Due Date: Tuesday, April 22, start of class

Q1: (2pts) Let \((X, \Sigma)\) be a measure space (i.e. \(\Sigma\) is a \(\sigma\)-algebra on \(X\)). Let \(\mu\) be a measure on \((X, \Sigma)\). Assume: (i) \(\mu(X) = \infty\); (ii) \(\mu\) is \(\sigma\)-finite; (iii) \(\mu(\{x\}) = 0\) all \(x \in X\) (that is to say, \(\mu\) is atomless).

Show that we can write \(X\) as a disjoint union of sets

\[ X = \bigcup_{n \in \mathbb{N}} X_n \]

each with measure one.

Q2: (2pts) Let \((X, \Sigma)\) be a measure space. Let \(f : X \to \mathbb{R}\) be such that

\[ f^{-1}([-\infty, q]) \in \Sigma \quad \text{all } q \in \mathbb{Q}. \]

Show that \(f\) is measurable with respect to \(\Sigma\) (i.e. the pullback of any open set along \(f\) is in \(\Sigma\)).

Q3: (2pts) Let \(X\) be the closed unit square, \([0, 1] \times [0, 1]\) equipped with the subspace topology (from the usual topology on \(\mathbb{R}^2\)). Let \(\Sigma\) be the resulting \(\sigma\)-algebra of Borel subsets of \(X\). Let \(\mu\) be Lebesgue measure on \(X\) (i.e. the restriction of the measure \(m\) on \(\mathbb{R}^2\) defined on page one of the course notes).

Let \(f : X \to \mathbb{R}\) be defined by

\[ (x, y) \mapsto x^2 y^2. \]

Let \(\Sigma_0\) be the \(\sigma\)-algebra consisting of all sets of the form \(A \times [0, 1]\) for \(A \subset [0, 1]\) Borel.

Calculate \(E(f|\Sigma_0)\), the conditional expectation of \(f\) with respect to \(\Sigma_0\).

Q4: (4pts) Let

\[ X = \prod_{n \in \mathbb{N}} \{0, 1\}, \]

with the product topology. Let \(\mu\) be the product measure on this space. (This is to say, for \(A = \{f \in X : f(1) = \ell_1, f(2) = \ell_2, \ldots, f(n) = \ell_n\}\), we have \(\mu(A) = 2^{-n}.\))

For each finite \(S \subset \mathbb{N}\) define

\[ \psi_S : X \to \mathbb{R} \quad f \mapsto (-1)^{-|\{n : f(n) = 0\}|}. \]

Show that \(\{\psi_S : S \subset \mathbb{N}, S \text{ finite}\}\) gives an orthonormal basis for the Hilbert space \(L^2(X, \mu)\).

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I have put the current version of the course notes on line at:

http://www.math.ucla.edu/~greg/measure.html

I will try to keep updating this site.

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1Remarks: \(\psi_{\emptyset}\) is the function with constant value 1. For orthoganility, try to show \(\langle \psi_S, \psi_T \rangle = \int \psi_S \Delta_T d\mu\). The final issue is to see that the linear combinations of these functions are dense in \(L^2(X, \mu)\).