Two generated groups are universal

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1 2

Abstract

For any countable Borel equivalence relation E on a standard Borel space X, there is a Borel function θ from X to the 2-generated groups such that $xEy \Leftrightarrow \theta(x) \cong \theta(y)$.

1 Introduction

Definition Let Grp denote the collection of $(\varphi, \psi) \in 2^{\mathbb{N}\times\mathbb{N}} \times 2^{\mathbb{N}}$ such that
(a) $\varphi(0, n) = \varphi(n, 0) = n$ all $n \in \mathbb{N}$ (identity);
(b)$\varphi(n, \varphi(m, \ell)) = \varphi(\varphi(n, m), \ell)$ (associativity);
(c) $\varphi(n, \psi(n)) = \varphi(\psi(n), n) = 0$ (inverse).

This space Grp can be thought of as the possible ways to place a group structure on $\mathbb{N}$. It forms a closed subset of $2^{\mathbb{N}\times\mathbb{N}} \times 2^{\mathbb{N}}$ in the product topology and hence is a Polish space in the subspace topology. Note that there is a natural equivalence relation on Grp: That of isomorphism. Thus we can set

$$(\varphi, \psi) \cong (\varphi', \psi')$$

if there is a bijection π of $\mathbb{N}$ such that at each n, m

$$\pi(\psi(n)) = \psi'(\pi(n)),$$

$$\pi(\varphi(m, n)) = \varphi'(\pi(m), \pi(n)).$$

The goal of this paper is to show the following:

Theorem 1.1 *Let E be a Borel equivalence relation on a Polish space X all of his classes are countable. Then there is a Borel function*

$$\theta : X \to \mathrm{Grp}$$

such that
(a) $\forall x_1, x_2 \in X$, x_1Ex_2 if and only if $\theta(x_1) \cong \theta(x_2)$;
(b) at all $x \in X$ we have $\theta(x)$ is a group which is generated by two of its elements.

The case for general finitely generated groups was solved by [3], who could obtain in (b) that each group was generated by *five* of its elements.

This notion of reduction is often referred to as *Borel reducibility.* The basic theory has been developed in [2], where in particular it is shown that:

[1]2000 Mathematics Subject Classification: Primary 20B07, 03E15

[2]The first author gratefully acknowledges support from the Australian Research Council in the form of an Australian Professorial Fellowship.

Theorem 1.2 *(Jackson, Kechris, Louveau) For any Borel equivalence relation on a Polish X with countable classes, there is a Borel function*

$$\rho : X \to 2^{\mathbb{F}_2}$$

such that

$$\forall x, y \in X (xEy \Leftrightarrow \rho(x) E_{\mathbb{F}_2} \rho(y)).$$

Here $E_{\mathbb{F}_2}$ refers to the orbit equivalence relation arising from the shift action: Thus for

$$f, g : \mathbb{F}_2 \to \{0, 1\},$$

we say that they are $E_{\mathbb{F}_2}$ equivalent if there is some $\sigma \in \mathbb{F}_2$ such that at every $\tau \in \mathbb{F}_2$

$$f(\tau) = g(\tau\sigma).$$

Thus in proving the main theorem it is only necessary to reduce the specific example $E_{\mathbb{F}_2}$ which they provide.

The proof takes two steps. First we produce an infinitely generated group $\mathcal{G}_{\vec{x}}$ which admits very canonical isomorphisms. Then we embed it into a 2-generated group $\mathcal{H}_{\vec{x}}$. This second part of the construction was inspired by some talks given by Simon Thomas at a set theory conference in Kyoto, were he in particular described Galvin's alternate proof from [1] that every countable group embeds in a 2-generated group and went on to comment on metamathematical issues related to that result.

The reductions

$$\vec{x} \mapsto \mathcal{G}_{\vec{x}}$$

and then

$$\mathcal{G}_{\vec{x}} \mapsto \mathcal{H}_{\vec{x}}$$

are both Borel. The verification of this is little more than unwinding the definitions and therefore I will not comment specifically on the effectiveness of the process.

2 The group $\mathcal{G}_{\vec{x}}$

Notation Let $\mathbb{F}_2 = \langle a, b\rangle$, $\mathbb{F}_3 = \langle a, b, c\rangle$. Let $\mathbb{F}_2$ act by right shift on $2^{\mathbb{F}_2}$ with

$$\sigma \cdot f(\tau) = f(\tau\sigma)$$

and $\mathbb{F}_3$ on $(2^{\mathbb{N}})^{\mathbb{F}_3}$ by

$$\sigma\vec{x}(\tau) = \vec{x}(\tau\sigma).$$

Lemma 2.1 *There is a Borel function*

$$2^{\mathbb{F}_2} \to (2^{\mathbb{N}})^{\mathbb{F}_3}$$
$$f \mapsto \vec{x}_f$$

such that

(a) $fE_{\mathbb{F}_2}g$ iff $\vec{x}_f E_{\mathbb{F}_3} \vec{x}_g$;

