The fine structure and Borel complexity of orbits

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§**0. Introduction** Recent work in the general theory of Polish groups has looked for inspiration from the study of countable models, where a major tool in analyzing isomorphism types is provided by the Scott analysis. One has many results that were first proved in the case of the "logic action" on countable models by an appeal to the Scott analysis, and only later generalized to general Polish group actions – usually in some *ad hoc* manner, and occasionally in a somewhat diluted form. On top of this there were even a small number of results known for closed subgroups of S_∞ by use of the Scott analysis but open for more general Polish groups.

I will go through three different parallel definitions, each of which has some moral right to be considered *the* Scott analysis, but only the second of which historically fills this role.

0.1 Definition S_∞ is the group of all permutations of $\mathbb{N}$ with the topology of pointwise convergence. For $\vec{a}, \vec{b}$ finite 1-1 sequences of the same length, we let

$$V_{\vec{a},\vec{b}} = \{\sigma \in S_\infty : \sigma(\vec{a}) = \vec{b}\},$$

and then

$$V_{\vec{a}} = V_{\vec{a},\vec{a}}.$$

Note that the $V_{\vec{a},\vec{b}}$ sets form a basis for the topology on S_∞ and the $V_{\vec{a}}$ sets form a neighborhood basis at the identity.

0.2 Definition Let X be a Polish S_∞-space – that is to say, a Polish space equipped with a continuous action of S_∞. Let $\{U_\ell : \ell \in \mathbb{N}\}$ be a basis for the topology on X.

Given $x \in X$, $\vec{a}, \vec{b}$ finite 1-1 sequences of the same length, we define $\hat{\varphi}^x_{\alpha,\vec{a},\vec{b}}$ by induction on α.

$$\hat{\varphi}^x_{0,\vec{a},\vec{b}} = \{\ell \in \mathbb{N} : V_{\vec{a},\vec{b}} \cdot x \cap U_\ell \neq \emptyset\},$$

and then

$$\hat{\varphi}^x_{\alpha+1,\vec{a},\vec{b}} = \{\hat{\varphi}^x_{\alpha,\vec{a}^\frown\vec{c},\vec{b}^\frown\vec{d}} : \vec{a}^\frown\vec{c}, \vec{b}^\frown\vec{d} \,\text{both sequences}\; 1-1,\ \text{same length}\},$$

and at λ a limit,

$$\hat{\varphi}^x_{\lambda,\vec{a},\vec{b}} = \{\hat{\varphi}^x_{\alpha,\vec{a},\vec{b}} : \alpha < \lambda\}.$$

We also let $\hat{\varphi}^x_\alpha$ simply denote $\hat{\varphi}^x_{\alpha,\emptyset,\emptyset}$, and $\hat{\varphi}^x_{\alpha,\vec{a}}$ denote $\hat{\varphi}^x_{\alpha,\vec{a},\vec{a}}$.

It follows from there only being countably many $\vec{a}, \vec{b}$ sequences to consider that there will be an ordinal $\delta < \omega_1$ such that for all $\vec{a}, \vec{b}, \vec{a}^*, \vec{b}^*$

$$\hat{\varphi}^x_{\delta,\vec{a},\vec{b}} = \hat{\varphi}^x_{\delta,\vec{a}^*,\vec{b}^*}$$

if and only

$$\forall \alpha < \omega_1 (\hat{\varphi}^x_{\alpha,\vec{a},\vec{b}} = \hat{\varphi}^x_{\alpha,\vec{a}^*,\vec{b}^*}).$$

We then let $\delta(x)$ be the least such ordinal δ, which we call the *Scott height* of x. We let

$$\hat{\varphi}^x = \hat{\varphi}^x_{\delta(x)+2}.$$

Note that all these $\hat{\varphi}$-invariants are hereditarily countable sets.

In some form the next theorem should be mostly credited to Dana Scott, though his original presentation was somewhat different and later authors have noticed various minor refinements.

0.3 Theorem Let X be a Polish S_∞-space. Then:

(0) there is a countable ordinal $\delta(x)$ such that $\hat{\varphi}^x_{\delta(x)}$ determines $\hat{\varphi}^x_\alpha$ for every ordinal α, and moreover is an invariant of $[x]_{S_\infty}$;

we then let $\hat{\varphi}^x = \hat{\varphi}^x_{\delta(x)+2}$ and obtain in turn:

(i) $\hat{\varphi}^x = \hat{\varphi}^y$ if and only if $xE_{S_\infty}y$ (that is to say, they have the same orbit);

(ii) for each countable ordinal α the set

$$\{(x,y) \in X^2 : \hat{\varphi}^x_{\alpha,} = \hat{\varphi}^y_\alpha\}$$

is Borel, and in fact $\underset{\sim}{\Pi}^0_{\alpha+\omega}$ (and here one shows by induction that for each countable ordinal α there is an n such that for all suitable $\vec{a}, \vec{b}, \vec{c}, \vec{d}$, the set

$$\{(x,y) \in X^2 : \hat{\varphi}^x_{\alpha,\vec{a},\vec{b}} = \hat{\varphi}^y_{\vec{c},\vec{d}}\}$$

is fact $\underset{\sim}{\Pi}^0_{\alpha+n}$);

(iii) for $A \subset X$ an S_∞ invariant $\underset{\sim}{\Sigma}^0_{1+\alpha}$ set, if $\hat{\varphi}^x_\alpha = \hat{\varphi}^y_\alpha$, then $x \in A \Leftrightarrow y \in A$ (and here one shows by induction on α that for $A \subset X$ a $\underset{\sim}{\Sigma}^0_{1+\alpha}$ set and $\hat{\varphi}^x_{\alpha,\vec{a},\vec{b}} = \hat{\varphi}^y_{\vec{c},\vec{d}}$,

$$x \in A^{\Delta V_{\vec{a},\vec{b}}} \Leftrightarrow y \in A^{\Delta V_{\vec{c},\vec{d}}});$$

(iv) the equivalence relation $\hat{\varphi}^x_\alpha = \hat{\varphi}^y_\alpha$ is reducible to a Borel equivalence relation induced by the continuous action of S_∞ on a Polish space.

