ if and only if

$$\bigwedge_{n \in \mathbb{N}} \exists y(y < x \wedge \psi_n(y)) \wedge \forall z(z < x \Rightarrow \exists y(y < x \wedge z < y \wedge \bigvee_{n \in \mathbb{N}} \psi_n(y).$$

(□Claim)

We are then only left with showing $\infty$ *not* definable over $\mathcal{L}_{\omega_1,\omega}$.

Let us fix some ordinal $\gamma$. It suffices to show that $\infty$ is not definable by a formula of rank less than $\gamma$.

We will define a new structure $\mathcal{M}^*$. The $Q$ part of $\mathcal{M}^*$ will be exactly the same as in $\mathcal{M}$: That is to say, it will consist of

(a*) $(0, \alpha, q)$ for $\alpha \in \omega_1, q \in \mathbb{Q}$;
(b*) $(1, \alpha, q)$ for $\alpha \in \omega_1, q \in \mathbb{Q}$;
(c*) an object $\infty$.

The linear ordering $<$ on $Q^{\mathcal{M}^*}$ will be the same as $<$ on $Q^{\mathcal{M}}$. The $P$ will again consist of all $(i, \alpha, q)$ for which we have $q \in A$. The difference will be in how we define the branches to the side.

Here we will have the further objects

(d*) $(\alpha, \beta)$ where $\alpha \in \mathcal{M}_\beta$ and $\beta < \gamma$ or $\alpha \in \mathcal{M}_\gamma$ and $\beta \geq \gamma$.

In other words, we have taken the original structure and replaced any copy of some $\mathcal{M}_\beta$, $\beta > \gamma$ by a copy of $\mathcal{M}_\gamma$.

**Claim:** Let $\delta \leq \gamma$. Let $a_1, a_2, ...a_n$ be elements of $Q^{\mathcal{M}} = Q^{\mathcal{M}^*}$. Let $\alpha_1, \alpha_2, ..., \alpha_m$ be distinct countable ordinals. At each $i \leq m$ suppose $\vec{a}^i = \langle a_1^i, a_2^i, ...\rangle \in \mathcal{M}_{\alpha_i}^{<\infty}$ and let $\vec{b}^i$ be chosen so that if $\alpha_i < \gamma$ then $\vec{b}^i \in \mathcal{M}_{\alpha_i}^{<\infty}$ while if $\alpha_i \geq \gamma$ then $\vec{b}^i \in \mathcal{M}_\gamma^{<\infty}$. Assume that at each $i \leq m$, if $\alpha_i < \gamma$ then

$$\varphi_\delta^{\vec{a}_i, \mathcal{M}_{\alpha_i}} = \varphi_\delta^{\vec{b}_i, \mathcal{M}_{\alpha_i}},$$

whilst if $\alpha_i \geq \gamma$ then

$$\varphi_\delta^{\vec{a}_i, \mathcal{M}_{\alpha_i}} = \varphi_\delta^{\vec{b}_i, \mathcal{M}_\gamma}.$$

Then

$$\varphi_\delta^{a_1 {}^\frown a_2 {}^\frown ... (a_1^1, \alpha_1) {}^\frown (a_2^1, \alpha_1) {}^\frown ... (a_1^2, \alpha_2) {}^\frown ...., \mathcal{M}} = \varphi_\delta^{a_1 {}^\frown a_2 {}^\frown ... (b_1^1, \alpha_1) {}^\frown (b_2^1, \alpha_1) {}^\frown ... (b_1^2, \alpha_2) {}^\frown ...., \mathcal{M}^*}.$$

**Proof of Claim:** Routine induction on $\delta$. (□Claim)

Thus if suffices to show that in $\mathcal{M}^*$ we do not have $\infty$ definable by a formula of rank less than $\gamma$.

But in fact, no formula of $\mathcal{L}_{\omega_1, \omega}$ defines $\infty$ in $\mathcal{M}^*$.

The simplest way to do this is to collapse $\omega_1$ by forcing and go to a generic extension $V[G]$ of $V$ in which $\omega_1^V$ becomes countable. Then we can find an open interval

$$\{x : a < x < b\}$$

interval in $(Q, <)$ containing $\infty$ but no elements $p_\alpha$ for $\alpha < \gamma$. Using Cantors back and forth argument we can find an automorphism

$$\pi : (\{x : a < x < b\}, <) \cong (\{x : a < x < b\}, <)$$

with

$$\pi(\infty) \neq \infty$$

and for all $c$ with $a < c < b$ we have

$$P(c) \Leftrightarrow P(\pi(c)).$$

It is routine to extend $\pi$ to an automorphism of $\mathcal{M}^*$. Since automorphisms preserve satisfaction of formulas in $\mathcal{L}_{\omega_1, \omega}$ and since satisfaction of formulas is absolute between $V$ and any of its generic extensions, it follows that $\infty$ is not definable in $\mathcal{M}^*$ by any formula in $\mathcal{L}_{\omega_1, \omega}$ in $V$. □

# References


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