(b) for all $F \subset \mathbb{F}_3$ finite and and assignment

$$F \to 2^{\mathbb{N}}$$

$$\tau \mapsto s_\tau$$

with each s_τ eventually zero (i.e. $\forall \tau \in \mathbb{F} \forall^\infty n(s_\tau(n) = 0)$) we have some $\sigma \in \mathbb{F}_3$ with

$$\vec{x}(\tau\sigma) = s_\tau$$

all $\tau \in F$;

(c) $\forall f, g \in 2^{\mathbb{F}_2} \forall \sigma_1, \sigma_2 \in \mathbb{F}^3$, if $\vec{x}_f(\sigma_1 =)\vec{x}_g(\sigma_2)$ and neither is eventually zero ($\exists^\infty n(\vec{x}_f(\sigma_1)(n) \neq 0)$ and similarly for g), then

$$fE_{\mathbb{F}_2}g.$$

Proof Let π map $\mathbb{N}$ onto $\mathbb{F}_2$. Let $(F_m)_{m\in\mathbb{N}}$ enumerate finite subsets of $\mathbb{F}_3$, each F_m consisting only of words with length $\leq m$; and at each m let $(\rho_{n,m})_{n\in\mathbb{N}}$ enumerate functions from F_m to the elements of $2^{\mathbb{N}}$ which are eventually 0.

Given $f \in 2^{\mathbb{F}_2}$ and $\sigma \in \mathbb{F}_2$,

$$\vec{x}_f(\sigma)(n) = f(\pi(n)\sigma).$$

For $\sigma \in \mathbb{F}_2, \tau \in \mathbb{F}_3$, $M \in \mathbb{N}$, τ having neither c nor c^{-1} as its right most element in the representation as a word, let

$$\vec{x}(\tau c^M \sigma) = \vec{0}$$

unless for some $N = 5^{m+1}3^n$

$$\tau c^{M-N} \in F_{m,n},$$

whence

$$\vec{x}(\tau c^M \sigma) = \rho_{n,m}(\tau c^{M-N}).$$

$\square$

Definition We define $\mathcal{G}_{\vec{x}}$ for $\vec{x}$ in the image of our above reduction – that is to say, $\vec{x} = \vec{x}_f$ some $f \in 2^{\mathbb{F}_2}$.

Fix an injection

$$2^{<\mathbb{N}} \to \mathbb{P},$$

$$s \mapsto p_s$$

from finite sequences to primes, with p_0, the prime associated to the empty string, equal to 2.

Partition $\mathbb{N}$ into disjoint sets $(A_\sigma)_{\sigma\in\mathbb{F}_3}$ and S. Each A_σ will be thought of as a copy of $\mathbb{Q}$ –

$$A_\sigma = \{q_\sigma : q \in \mathbb{Q}\}.$$

We let $\tau \in \mathbb{F}_3 = \langle a, b, c\rangle$ act on these by

$$\tau : q_\sigma \mapsto q_{\tau\sigma}.$$

(Thus so far the permutation group does not depend on $\vec{x}$. I will in the following identify each τ with the permutation of $\mathbb{N}$ it induces by this description above.) We define

$$d_s^{\vec{x}} : A_\sigma \to A_\sigma,$$

$$q_\sigma \mapsto p_s \cdot q_\sigma$$

if $\vec{x}(\sigma) \supset s$, and

$$q_\sigma \mapsto q_\sigma$$

if $\vec{x}(\sigma)$ does not extend s. We define

$$e_{\vec{x}} : q_e \mapsto (1+q)_e,$$

and the identity at all other points inside $\mathbb{N}$.

It remains to define how the a, b, c and various $d_s^{\vec{x}}$ act on S, the set left to one side. Here we let those group elements generate an infinite rank free group in the indicated basis. We have already indicated that $e_{\vec{x}}$ acts trivially on S.

Now we simply let $\mathcal{G}_{\vec{x}}$ be the permutation group generated by the elements indicated: $a, b, c, e_{\vec{x}}, d_s^{\vec{x}}, s \in 2^{<\infty}$.

Note that only the $d_s^{\vec{x}}$'s depend on $\vec{x}$. Later on when considering isomorphisms between these groups, we will simply want to move only the $e_{\vec{x}}$, and to underscore this I am choosing to write the $\vec{x}$ as a subscript.

Notation For $\vec{x}$ as above, $\tau \in \mathbb{F}_3$

$$e_{\vec{x}}^\tau = \tau e_{\vec{x}} \tau^{-1},$$

$$q_\tau \mapsto (1+q)_\tau;$$

for $s \in 2^{<\infty}$,

$$d_s^{\vec{x},\tau} = \tau d_s^{\vec{x}} \tau^{-1},$$

$$q_\sigma \mapsto (p_s \cdot q)_\sigma$$

iff $\vec{x}(\tau^{-1}\sigma) \supset s$.

