

# There are more co-analytic sets than Borel

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September 18, 2008

## Abstract

This paper gives a proof based on large cardinal ideas that there is no injection inside  $L(\mathbb{R})$  from  $\mathbb{Q}_1^1$  to Borel.

## 1 Introduction

This paper concerns cardinalities inside  $L(\mathbb{R})$  under determinacy assumptions, in particular giving another proof of a previous established result that the cardinality of  $\mathbb{Q}_1^1$  is greater than that of  $\mathbb{A}_1^1$  inside  $L(\mathbb{R})$ .

**Definition** For  $A, B \in L(\mathbb{R})$  we write

$$|A|_{L(\mathbb{R})} \leq |B|_{L(\mathbb{R})}$$

if there is an injection from  $A$  to  $B$  in  $L(\mathbb{R})$ . We write

$$|A|_{L(\mathbb{R})} < |B|_{L(\mathbb{R})},$$

and say that —it the  $L(\mathbb{R})$ -cardinality of  $A$  is less than that of  $B$  if there is an injection in  $L(\mathbb{R})$  from the first set to the second, but not from the second to the first.

From [2]:

**Theorem 1.1** (*Hjorth*) Assuming  $AD^{L(\mathbb{R})}$ , for all  $\alpha < \beta < \omega_1$

$$|\mathbb{Q}_\alpha^0|_{L(\mathbb{R})} < |\mathbb{Q}_\beta^0|_{L(\mathbb{R})}.$$

The exact computations in the Wadge hierarchy were given in [1], which in particular gave:

**Theorem 1.2** (*Andretta, Hjorth, Neeman*) Assuming  $AD^{L(\mathbb{R})}$ , for all  $n > 1$

$$|\mathbb{A}_1^1|_{L(\mathbb{R})} < |\mathbb{Q}_1^1|_{L(\mathbb{R})} < |\mathbb{Q}_n^1|_{L(\mathbb{R})}.$$

The proof given there was exacting a technical. In this short note I will sketch a simpler proof based on large cardinal concepts that

$$|\mathbb{A}_1^1|_{L(\mathbb{R})} < |\mathbb{Q}_1^1|_{L(\mathbb{R})}.$$

## 2 Proof

For conceptual simplicity, let's start by assuming there are enough large cardinals ensure determinacy and absoluteness of the theory of  $L(\mathbb{R})$  through all forcing extensions. See for instance [6].

Let

$$U \subset 2^\omega \times 2^\omega$$

be a universal  $\Pi_1^1$  set. Assume for a contradiction

$$|\underline{\Delta}_1^1|_{L(\mathbb{R})} \geq |\underline{\Pi}_1^1|_{L(\mathbb{R})}.$$

Then in  $L(\mathbb{R})$  we can find a relation

$$R \subset 2^\omega \times 2^\omega \times 2^\omega$$

such that:

- (a)  $(x, y_1, y_2) \in R \Rightarrow U_{y_1} = 2^\omega \setminus U_{y_2}$ ;
- (b)  $\forall x \exists y_1, y_2 R(x, y_1, y_2)$ ;
- (c) if  $(x, y_1, y_2) \in R$ ,  $(x', y'_1, y'_2) \in R$ , and  $U_x = U_{x'}$ , then  $U_{y_1} = U_{y'_1}$ .

By Basis( $\underline{\Sigma}_1^2, \underline{\Delta}_2^1$ ) in  $L(\mathbb{R})$  and [3], we can find tree representatives for such a relation  $R$  in all generic extensions. By considering homogeneous forcing notions collapsing various cardinals and appealing to the stabilization of the theory we can find some choice of the relation  $R \in L(\mathbb{R})$  above, and a measurable cardinal  $\kappa$ , an inaccessible

$$\theta > \kappa$$

which is a limit of measurable cardinals and tree

$$T \subset 2^{<\omega} \times 2^{<\omega} \times 2^{<\omega} \times \delta^{<\omega},$$

on some  $\delta$  such that in all forcing extensions of size less than  $\beth_\omega(\kappa)^+ = |V_{\kappa+\omega}|^+$  we have that  $T$  continues to have  $p[T] = R$ , where  $R$  is now interpreted as its canonical extension in  $L(\mathbb{R})$  of the generic extension, and thus we continue to have

- (a)  $(x, y_1, y_2) \in p[T] \Rightarrow U_{y_1} = 2^\omega \setminus U_{y_2}$ ;
- (b)  $\forall x \exists y_1, y_2 p[T](x, y_1, y_2)$ ;
- (c) if  $(x, y_1, y_2) \in p[T]$ ,  $(x', y'_1, y'_2) \in p[T]$ , and  $U_x = U_{x'}$ , then  $U_{y_1} = U_{y'_1}$ .

