Measure Theory, 2008, Homework Two

Tuesday, May 13, start of class

Q1: (2pts) Let $I = [0, 1]$ be the closed unit interval in its usual compact topology and let $C(I)$ the collection of continuous functions on $I$ with the metric $d(f, g) = \sup_{z \in I} |f(z) - g(z)|$.

Define

$$\Phi : C(I) \to C(I)$$

by

$$\Phi(f)(x) = \int_0^x f(x) d\lambda(x),$$

where $\lambda$ is Lebesgue measure.

Show that $\Phi$ is a continuous function in the indicated topology on $C(I)$.

Q2:

**Notation** For $K$ a compact metric space, let $P(K)$ be the probability measures on $K$ equipped with the topology generated by the basic open sets

$$\{\mu : s_1 < \mu(f_1) < r_1, s_2 < \mu(f_2) < r_2, ..., s_n < \mu(f_n) < r_n\}$$

for $f_1, f_2, ..., f_n \in C(K)$.

(a) (2pts) Let $K$ be a compact metric space. Let $C(K, [-1, 1])$ be the subspace of $C(K)$ consisting of continuous functions with norm at most one – that is to say, the range included in $[1, -1]$ Show that if $\{f_i : i \in \mathbb{N}\}$ is a countable dense subset of $C(K, [-1, 1])$ then the function

$$\pi : P(K) \to \prod_{i \in \mathbb{N}} [0, 1]$$

given by

$$(\pi(\mu))(n) = \mu(f_n)$$

is continuous and open onto its image (i.e. $\pi$ effects a homeomorphism between $P(K)$ and $\pi[P(K)]$).

(b) (2pts) Show that $P(K)$ is a compact metrizable space.

(c) (2pts) Let $\psi : K \to K$ be a homeomorphism. At each $n$ let $F_n$ be the collection of $\mu \in P(K)$ for which

$$|\int f d\mu - \int f \circ \psi d\mu| \leq \frac{1}{n}$$

for each $f \in C(K)$ with $-1 \leq f \leq 1$.

Show that each $F_n$ is a closed, non-empty subset of $P(K)$.

(d) (2pts) Use (b) and (c) to conclude that there is a $\psi$-invariant Borel probability measure on $K$.

\[1\] For non-empty: Given $n$, choose any $\nu \in P(K)$ and define $\Lambda \in C(K)$ by

$$f \mapsto \frac{1}{2n + 1} \sum_{-n \leq \ell \leq n} \int f \circ \psi^\ell d\nu.$$ 

Observe that $\Lambda$ is a positive, linear functional, with $\Lambda(1) = 1$, and then argue the measure corresponding to $\Lambda$ is as required.