Lemma 2.2 *If $\vec{x}E_{\mathbb{F}_3}\vec{y}$ then there exists a permutation π of $\mathbb{N}$ such that*

$$\pi^{-1}\tau\pi = \tau$$

all $\tau \in \mathbb{F}_3$ and

$$\pi^{-1}d_s^{\vec{x}}\pi = d_s^{\vec{y}},$$
$$\pi^{-1}e_{\vec{x}}\pi = e_{\vec{y}}^{\sigma}(= e_{\vec{x}}^{\sigma})$$

for some $\sigma \in \mathbb{F}_3$.

Proof Fix some $\sigma \in \mathbb{F}_3$ such that

$$\vec{y}(\tau) = \vec{x}(\tau\sigma)$$

all $\tau \in \mathbb{F}_3$. Define

$$\pi_\sigma : \mathbb{N} \to \mathbb{N}$$

to be the identity on S and on the various A_τ's we let

$$\pi_\sigma : q_\tau \mapsto q_{\tau\sigma}.$$

By multiplying on the right, we commute with the $\mathbb{F}_3$ action, and clearly we are okay for the generators a, b, c.

For $q_\tau \in A_\tau$ we have

$$\pi_\sigma^{-1}d_s^{\vec{x}}\pi_\sigma(q_\tau) = (p_s q)_\tau$$

if $\vec{x}(\tau\sigma) \supset s$, in otherwords if $\vec{y}(\tau) \supset s$, and equal to the identity otherwise. Thus

$$\pi_\sigma^{-1}d_s^{\vec{x}}\pi_\sigma = d_s^{\vec{y}}.$$

Finally for $q_{\sigma^{-1}} \in A_{\sigma^{-1}}$ we have

$$\pi_\sigma^{-1}e_{\vec{x}}\pi_\sigma : q_{\sigma^{-1}} \mapsto q_e \mapsto (1+q)_e \mapsto (1+q)_\sigma.$$

□

Notation Let $H_{\vec{x}}$ be the subgroup of $\mathcal{G}_{\vec{x}}$ generated by the elements of the form

$$\eta \cdot e_{\vec{x}}^{\tau},$$

$\tau \in \mathbb{F}_3, \eta \in \mathbb{Q}$, with $\eta \cdot e_{\vec{x}}^{\tau}$ well defined.

Lemma 2.3 *$H_{\vec{x}}$ is the set of $\gamma \in \mathcal{G}_x$ for which*

$$\gamma^{\frac{1}{2^n}}$$

exists for all n (that is to say, for all n there exists δ with $\delta\delta...(2n$ times$)...\delta = \gamma$).

Proof Clearly $d_0^{\vec{x}}e_{\vec{x}}(d_0^{\vec{x}})^{-1} = \frac{1}{2}e_{\vec{x}}$, so $H_{\vec{x}}$ is included in the set of such elements.

On the other hand, note that $H_{\vec{x}}$ is a normal subgroup of $\mathcal{G}_{\vec{x}}$ and, by our choice of S to one side, $\mathcal{G}_{\vec{x}}/H_{\vec{x}} \cong \mathbb{F}_\infty$. □

Notation Let $B_{\vec{x}}$ be the set of γ such that there exists $h \neq e$ in $H_{\vec{x}}$ with

$$\gamma^{-1}h\gamma = \frac{1}{p}h$$

for some $p \in \mathbb{P}, p \neq 1$. Let $G_{\vec{x}}$ the set of elements of the form $\gamma_1\gamma_2$ with $\gamma_2 \in H_{\vec{x}}$ and γ_1 in the subgroup generated by the $d_s^{\vec{x},\sigma} = \sigma d_s^{\vec{x}}\sigma^{-1}$, $s \in 2^{<\infty}, \sigma \in \mathbb{F}_3$.

Lemma 2.4 *$G_{\vec{x}}$ is the subgroup generated by $B_{\vec{x}}$.*

Proof Clearly $G_{\vec{x}}$ is included in the generated subgroup. Conversely, we need to show that every element of $B_{\vec{x}}$ is in $G_{\vec{x}}$.

For this purpose, write $\gamma \in B_{\vec{x}}$ as

$$\gamma = \gamma_1 \gamma_2$$

where $\gamma_1 \in \mathbb{F}_3$ and $\gamma_2 \in G_{\vec{x}}$. Let $h \in H_{\vec{x}}$ witness γ's membership in $B_{\vec{x}}$.

$h \neq e$ and so there finite set $F \subset \mathbb{F}_3$ such that $h \cdot q_\sigma \neq q_\sigma$ if and only if $\sigma \in F$. But now $h^{\gamma_2} =_{df} \gamma_2 h \gamma_2^{-1}$ has the same property on the set $\gamma_2 \cdot F$. Since the action of $\mathbb{F}_3$ on its finite, non-empty subsets has no fixed points, we have $\gamma_2 = e$, as required. □

For future reference, note that the proof of the last lemma in fact gives:

Lemma 2.5 *If $h \in H_{\vec{x}} \neq e$ and $g \in \mathcal{G}_{\vec{x}}$ with*

$$g \cdot h = \frac{1}{p} h$$

for some $p \in \mathbb{P}$, then $g \in G_{\vec{x}}$.

