An anticompleteness theorem

Greg Hjorth and Andre Nies

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[1] [2]

Harvey Friedman has shown the following completeness theorem:

Theorem 0.1 *(Friedman) Any countable consistent theory has a Borel model of size continuum.*

We show a kind of incompleteness theorem at the Borel level:

Theorem 0.2 *There is a complete, consistent Borel theory of size continuum with no Borel model.*

The proof works in stages: First we construct a Borel theory $\mathbb{T}$; then we show that all its omega models have the same theory; then we let $\mathbb{T}^*$ be the theory of those omega models; then we observe that $\mathbb{T}^*$ has no Borel model.

It might be relevant to point out that the proof gives a slightly stronger incompleteness result. Usually logicians require all models to have the equality relation inside models equal to true equality, however it does make sense to think in terms of models $\mathcal{M}$ where $(=)^{\mathcal{M}}$ is an equivalence relation with more than one point in each equivalence class. Even with this more generous notion of model, our theory $\mathbb{T}^*$ will have no Borel model.

Definition E_0 is the equivalence relation of eventual agreement on infinite Binary sequences.

In the following we identify 2^ω with the collection of all subsets of ω.

1 The theory $\mathbb{T}$

We first define a Borel theory with several types.

(1): $\mathcal{C}$. This include interpretations of the constant symbols $\{c_x : x \in 2^\omega\}$. There will be a relation E defined on $\mathcal{C}$ with

$$c_x E c_y$$

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if and only if xE_0y. E will be an equivalence relation on $\mathcal{C}$ with infinite equivalence classes.

(2): $\mathcal{F}$: Every element of $\mathcal{F}$ will be an onto function from $\mathcal{C}$ and B (where B is discussed in the section on $\mathcal{T}$ below). We require that

$$\forall f \in \mathcal{F} \forall x, y \in \mathcal{C}(xEy \Leftrightarrow f(x) = f(y).$$

We require that identity in f is determined by its value as a function:

$$\forall f, g \in \mathcal{F}(\mathcal{F} \neq g \Rightarrow \exists d(f(d) \neq g(d)).$$

Moreover for each n we introduce the axiom which states that given $d_1, d_2, ..., d_n$ in $\mathcal{C}$ which are E-inequivalent and given $b_1, ..., b_n \in B$ we have

$$\exists f \in \mathcal{F}(f(d_1) = b_1 \wedge f(d_2) = b_2 \wedge ... \wedge f(d_n) = b_n).$$

(3): $\mathcal{A}$: Will have constant symbols $(e_n)_{n \in \omega}$ and unary predicates $(U_x)_{x \in 2^\omega}$ defined over elements of $\mathcal{A}$. We will have the further sentences in our theory:-

(a) Whenever $x, y, z \in 2^\omega$ with $z = x \cap y$ then

$$\forall e \in \mathcal{A}((U_x(e) \wedge U_y(e)) \Leftrightarrow U_z(e)).$$

(b) Whenever $z = \omega \setminus x$ we have

$$\forall e \in \mathcal{A}(U_x(e) \Leftrightarrow \neg U_z(e)).$$

(c) Whenever $z \subset y$ we have

$$\forall e \in \mathcal{A}(U_z(e) \Rightarrow U_y(e)).$$

(d) If $n \notin x$ then

$$\neg U_x(n).$$

(e) If $n \in x$ then

$$U_x(n).$$

(4): $\mathcal{T}$: This in turn will consist of two types, T and B. There will be a function L defined on T, taking values in $\mathcal{A}$. We will also have a relation Q defined between elements of B and elements of T. There will be binary on S on T. (Intuitively one should think of T as elements of a tree and L assigning levels to those elements. B should be thought of as a collection of infinite branches. Q tells us which of the branches are above which of the nodes. S provides the successor relation.)

$\exists! t \in T(L(t) = e_0)$.
$\forall t \in T \exists t_0, t_1(t_0 St \wedge t_1 St \wedge t_0 \neq t_1 \wedge \forall t'(t' St \Rightarrow (t' = t_0 \vee t' = t_1)))$.
$\forall t, t' \in T((t' St \wedge L(t) = e_n) \Rightarrow L(t') = e_{n+1})$.
$\forall b \in B \forall e \in \mathcal{A} \exists! t \in T(L(t) = e \wedge Q(b, t))$.
$\forall t \in T \forall b \in B(Q(b, t) \Rightarrow \exists! t'(t' St \wedge Q(b, t)))$.

$\forall t \in T \exists b \in B(Q(b,t))$.
$\forall b, b' \in B(b \neq b' \Rightarrow \exists t \in T(Q(b,t) \wedge \not Q(b',t)))$.