$T$  will arise from the Scale on  $\Sigma_1^2$  in  $L(\mathbb{R})$  of some suitable massive generic extension. Thus we can assume there is a *function*  $\pi$  uniformly definable over all such  $L(\mathbb{R})$ 's with

$$\pi(x) = (\pi_0(x), \pi_1(x)),$$

and

$$(x, \pi_0(x), \pi_1(x)) \in R$$

all  $x \in 2^\omega$ .

For future reference, let us fix now a measure  $\mu$  on  $\kappa$ .

**Definition** A countable, transitive structure

$$\mathcal{M} = (M; \in, \kappa_0, \mu_0, T_0)$$

is a *frog* if there exists

$$\rho : \mathcal{M} \rightarrow V_\theta$$

with

$$\begin{aligned} \kappa_0 &\mapsto \kappa, \\ \mu_0 &\mapsto \mu, \\ T_0 &\mapsto T. \end{aligned}$$

Note that this final clause ensures that any element of  $p[T_0]$  is in  $p[T]$  and hence  $R$ .

A countable transitive structure

$$\mathcal{N} = (N; \in, \kappa_0, \mu_0, A_0)$$

is a *tadpole* if it satisfies powerset, comprehension, and all other axioms of ZFC except possible replacement, and it is iterable against the measure  $\mu_0$ , and  $A_0 \subset \kappa_0$ .

Given  $\mathcal{M} = (M; \in, \kappa_0, \mu_0, T_0)$  a frog and  $A_0 \in \mathcal{P}(\kappa_0)^{\mathcal{M}}$ , we let

$$\mathcal{N} = (V_{\kappa_0+\omega}; \in, \kappa_0, \mu_0, A_0)$$

be the *tadpole induced from  $\mathcal{M}$  by  $A$* .

Note that any frog has unboundedly many measurables, and it will all generic extensions of the frog will be iterable against the surviving measurables in light of the embedding into a large rank initial segment of  $V$ .

**Definition** For  $\mathcal{N} = (N; \in, \kappa_0, \mu_0, A_0)$  a tadpole, we let  $V_{\mathcal{N}}$  be the set of codes for ordinals  $\alpha < \omega_1$  such that if we take the iteration

$$i_{0,\alpha} : \mathcal{N} \rightarrow \mathcal{N}_\alpha$$

of length  $\alpha$  against the measure  $\mu_0$ , then

$$\alpha \in i_{0,\alpha}[A_0].$$

For  $x \in 2^\omega$  coding a tadpole  $\mathcal{N}$ , we let  $a(x)$  be chosen canonically, and uniformly recursively in  $x$ , with

$$U_{a(x)} = V_{\mathcal{N}}.$$

Note that any two codes for the same tadpole give rise to the same  $\Pi_1^1$  set.

Thus given a tadpole  $\mathcal{N}$  there will be a term  $\tau_{\mathcal{N}}$  in

$$\mathbb{P}_{\mathcal{N}} = \text{Coll}(\omega, \mathcal{N})$$

such that if  $\sigma_{\mathcal{N}}$  is the canonical term for an element of  $2^\omega$  coding  $\mathcal{N}$  then  $\mathbb{P}_{\mathcal{N}}$  forces that  $\sigma_{\mathcal{N}}[\dot{G}]$  is a code for a Borel  $B[\dot{G}]$  set of least possible rank with

$$\mathbb{P}_{\mathcal{N}} \Vdash B[\dot{G}] = U_{\pi_0(a(\sigma_{\mathcal{N}}[\dot{G}]))}.$$

**Lemma 2.1**

$$\mathbb{P}_{\mathcal{N}} \times \mathbb{P}_{\mathcal{N}} \Vdash B[\dot{G}_l] = B[\dot{G}_r].$$

**Proof** Since

$$\mathbb{P}_{\mathcal{N}} \times \mathbb{P}_{\mathcal{N}} \Vdash U_{\pi_0(a(\sigma_{\mathcal{N}}[\dot{G}_l]))} = U_{\pi_0(a(\sigma_{\mathcal{N}}[\dot{G}_r]))} = V_{\mathcal{N}}.$$

□

So for any  $\mathcal{N}$  we can define a corresponding  $\alpha_{\mathcal{N}}$  such that

$$\mathbb{P}_{\mathcal{N}} \Vdash B[\dot{G}] \text{ is a Borel set of rank } \alpha_{\mathcal{N}}.$$

By appealing to Wadge determinacy, the calculation of  $\alpha_{\mathcal{N}}$  is absolute to inner model containing uncountably many ordinals and satisfying  $\Sigma_1^1$  determinacy. Since every generic extension of a frog can be subject to an iteration of length  $\omega_1$ , it will continue to correctly calculate  $\alpha_{\mathcal{N}}$  for all its tadpoles through all generic extensions.