Notation From now on I will write

$$G_{\vec{x}} = K_{\vec{x}} \ltimes H_{\vec{x}},$$

as the semi-direct product of $H_{\vec{x}}$ with $K_{\vec{x}}$, the subgroup generated in $\mathcal{G}_{\vec{x}}$ by elements of the form $d_s^{\vec{x},\sigma}$, $\sigma \in \mathbb{F}_3, s \in 2^{<\infty}$. Recall that $d_s^{\vec{x},\sigma}$ is the identity on A_τ if $\vec{x}(\sigma^{-1}\tau) \perp s$ and corresponds to p_s multiplication if $\vec{x}(\sigma^{-1}\tau) \supset s$.

In future I will use abelian and additive notation to refer to the group operations inside $K_{\vec{x}}$ and inside $H_{\vec{x}}$. Given $g \in K_{\vec{x}}, h \in H_{\vec{x}}$,

$$g \cdot h$$

denotes $h^g = ghg^{-1}$, the effect of conjugating h by g.

For $r \in \mathbb{Q}, r \neq 0, 1$, let $B_{r,\vec{x}}$ the set of elements in $G_{\vec{x}}$ such that there exists some $h \in H_{\vec{x}}, h \neq 0$, with

$$g \cdot h = \frac{1}{r} h$$

(that is to say, if $r = \frac{m}{n}$, then adding $g \cdot h$ to itself m many times yields the same group element if we add h to itself n many times) and for every $h \in H$ we have that if $g \cdot h \in \mathbb{Q} \cdot h$ then either

$$g \cdot h = \frac{1}{r} h$$

or

$$g \cdot h = h.$$

Lemma 2.6 *If*

$$g = \sum_{i \leq N} n_i d_{s_i}^{\vec{x},\tau} \in B_r,$$

with each $lh(s_i) \leq L$ then at each $\sigma \in \mathbb{F}_2$, $s \in 2^L$,

$$\prod_{\{i : \tau_i = \sigma, s_i \subset s\}} (p_{s_i})^{n_i} \in \{1, r\}.$$

Proof Choose some τ with $\vec{x}(\sigma^{-1}\tau) \supset s$. Then

$$e_{\vec{x}} : q_e \mapsto (1+q)_e$$

$$\therefore e_{\vec{x}}^\tau : q_\tau \mapsto q_e \mapsto (1+q)_e \mapsto (1+q)_\tau.$$

For any $s_i \subset s$ we have

$$d_{s_i}^{\sigma,\vec{x}} : q_\tau \mapsto q_{\sigma^{-1}\tau} \mapsto (p_{s_i} \cdot q)_{\sigma^{-1}\tau} \mapsto (p_{s_i} \cdot q)_\tau,$$

since $\vec{x}(\sigma^{-1}\tau) \supset s_i$, and hence

$$g : q_\tau \mapsto (\prod_{\{i : \tau_i = \sigma, s_i \supset s\}} (p_{s_i})^{n_i} \cdot q)_\tau.$$

From this the claim follows. □

Lemma 2.7 *If $g \in K_{\vec{x}} \cap B_p$ for some $p \in \mathbb{P}, p \neq 2$, then it can be written uniquely as*

$$\sum_{\tau \in F} n_\tau \cdot d_s^{\vec{x},\tau}$$

for some s with $p_s = p$.

Proof We can find a finite set F such that

$$g = \sum_{\tau \in S} g_\tau,$$

where each g_s has the form

$$\sum_{s \in S_\tau} n_s d_s^{\vec{x},\tau},$$

for some finite $S_\tau \subset 2^{<\infty}$. The last lemma gives that in each case $p_s = p$, since p is prime. This gives the representation as indicated. Uniqueness follows since for any two choices $F \neq F'$ we can find a $\sigma_0 \in F \Delta F'$ and $\tau \in \mathbb{F}_3$ such that

$$\vec{x}(\sigma_0^{-1}\tau) \supset s$$

but

$$\vec{x}(\sigma^{-1}\tau) \perp s$$

all $\sigma \neq \sigma_0, \sigma \in F \cup F'$. □

Sharper representation theorems are possible, but this lemma will suffice for our purposes.

Definition $g, g \in G_{\vec{x}}$ are p, p' *related* if $g \in B_p, g' \in B_{p'}$ and for all $h \in H_{\vec{x}}$,

$$g \cdot h = \frac{1}{p} h \Rightarrow g' \cdot h = \frac{1}{p'} h.$$

Lemma 2.8 *If $g, g' \in K_{\vec{x}}$ are p, p' related, $p, p' \neq 2$, and we write them in the normal form*

$$g = \sum_{\tau \in F} n_\tau \cdot d_s^{\vec{x},\tau},$$

$$g' = \sum_{\tau \in F'} n'_\tau \cdot d_{s'}^{\vec{x},\tau},$$

$p_s = p, p' = p_{s'}$ indicated by the previous lemma, then

$$F \subset F'.$$

Proof Suppose instead that $\sigma_0 \in F \setminus F'$. By assumption on $\vec{x}$ being in the image of $f \mapsto \vec{x}_f$, we can get some $\sigma_1 \in \mathbb{F}_3$ such that

