Measure Theory, 2007, Homework Two

Solutions

People seemed to have no trouble at all with the applications of Radon-Nikodym in the first two questions, so I won't comment on those. Here are some remarks about the rest.

Q3: Let K be a compact metric space and equip $P(K)$ with the topology described in class (arising from viewing it as a closed subset of $C(K)^*$ in the weak star topology). Show that if $U \subset K$ is open, the set $\{\mu \in P(K) : \mu(U) = 1\}$ is G_δ. (Recall: A set in a space is said to be G_δ if it can be represented as a countable intersection of open sets.)

Some of the answers were a bit unsteady – as if people are very unsure about the whole nature of the spaces and objects we are dealing with. (A very reasonable reaction at this stage by the way. After all the notion of a "space of measures" involves several degrees of abstraction.)

Here's what you have to do.

At each n we can let $C_n \subset U$ be the set of x with $d(x, K \setminus U) \geq \frac{1}{n}$. We have $U = \bigcup C_n$, and so $\mu(U) = 1 \Leftrightarrow \mu(C_n) \to 1$.

It is a basic topological lemma, used in the course and proved in the notes, that at each n we can find a continuous function $f_n : K \to [0,1]$ with $f_n(x) = 1$ all $x \in C_n$ and the support of f included in U.

Note that $\mu(U) = 1$ implies $\mu(C_n) \to 1$, which in turn implies $\int f_n d\mu \to 1$. Conversely, since each f_n has its support included in U, and $\mu(U)$ equals the supremum of $\{\int f d\mu : f \prec U\}$, we have $\int f_n d\mu \to 1$ entailing $\mu(U) = 1$.

Thus $\mu(U) = 1$ if and only if

$$\forall m \exists n \int f_n \mu > 1 - \frac{1}{m}.$$

Thus if we let $V_{m,n}$ be the set of measures for which $\int f_n \mu > 1 - \frac{1}{m}$, then each $V_{m,n}$ is open and

$$\{\mu \in P(K) : \mu(U) = 1\} = \bigcap_m \bigcup_n V_{m,n},$$

demonstrating it to be a countable intersection of the open sets

$$W_m = \bigcup_n V_{m,n}.$$

Q4: For $S \subset \mathbb{N}$ finite, define

$$\psi_S : 2^{\mathbb{N}} \to \{-1, 1\}$$

by $\psi_S(x) = (-1)^{|\{n \in S : x(n) = 1\}|}$. Let μ be the usual product measure on $2^{\mathbb{N}}$.

Show that $\{\psi_S : S \subset \mathbb{N} \text{ finite }\}$ forms an orthonormal basis for the Hilbert space $L^2(2^{\mathbb{N}}, \mu)$.

Let X be used to denote $2^{\mathbb{N}}$. This is a compact metric space equipped with a probability measure.

On easily calculates that for S, T finite that $f_S \cdot f_T = f_{S\Delta T}$ and moreover

$$\int f_S d\mu = 0$$

unless $S = \emptyset$ in which case it equals 1. Thus

$$\langle f_S, f_T \rangle = \int f_{S\Delta T} d\mu,$$

which in turn equals 0 if $S \neq T$ and 1 if $S = T$. Thus this is certainly an orthonormal set in $L_2(X, \mu)$.

These functions are all not only square integrable but continuous. For two continuous (real valued) functions f, g on X we have

$$\langle f - g, f - g \rangle = \int (f - g)^2 d\mu < \sup_{x \in K} |f(x) - g(x)|^2,$$

and thus it certainly suffices to show the finite linear combinations of the f_S functions are dense in $C(K)$ with respect to the sup norm topology. For this it suffices to check that the hypotheses of Stone-Weierstrass are satisfied when we let $\mathcal{A}$ be the finite linear combinations of the f_S functions.

Immediately $\mathcal{A}$ is an algebra. It clearly separates points, since if $x \neq y$ in X there exists some n with $x(n) \neq y(n)$ and then $f_{\{n\}}$ differs on these two points. It contains the constant functions, since $f_\emptyset \equiv 1$ and we have thrown in all scalar multiples of all these functions, in particular $f_\emptyset$.

Thus by Stone-Weierstrass, $\mathcal{A}$ is dense in $C(X)$ (wrt sup norm) and hence in $L_2(X, \mu)$ (wrt L_2-norm).