We will have one last list of sentences in our theory which will be designed to make the functions in $\mathcal{F}$ behave in highly homogeneous manner. For this we introduce one further relation ternary R which should be thought of as measuring the disagreement between elements of $\mathcal{F}$. R will only hold when the first two coordinates are in $\mathcal{F}$ and the last in $\mathcal{A}$.

$\forall f, g \in \mathcal{F} \exists! e \in \mathcal{A}(R(f,g,e))$.
$\forall f, g \in \mathcal{F} \forall e \in \mathcal{A}(R(f,g,e) \Rightarrow R(g,f,e))$.
$\forall f, g, h \in \mathcal{F} \forall e \in \mathcal{A}((R(f,g,e) \wedge R(g,h,e) \Rightarrow R(f,h,e))$.

For each n we further introduce the sentence which describes that for all $f, g \in \mathcal{F}$ we have that if $R(f,g,e_n)$ holds then there are exactly n-equivalence classes on which f and g disagree.

Definition Let $\mathbb{T}$ be the above theory. $\mathcal{L}(\mathbb{T})$ denotes its language.

2 Some properties of the theory $\mathbb{T}$

Lemma 2.1 *If $\mathcal{M}$ is a Borel model of $\mathbb{T}$, then*

$$(\mathcal{A})^{\mathcal{M}} = \{(e_n)^{\mathcal{M}} : n \in \omega\}.$$

Proof Otherwise choose $e \in (\mathcal{A})^{\mathcal{M}}$ which is outside this set. We obtain a Borel non-principal ultrafilter on 2^ω with the specification

$$x \in \mu \Leftrightarrow \mathcal{M} \models U_x(e),$$

which is well known to be impossible. □

Lemma 2.2 *(Malitz) Let T be a perfect binary tree (that is to say, $T \cong 2^{<\omega}$ under inclusion). Let $[T]$ denote the branches through T. If $S, S' \subset [T]$ are dense in the sense that for all $b \in T \exists b \in S, b' \in S'$*

$$b \subset b, b',$$

then there is an automorphism

$$\pi : T \cong T$$

with $\{\pi(b) : b \in S\} = S'$.

Lemma 2.3 *Let $\mathcal{L}_0$ be a finite subset of $\mathcal{L}(\mathbb{T})$ including $\mathcal{C}, \mathcal{F}, \mathcal{A}, \mathcal{T}, E, R, L, S, Q$. Let $\mathcal{L}_1 = \mathcal{L}_0 \cup \{e_n : n \in \omega\}$. Let $\mathcal{M}_0, \mathcal{M}_1$ be models of $\mathbb{T}|_{\mathcal{L}_1}$ which are "standard" in the sense that*

$$(\mathcal{A})^{\mathcal{M}_i} = \{(e_n)^{\mathcal{M}_i} : i \in \omega\}$$

for $i = 0, 1$. Then

$$\mathcal{M}_0 \cong \mathcal{M}_1.$$

Proof Let $[x_0]_{E_0}, [x_1]_{E_1}, ..., [x_M]_{E_0}$ enumerate the equivalence classes of $\{x : c_s \in \mathcal{L}\}$. Choose $f \in \mathcal{F}^{\mathcal{M}_0}$. Choose N such that for all $i \neq j \leq M$ there exists $t_0 \neq t_1 \in T^{\mathcal{M}_0}$ with

$$\mathcal{M}_0 \models L(t_0) = e_N, L(t_1) = e_N,$$

$$\mathcal{M}_0 \models Q(f(c_{x_i}), Q(f(c_{x_j})).$$

Let $T_N^{\mathcal{M}_0}, T_N^{\mathcal{M}_1}$ respectively be the set

$$\{t \in T^{\mathcal{M}_0} : \exists n \leq N \mathcal{M}_0 \models L(t) = e_n\},$$

$$\{t \in T^{\mathcal{M}_1} : \exists n \leq N \mathcal{M}_1 \models L(t) = e_n\}.$$

Let

$$\sigma : T_N^{\mathcal{M}_0} \cong T_N^{\mathcal{M}_1}$$

preserve the successor relation. Choose $b_1, ..., b_N \in B^{\mathcal{M}_1}$ such for all t with

$$\mathcal{M}_0 \models L(t) = e_N,$$

and all $i \leq M$, we have

$$\mathcal{M}_0 \models Q(f(c_{x_i}), t)$$

iff

$$\mathcal{M}_1 \models Q(b_i, \sigma(t)).$$

Now let $(t_i)_{i\in\omega}$ enumerate nodes in $T^{\mathcal{M}_0}$ such that

(a) t_i not in used in any of the branches associated to any of the c_{x_j}'s (i.e. $\forall j \leq N \mathcal{M}_0 \models \neg Q(f(c_{x_j}), t_i)$).