$$\vec{x}(\sigma_0^{-1}\sigma_1) \supset s,$$

but for all $\tau \in F \cup F', \tau \neq \sigma_0$,

$$\vec{x}(\tau^{-1}\sigma_1) \perp s'.$$

$$e_{\vec{x}}^{\sigma_1} : q_{\sigma_1} \mapsto (1+q)_{\sigma_1},$$

and is the identity off of A_{σ_1}. On the other hand,

$$d_{s'}^{\vec{x},\tau} : q_{\sigma_1} \mapsto q_{\sigma_1}$$

if $\vec{x}(\tau^{-1}\sigma_1)$ does *not* extend s' while

$$d_s^{\vec{x},\sigma_0} : q_{\sigma_1} \mapsto (p_s q)_{\sigma_1}$$

since $\vec{x}(\sigma_0^{-1}\sigma_1) \supset s$. Hence

$$g \cdot e_{\vec{x}}^{\sigma_1} = \frac{1}{p} e_{\vec{x}}^{\sigma_1},$$

$$g' \cdot e_{\vec{x}}^{\sigma_1} = e_{\vec{x}}^{\sigma_1}.$$

□

Lemma 2.9 *For $s \supset s'$, $d_s^{\vec{x}}$ and $d_{s'}^{\vec{x}}$ are always $p_s, p_{s'}$ related.*

Proof $e_{\vec{x}}$ witnesses their membership in $B_{p_s}, B_{p_{s'}}$. To see they are related, note that $d_s^{\vec{x}}$ skews A_σ by a factor of p_s exactly when $\vec{x}(\sigma) \supset s$, which always implies $\vec{x}(\sigma) \supset s'$ and then that $d_{s'}^{\vec{x}}$ acts by $p_{s'}$ multiplication on A_σ.

For any $h \in H$, we can let F be the finite subset of $\mathbb{F}_3$ on which it is supported – i.e.

$$h \cdot q_\sigma \neq q_\sigma \Leftrightarrow \sigma \in F.$$

Then $d_s^{\vec{x}} \cdot h = \frac{1}{p_s} h$ if and only if $\vec{x}(\sigma) \supset s$ all $\sigma \in F$, which would in turn imply $\vec{x}(\sigma) \supset s'$ all $\sigma \in F$, which in turn yields $d_{s'}^{\vec{x}} \cdot h = \frac{1}{p_{s'}} h$. □

Definition For $z \in 2^{\mathbb{N}}$, a sequence $(g_n)_{n \in \mathbb{N}}$ in $G_{\vec{x}}$ is said to be *z-connected* if each $g_n \in B_{p_{z|n}}$ and g_{n+1}, g_n is always $p_{z|n+1}, p_{z|n}$ related and there is a single $h \in H_{\vec{x}}$ such that

$$g_n \cdot h = \frac{1}{p_{z|n}} h$$

at every n.

Lemma 2.10 *If $(g_n)_n$ is z-connected, then for some σ we have*

$$\vec{x}(\sigma) = z.$$

Proof By considering the semi-direct product structure of $G_{\vec{x}}$, we can assume each $g_n \in K_{\vec{x}}$. Using the previous couple of lemmas we can write each

$$g_n = \sum_{\tau \in S_n} n_{\tau,n} d_{p_{z|n}}^{\vec{x},\tau},$$

where at each n we have $S_{n+1} \subset S_n$, both finite and non-empty. Thus there is a single S such $S_n = S$ all sufficiently large n.

Let $h \in H_{\vec{x}}$ be the witness to z-connectedness. Let $F \subset \mathbb{F}_3$ be the *support* of h, in the sense

$$h \cdot q_\sigma \neq q_\sigma$$

exactly when $\sigma \in F$.

$K_{\vec{x}}$ acts by multiplying each of the A_σ's separately, and hence at each n and $\sigma \in F$

$$g_n \cdot q_\sigma = (p_{z|n} q)_\sigma,$$

and hence for some $\tau \in S_n = S$ we have

$$\vec{x}(\tau^{-1}\sigma) \supset s_n.$$

S is finite and hence we obtain a single $\tau \in S$ such that

$$\vec{x}(\tau^{-1}\sigma) \supset s_n$$

at infinitely many n, which completes the proof. □

Lemma 2.11 *$e_{\vec{x}}$ witnesses that $(d_{\vec{x}(e)|n}^{\vec{x}})_{n \in \mathbb{N}}$ is $\vec{x}(e)$-connected.*

Proof Clearly $e_{\vec{x}} \in H_{\vec{x}}$ and at each n we have

$$d_{\vec{x}(e)|n}^{\vec{x}} : q_e \mapsto (p_{\vec{x}(e)|n} q)_e,$$

and so $d_{\vec{x}(e)|n}^{\vec{x}}$ acts on $e_{\vec{x}}$ by sending it to $\frac{1}{p_{\vec{x}(e)|n}} e_{\vec{x}}$. Now the lemma follows by 2.9. □

3 The group $\mathcal{H}_x$

Now we will define for each $\vec{x}$ in the set $\{\vec{x}_f : f \in 2^{\mathbb{F}_2}\}$ a second group $\mathcal{H}_{\vec{x}}$ which will be generated by two elements.