**Lemma 2.2** Let  $\mathcal{M} = (M; \in, \kappa_0, \mu_0, T_0)$  be a frog,  $A \in \mathcal{P}(\kappa)^{\mathcal{M}}$ , and  $\mathcal{N}$  the tadpole induced by  $A$ . Then

$$\mathcal{M} \models \alpha_{\mathcal{N}} < \kappa_0.$$

**Proof** Take the iteration of  $\mathcal{M}$  of length  $\alpha_{\mathcal{N}} + 1$  mapping

$$i_{0,\alpha_{\mathcal{N}}+1}^{\mathcal{M}} : \mathcal{M} \rightarrow \mathcal{M}_{\alpha_{\mathcal{N}}+1},$$

$$\kappa_0 \mapsto \kappa_{\alpha_{\mathcal{N}}+1}.$$

The important point about this iteration is that it moves  $\kappa_0$  to an ordinal above  $\alpha_{\mathcal{N}}$ .  $i_{0,\alpha_{\mathcal{N}}+1}^{\mathcal{M}} \upharpoonright \mathcal{N}$  equals the internal iterate of  $\mathcal{N}$  along its measure, since  $\mathcal{N}$  is closed under power set. Thus  $V_{\mathcal{N}} = V_{i_{0,\alpha_{\mathcal{N}}+1}^{\mathcal{M}}(\mathcal{N})}$ .

Thus,

$$V_{\mathcal{N}} = V_{i_{0,\alpha_{\mathcal{N}}+1}^{\mathcal{M}}(\mathcal{N})},$$

and hence

$$\alpha_{\mathcal{N}} = \alpha_{i_{0,\alpha_{\mathcal{N}}+1}^{\mathcal{M}}(\mathcal{N})},$$

and thus by the appeal to Wadge determinacy mentioned above,

$$\mathcal{M}_{\alpha_{\mathcal{N}+1}} \models \alpha_{i_{0, \alpha_{\mathcal{N}+1}}(\mathcal{N})} < \kappa_{\alpha_{\mathcal{N}+1}},$$

and hence by elementarity

$$\mathcal{M} \models \alpha_{\mathcal{N}} < \kappa_0.$$

□

Thus by cardinality considerations inside  $\mathcal{M}$  we can find a single  $\alpha < \kappa_0$  such for some sequence  $(A_\beta)_{\beta \in \kappa_0}$  we have that for  $\mathcal{N}_\beta$  the tadpole induced from  $A_\beta$

$$\mathcal{M} \models \mathbb{P}_{\mathcal{N}_\beta} \Vdash B_\beta[\dot{G}] \text{ is a Borel set of rank } \alpha,$$

$$\mathcal{M} \models \mathbb{P}_{\mathcal{N}_\beta} \Vdash B_\beta[\dot{G}] = U_{\pi_0(a(\sigma_{\mathcal{N}}[\dot{G}]))},$$

and hence

$$\mathcal{M} \models \mathbb{P}_{\mathcal{N}_\beta} \times \mathbb{P}_{\mathcal{N}_\beta} \Vdash B_\beta[\dot{G}_l] = B_\beta[\dot{G}_r]$$

and

$$\mathcal{M} \models \mathbb{P}_{\mathcal{N}_\beta} \times \mathbb{P}_{\mathcal{N}_\gamma} \Vdash B_\beta[\dot{G}_l] \neq B_\gamma[\dot{G}_r]$$

for  $\beta \neq \gamma$ .

Thus we obtain, inside  $\mathcal{M}$ , more than  $\beth_{1+\alpha+1}$  many inequivalent codes for invariant Borel sets – which is exactly the situation ruled out by the paper [5], and hence a contradiction.

So much for the argument under the simplifying assumptions indicated, now for a proof under only  $\text{AD}^L(\mathbb{R})$ .

This part uses some largely unpublished work of Hugh Woodin's, who showed that for any  $S \subset \text{Ord}$  in  $L(\mathbb{R})$  we have that on a cone of  $x \in 2^\omega$

$$\text{HOD}_S^{L[x, S]} \models (\omega_2)^L[x, S] \text{ is a Woodin cardinal,}$$

where here  $\text{HOD}_S^{L[x, S]}$  is the collection of all sets in  $L[x, S]$  which (inside  $L[x, S]$ ) are hereditarily definable from  $S$  and the ordinals. Working inside such a model where  $S$  codes up the tree  $T$  for the complete  $\Sigma_1^2$  set, the argument passes through as above.

## References

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