(b) t_i is an immediate successor of a branch used in some c_{x_j} (i.e. $\exists t, j \mathcal{M}_0 \models t_i S t, Q^{\mathcal{M}_0}(f(c_{x_j}), t)$).

Let $(\hat{t}_i)_{i\in\omega}$ be defined similarly in $T^{\mathcal{M}_1}$ but for the branches $b_1, ..., b_M$:

(a) $\forall j \leq N \mathcal{M}_1 \models \neg Q(b_j, \hat{t}_i$.

(b) $\exists t, j \mathcal{M}_1 \models \hat{t}_i S t, Q^{\mathcal{M}_0}(b_j, t)$.

We can do this so that

$$(\mathcal{M}_0 \models L(t_i) = e_n) \Rightarrow (\mathcal{M}_1 \models L(\hat{t}_i, e_n)),$$

and if there exists t with

$$\mathcal{M}_0 \models Q(f(c_{x_j}), t), t_i S t$$

then there exists $\hat{t}$ with

$$\mathcal{M}_1 \models Q(b_j, \hat{t}), \hat{t}_i S \hat{t}.$$

At each i, let T^i be the set of t in $T^{\mathcal{M}_0}$ which has t_i as an ancestor. With the structure endowed by $\mathcal{M}_0$ this becomes a perfect binary tree – think of this intuitively as the tree of points whose last contact with one of the branches associated to one of the x_j's is equal to t_i. Similarly we let $\hat{T}^i$ be the set of $\hat{t} \in T^{\mathcal{M}_1}$ such that $\hat{t}_i$ is an ancestor of $\hat{t}$. Let

$$B_i = \{b : \mathcal{M}_0 \models Q(b, t_i)\},$$
$$\hat{B}_i = \{b : \mathcal{M}_1 \models Q(b, \hat{t}_i)\}.$$

We can then apply Malitz's lemma to find

$$\pi : T^i \cong \hat{T}^i$$

with induced

$$\pi_i[B^i] = \hat{B}_i.$$

Let

$$\pi : \mathcal{T}^{\mathcal{M}_0} \cong \mathcal{T}^{m_1}$$

be the result of patching these together and assigning

$$\pi(f(c_{x_j})) = b_j.$$

Fix $\hat{f} \in \mathcal{F}^{\mathcal{M}_1}$ with $\hat{f}(c_{x_j}) = b_j$ all $j \leq M$. (The axioms listed under (2) above make this possible.)

We now extend π to become an isomorphism of structures. $c \in \mathcal{C}^{\mathcal{M}_0}$ we let

$$\pi(c) = (c_x)^{\mathcal{M}_1}$$

if $c = (c_x)^{\mathcal{M}_0}$ some x, and otherwise simply choose $\pi(c)$ so that

$$\hat{f}^{\mathcal{M}_1}(\pi(c)) = \pi(f^{\mathcal{M}_0}(c));$$

this is possible since π at the level of $B^{\mathcal{M}_0} \to B^{\mathcal{M}_1}$ is one to one and onto, and in the axioms at (1) give $\hat{f}$ as an isomorphism between $\mathcal{C}/E$ and B. Since each E class is infinite, we can do this so that π provides a bijection

$$([c]_E)^{\mathcal{M}_0} \cong ([\pi(c)]_E)^{\mathcal{M}_1}.$$

The fact that $\mathcal{M}_0, \mathcal{M}_1$ are ω-models in terms of $\mathcal{A}$ (that is to say $\mathcal{A}^{\mathcal{M}_i} = \{(e_n)^{\mathcal{M}_i} : n \in \omega\}$), along with axioms at (2) ensure that $\mathcal{F}^{\mathcal{M}_0}$ and $\mathcal{F}^{\mathcal{M}_1}$ consist of exactly all the functions

$$(\mathcal{C}/E)^{\mathcal{M}_0} \to B^{\mathcal{M}_0}$$

and

$$(\mathcal{C}/E)^{\mathcal{M}_1} \to B^{\mathcal{M}_1}$$

which agree with $\mathcal{F}$ and $\hat{f}$ respectively on all but finitely many values. From this it is clear how to extend π to an isomorphism between $\mathcal{F}^{\mathcal{M}_0}$ and $\mathcal{F}^{\mathcal{M}_1}$.