Definition Fix an enumeration $(s_n)_{n\in\mathbb{N}}$ of $2^{<\infty}$. We let ψ be a permutation on $\mathbb{N} \times \mathbb{Z}^2$, defined by

$$\psi : (m, n, \ell) \mapsto (m, n - 1, \ell).$$

Keeping in mind the group elements defined in the previous section, and viewing them as permutations on $\mathbb{N}$ in the way they were defined at that time, we let

$$\phi_{\vec{x}} : (m, 1, 0) \mapsto (e_{\vec{x}} \cdot m, 1, 0),$$

$$\phi_{\vec{x}} : (m, 1, \ell) \mapsto (m, 1, \ell)$$

for $\ell \neq 0$,

$$\phi_{\vec{x}} : (m, 3, \ell) \mapsto (a \cdot m, 3, \ell),$$
$$\phi_{\vec{x}} : (m, 5, \ell) \mapsto (b \cdot m, 5, \ell),$$
$$\phi_{\vec{x}} : (m, 7, \ell) \mapsto (c \cdot m, 7, \ell),$$
$$\phi_{\vec{x}} : (m, 2n + 9, \ell) \mapsto (d^{\vec{x}}_{s_n} \cdot m, 2n + 9, \ell),$$

and at all $\ell \geq 0$

$$\phi_{\vec{x}} : (m.n, \ell) \mapsto (m, n, \ell);$$

for $n = 0$,

$$\phi_{\vec{x}} : (m, 0, \ell) \mapsto (m, 0, \ell + 1),$$

and

$$\phi_{\vec{x}} : (m, n, \ell) \mapsto (m, n, \ell)$$

for $n < 0$ and n an even number > 0.

We then let $\mathcal{H}_{\vec{x}}$ be the permutation group defined by these elements.

Lemma 3.1 *For each of $g \in \{a, b, c, d^{\vec{x}}_s : s \in 2^{<\infty}\}$ we can find some $\phi \in \mathcal{H}_{\vec{x}}$ such that*

$$\phi : (n, 0, 0) \mapsto (g \cdot n, 0, 0)$$

and $\phi(n, m, \ell) = (n, m, \ell)$ for $(m, \ell) \neq (0, 0)$.

Proof Say that g appears in the $2n + 1$ row (i.e. $\phi_{\vec{x}}(m, 2n + 1, \ell) = (g \cdot m, 2n + 1, \ell)$.

Then

$$\phi_1 = (\phi_{\vec{x}})^{\psi^{2n+1}} =_{df} \psi^{2n+1}\phi_{\vec{x}}\psi^{-2n-1}$$

places g on the blocks of the form $\{(m, 0, \ell) : m \in \mathbb{N}\}$ for $\ell \geq 0$ whilst

$$\phi_2 = (\phi_{\vec{x}})^{\phi_{\vec{x}}\psi^{2n+1}} =_{df} \phi_{\vec{x}}\psi^{2n+1}\phi_{\vec{x}}\psi^{-2n-1}\phi_{\vec{x}}^{-1}$$

places it on the blocks of the form $\{(m, 0, \ell) : m \in \mathbb{N}\}$ for $\ell > 0$; on all other blocks of the form Thus $\phi_2^{-1}\phi_1$ is as required. □

Lemma 3.2 *There exists some $\phi \in \mathcal{H}_{\vec{x}}$ with*

$$\phi : (n, 0, 0) \mapsto (e_{\vec{x}} \cdot n, 0, 0)$$

and $\phi(n, m, \ell) = (n, m, \ell)$ for $(m, \ell) \neq (0, 0)$.

Proof Applying the same construction as before with $0 = 1$, letting

$$\phi_1 = (\phi_{\vec{x}})^{\psi},$$

$$\phi_2 = (\phi_{\vec{x}})^{\phi_{\vec{x}}\psi}$$

we obtain with $\phi_3 = \phi_2^{-1}\phi_1$ an element which looks like $e_{\vec{x}}$ on the block $\{(n, 0, 0) : n \in \mathbb{N}\}$ and $-e_{\vec{x}}$ on the block $\{(n, 1, 0) : n \in \mathbb{N}\}$. By applying the last lemma to some $g \in \mathcal{G}_{\vec{x}}$ which has

$$g \cdot e_{\vec{x}} = \frac{1}{2} e_{\vec{x}}$$

we can find an element ϕ_g of $\mathcal{H}_{\vec{x}}$ which looks like g on the $\{(n, 0, 0) : n \in \mathbb{N}\}$ block and the identity elsewhere. Conjugating ϕ_3 by ϕ_g we obtain $(\phi_3)^{\phi_g}$ which acts like $\frac{1}{2}e_{\vec{x}}$ on the block $\{(n, 0, 0) : n \in \mathbb{N}\}$ and $-e_{\vec{x}}$ on the block $\{(n, 1, 0) : n \in \mathbb{N}\}$. Then the required ϕ is in the group generated by ϕ_3 and $(\phi_3)^{\phi_g}$. □

Lemma 3.3 *If $\vec{x} E_{\mathbb{F}_3} \vec{y}$, then*

$$\mathcal{H}_{\vec{x}} \cong \mathcal{H}_{\vec{y}}.$$

Proof Start with $\mathcal{H}_x$ in the indicated generating set. Suppose $\sigma \cdot \vec{x} = \vec{y}$. By the arguments given in the last two lemmas, we can find some $\psi_\sigma \in \mathcal{H}_{\vec{y}}$ which is exactly the same as $\psi_{\vec{y}}$ except on the block $\{(m, 1, 0) : m \in \mathbb{N}\}$ it has

$$\psi_\sigma : (m, 1, 0) \mapsto e^{\sigma}_{\vec{y}}.$$

The proof of 2.2 gives a permutation π of $\mathbb{N}$ with

$$\pi^{-1} g \pi = g$$

for all $g \in \{a, b, c\}$, has

$$\pi^{-1} d^s_{\vec{x}} \pi = d^s_{\vec{y}}$$

for $s \in 2^{<\infty}$, and has

$$\pi^{-1} e_{\vec{x}} \pi = e^{\sigma}_{\vec{x}} = e^{\sigma}_{\vec{y}}.$$

This then extend to an isomorphism of $\mathcal{H}_{\vec{x}}$ to $\mathcal{H}_{\vec{y}}$ via

$$\pi \oplus 1 \oplus 1 : (m, n, \ell) \mapsto (\pi(m), n, \ell),$$

and then

$$(\pi \oplus 1 \oplus 1)^* : \mathcal{H}_{\vec{x}} \to \mathcal{H}_{\vec{y}},$$
$$\psi \mapsto (\pi \oplus 1 \oplus 1)^{-1} \psi (\pi \oplus 1 \oplus 1).$$

□

Definition Define

$$\theta : \mathcal{H}_{\vec{x}} \to \mathrm{Sym}(\mathbb{Z} \times \mathbb{Z})$$

by the specification that if $\varphi(m, n, \ell) = (m', n', \ell')$, then

$$\theta(\varphi)(n, \ell) = (n', \ell').$$

This is well defined, since $\psi_{\vec{x}}$ and ϕ never split between the blocks of the form $\{(m, n, \ell) : m \in \mathbb{N}\}$.

Lemma 3.4 *If $\psi \in \mathcal{H}_{\vec{x}}$ is divisible by all powers of 2, then it is in the kernel of θ.*

Proof The image of θ is isomorphic to

$$\mathbb{Z} \ltimes \bigoplus_{\mathbb{Z}} \mathbb{Z}.$$

□

Notation From now on I will identify elements of $\mathrm{Ker}(\theta)$ with the corresponding element in

$$\prod_{\mathbb{Z}\times\mathbb{Z}} \mathcal{G}_{\vec{x}}.$$

That is to say, if ψ is in the kernel, then we identify ψ with the corresponding function

$$(n,\ell) \mapsto \gamma$$

if $\psi(m,n,\ell) = (\gamma \cdot m, n, \ell)$.

Lemma 3.5 *If $\psi \in \mathcal{H}_{\vec{x}} \cap \mathrm{Ker}(\theta)$ is divisible by all powers of 2, then it has finite support.*

Proof Let $\mathcal{G}_R$ be the subgroup of $\mathcal{G}_{\vec{x}}$ generated by $\{a, b, c, d_s^{\vec{x}} : s \in 2^{<\infty}\}$. As we previously observed in the last section, this group is naturally isomorphic to $\mathbb{F}_\infty$, and hence has no elements divisible by all powers of 2.

Now by the choice of our generating set, every element in $h_{\vec{x}}$ behaves like some element in $\mathcal{G}_{\vec{x}} \setminus \mathcal{G}_R$ on only finitely many blocks of the form $\{(m,n,\ell) : \mathcal{M} \in \mathbb{N}\}$. □

Definition Let $H^+_{\vec{x}}$ be the elements of $\mathcal{H}_{\vec{x}}$ which are divisible by all powers of 2. For $r \in \mathbb{Q}, r \neq 0, 1$, let $B^+_{r,\vec{x}}$ the set of elements in $\mathcal{H}_{\vec{x}}$ such that there exists some $h \in H^+_{\vec{x}}, h \neq 0$, with

$$g \cdot h = \frac{1}{r} h$$

and for every $h \in H^+_{\vec{x}}$ we have that if $g \cdot h \in \mathbb{Q} \cdot h$ then either

$$g \cdot h = \frac{1}{r} h$$

or

$$g \cdot h = h.$$

Let $G^+_{\vec{x}}$ be the subgroup generated by the $B^+_{p,\vec{x}}$ as p ranges over the primes. $g, g \in G^+_{\vec{x}}$ are p, p' *related* if $g \in B^+_{p,\vec{x}}, g' \in B^+_{p',\vec{x}}$ and for all $h \in H^+_{\vec{x}}$,

$$g \cdot h = \frac{1}{p} h \Rightarrow g' \cdot h = \frac{1}{p'} h.$$

For $z \in 2^{\mathbb{N}}$, a sequence $(g_n)_{n\in\mathbb{N}}$ in $G_{\vec{x}}$ is said to be *z-connected* if each $g_n \in B^+_{p_{z|n}}$ and g_{n+1}, g_n is always $p_{z|n+1}, p_{z|n}$ related and there is a single $h \in H^+_{\vec{x}}$ such that

$$g_n \cdot h = \frac{1}{p_{z|n}} h$$

at every n.