It only remains to define π on $\mathcal{A}^{\mathcal{M}_0}$ – but here we simply send

$$(e_n)^{\mathcal{M}_0} \mapsto (e_n)^{\mathcal{M}_1},$$

and for the predicate symbols of the form U_x in the common language, our axiomatization at (3) ensures we have at each x, n

$$\mathcal{M}_0 \models U_x(e_n) \Leftrightarrow \mathcal{M}_1 \models U_x(e_n).$$

□

3 The theory $\mathbb{T}^*$

Notation Let $A_{\mathbb{T}}$ be the set of $\mathcal{M}$ as described in lemma 2.3: That is to say, for some finite subset $\mathcal{L}_0$ of $\mathcal{L}(\mathbb{T})$ including $\mathcal{C}, \mathcal{F}, \mathcal{A}, \mathcal{T}, E, R, L, S, Q$ and $\mathcal{L}_1 = \mathcal{L}_0 \cup \{e_n : n \in \omega\}$ we have that $\mathcal{M}$ is a model of $\mathbb{T}|_{\mathcal{L}_1}$ which has

$$(\mathcal{A})^{\mathcal{M}} = \{(e_n)^{\mathcal{M}} : i \in \omega\}.$$

Definition Let $\mathbb{T}^*$ be the set of $\phi \in \mathcal{L}(\mathbb{T})$ such that there exists $\mathcal{M} \in A_{\mathbb{T}}$ with

$$\mathcal{M} \models \phi.$$

Lemma 3.1 $\mathbb{T}^*$ *is Borel.*

Proof Clearly $A_{\mathbb{T}}$ is Borel, and then in light of 2.3 we have $\phi \in \mathbb{T}^*$ if and only if

$$\exists \mathcal{M} \in A_{\mathbb{T}}(\mathcal{M} \models \phi),$$

if and only if

$$\forall \mathcal{M} \in A_T(\mathcal{L}(\phi) \subset \mathcal{L}(\mathcal{M}) \Rightarrow \mathcal{M} \models \phi).$$

Thus $\mathbb{T}^*$ is $\underset{\sim}{\Delta}^1_1$ and hence Borel. □

Lemma 3.2 $\mathbb{T}^*$ *is complete.*

Proof From the structure of the definition of $\mathbb{T}^*$. □

Lemma 3.3 $\mathbb{T}^*$ *is consistent.*

Proof From 2.3 and the definition of $\mathbb{T}^*$, we have that for any finite $\mathcal{L}_0 \subset \mathcal{L}(\mathbb{T}^*)$ there is a countable model of $\mathbb{T}^*|_{\mathcal{L}_0}$. □

Theorem 3.4 $\mathbb{T}^*$ *has no Borel model.*

Proof For a contradiction, suppose $\mathcal{M}$ is a Borel model of $\mathbb{T}^*$. By lemma 2.1 and $\mathbb{T}^* \supset \mathbb{T}$,

$$(\mathcal{A})^{\mathcal{M}} = \{(e_n)^{\mathcal{M}} : n \in \omega\}.$$

Let $(t_n)_{n \in \omega}$ enumerate elements of $(T)^{\mathcal{M}}$. For each $b \in B^{\mathcal{M}}$, we let

$$\rho(b) = \{n : \mathcal{M} \models Q(b, t_n)\}.$$

Fix $f \in (\mathcal{F})^{\mathcal{M}}$. Define

$$\theta : 2^\omega \to 2^\omega$$

by

$$\theta(x) = \rho((f(c_x))^{\mathcal{M}}).$$

Then for all $x, y \in 2^\omega$,

$$xE_0y \Leftrightarrow \mathcal{M} \models c_x E c_y,$$

by the axioms at (1),

$$\Leftrightarrow \mathcal{M} \models f(c_x) = f(c_y),$$

by the axioms at (2),

$$\Leftrightarrow \rho((f(c_x))^{\mathcal{M}}) = \rho((f(c_y))^{\mathcal{M}}),$$

by the axioms at (4) describing B as a collection of branches through T. Thus we obtain a Borel function θ with

$$xE_0y \Leftrightarrow \theta(x) = \theta(y),$$

which is well known to be impossible. □

The proof gives a slightly stronger anticompleteness result. Even if we allow Borel models where equality in $\mathcal{M}$, $(=)^{\mathcal{M}}$, does not actually correspond to true $=$ in the outside world, we still obtain no Borel model of $\mathcal{M}$. The only adjustment to the above argument for this further result is to let $(t_n)_{n\in\omega}$ be a complete sequence of *representatives* for elements of $(T)^{\mathcal{M}}$.