Lemma 3.6 *If $h \in H^+_{\vec{x}}$ with non-trivial support F, and $\varphi \in \mathcal{H}_{\vec{x}}$ has*

$$\varphi \cdot h = \frac{1}{p} h$$

for some prime, then

(a) $\varphi(m,n,\ell)$ is of the form (m', n, ℓ) for all $(n,\ell) \in F$;

(b) for each $(n,\ell) \in F$, there is some $g_{n,\ell} \in G_{\vec{x}}$ such that

$$\forall m (\varphi(m,n,\ell) = (g_{n,\ell}(m), n\ell).$$

Proof (a) follows by considering the homorphism $\theta : \mathcal{H}_{\vec{x}} \to \mathrm{Sym}(\mathbb{Z}\times\mathbb{Z})$ and noting that every $\gamma \in \mathrm{Ran}(\theta)$ we have that $\gamma \cdot F = F$ only if $\gamma(n,\ell) = (n,\ell)$ all $(n,\ell) \in F$. Then (b) follows from 2.5. □

Lemma 3.7 *If $z \in 2^{\mathbb{N}}$ and $(\varphi_i)_{i\in\mathbb{N}}$ are z-connected, then at some (n,ℓ) we have a sequence $(g_i)_{i\in\mathbb{N}}$ such that*

(a) $\varphi_i(m,n,\ell) = (g_i(m),n,\ell)$ at at all i,m;
(b) $(g_i)_{i\in\mathbb{N}}$ are z-connected.

Proof Let $\psi \in H^+_{\vec{x}}$ be a witness that the φ_i's are z-connected. Fix some (n,ℓ) in the support of h and apply the last lemma to obtain g_i's in $G_{\vec{x}}$ as indicated at (a). We need to check that at each i $g_{i+1} \in B_{p_{z|(i+1)}}, g_i \in B_{p_{z|i}}$ and for all $h \in H_{\vec{x}}$,

$$g_i \cdot h = \frac{1}{p}h \Rightarrow g_{i+1} \cdot h = \frac{1}{p'}h.$$

The implication regarding all $h \in H_{\vec{x}}$ holds, since if it did not then we could define some $\psi^+ \in H^+_{\vec{x}}$ with $\psi^+(m,n,\ell) = (h(m),n,\ell)$ and $\psi^+(m,n',\ell') = (m,n',\ell')$ for $(n',\ell') \neq (n,\ell)$. The same consideration then also shows them to be $p_{z|(i+1)}, p_{z|i}$ related. □

Corollary 3.8 *If $\mathcal{H}_{\vec{x}} \cong \mathcal{H}_{\vec{y}}$, then $\vec{x}E_{\mathbb{F}_3}\vec{y}$.*

Proof Assume the groups are isomorphic.

We can define $\varphi \in \mathcal{H}_{\vec{x}}$ to act like a copy of $e_{\vec{x}}$ on $\{(m,0,0) : m \in \mathbb{N}\}$ and the identity on all other (m,n,ℓ), $(n,\ell) \neq (0,0)$. Similarly we can produce φ_n which behave like $d^{\vec{x}}_{\vec{x}(e)|n}$ on $\{(m,0,0) : m \in \mathbb{N}\}$ and the identity outside.

We need to check $\vec{x}(e)$-connectedness, but the structure of the definitions gives that this will hold since it is happening on the one block, $\{(m,0,0) : m \in \mathbb{N}\}$, on which φ_n is acting non trivially.

But now over on the $\mathcal{H}_{\vec{y}}$ we must have a sequence of $(\varphi^*_n)_{n\in\mathbb{N}}$ which are $\vec{x}(e)$-connected. But then by the last lemma, such a $\vec{x}(e)$-connected sequence will exist in $\mathcal{G}_{\vec{y}}$, which is sufficient to ensure $\vec{y}E_{\mathbb{F}_3}\vec{x}$. □

References

[1] F. Galvin, Fred, *Embedding countable groups in 2-generator groups,* **American Mathematical Monthly,** 100 (1993), no. 6, 578–580.

[2] S. Jackson, A.S. Kechris, A. Louveau, *Countable Borel equivalence relations,* **Journal Mathematical Logic,** vol. 2 (2002), no. 1, 1–80

[3] S. Thomas, B. Velickovic, *On the complexity of the isomorphism relation for finitely generated groups*, **Journal of Algebra**, (1999), no. 1, 352–373.

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