I take the view that any analysis which has properties (i)-(iv) for the continuous actions of a Polish group G in replace of S_∞ deserves to be called a *Scott analysis for G*, and we should not quibble over the slight subtleties which arise out of considering minor differences in which the process can be phrased. The problem this paper wants to consider is the extent to which there is a generalized Scott analysis for arbitrary Polish groups, or, rather, for which Polish groups can we hope to obtain a version of 0.3.

For $\mathcal{L}$ a countable language, we can consider the Polish space of $\mathcal{L}$-structures whose underlying set is $\mathbb{N}$, and in particular we obtain Scott's original application, to the effect that the isomorphism class of each structure will be Borel. In fact, Scott's original presentation was totally specific to the class of such countable $\mathcal{L}$-structures.

0.4 Definition Let $\mathcal{L}$ be a language, and let $\mathcal{M}$ be an $\mathcal{L}$-structure. For α an ordinal and $\vec{a}$ a finite sequence in $\mathcal{M}$, we define

$$\varphi^{\mathcal{M}}_{\alpha,\vec{a}}$$

by induction on α, with $\varphi^{\mathcal{M}}_{0,\vec{a}}(\vec{x})$ the quantifier free type of $\vec{a}$ in $\mathcal{M}$, and

$$\varphi^{\mathcal{M}}_{\alpha+1}(\vec{x})$$

being

$$(\bigwedge_{\vec{b}\in\mathcal{M}} \exists\vec{y}\varphi^{\mathcal{M}}_{\alpha,\vec{a}^\frown\vec{b}}(\vec{x},\vec{y})) \wedge (\forall\vec{y} \bigvee_{\vec{b}\in\mathcal{M}} \varphi^{\mathcal{M}}_{\alpha,\vec{a}^\frown\vec{b}}(\vec{x},\vec{y})),$$

and at limit λ we let

$$\varphi^{\mathcal{M}}_{\lambda,\vec{a}}(\vec{x}) = \bigwedge_{\alpha<\lambda} \varphi^{\mathcal{M}}_{\alpha,\vec{a}}(\vec{x}).$$

Again we come to some least $\delta(\mathcal{M}) < |\mathcal{M}|^+$ such that for all $\vec{a}$ the formula

$$\varphi^{\mathcal{M}}_{\delta(x),\vec{a}}$$

determines every

$$\varphi^{\mathcal{M}}_{\alpha,\vec{a}},$$

and we again let

$$\varphi^{\mathcal{M}} = \varphi^{\mathcal{M}}_{\delta(\mathcal{M}),\emptyset}.$$

It can be show that for any formula $\psi \in \mathcal{L}_{\infty,\omega}$ we have that if the rank of ψ is α, then $\varphi^{\mathcal{M}}_{\alpha+1,\vec{a}}$ determines whether $\mathcal{M} \models \psi(\vec{a})$.

In the case that $\mathcal{M}$ and $\mathcal{L}$ are countable, these formulas are all in $\mathcal{L}_{\omega_1,\omega}$ and $\varphi^{\mathcal{M}}$ is a complete invariant of its isomorphism type among *other countable* $\mathcal{L}$-structures. If we consider the Polish-S_∞ space of $\mathcal{L}$-structures in the topology generated by quantifier free formulas, then the analogs of 0.3(i)-(iv) all hold for the φ-invariants.

Finally, we can view the Scott analysis as actually describing a sequence of refining topologies around the various orbits.

0.5 Definition Let X be a Polish G-space. We let $\tau_{0,x}$ be the original topology on X. For $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ finite 1-1 sequences in $\mathbb{N}$ of the same length, we let $U_{\alpha,x,\vec{a},\vec{b},\vec{c},\vec{d}}$ be the collection of y such that

$$\overline{V_{\vec{a},\vec{b}} \cdot x}^{\tau_{\alpha,x}} = \overline{V_{\vec{c},\vec{d}} \cdot y}^{\tau_{\alpha,x}},$$

and we let $\tau_{\alpha+1,x}$ be the topology generated by the $U_{\alpha,x,\vec{a},\vec{b},\vec{c},\vec{d}}$ sets of this form. At limit stages we take the unions of the topologies generated thus far.

The analogy to 0.3 is given by the fact that if $xE_{S_\infty}y$ then $\tau_{\alpha,x} = \tau_{\alpha,y}$ at every α and that we always come to some countable δ such that $[x]_{S_\infty}$ is closed with respect to the topology $\tau_{\delta,x}$.

One can also cast 0.5 in more model theoretic terms. $\tau_{1+\alpha,x}$ corresponds to taking us our topology on the $\mathcal{L}$-structures produced by the fragment generated by the invariants of the form $\phi^x_{\alpha,\vec{a}}(\vec{x})$.

Before going forwards with the more general analysis presented in this paper, I wish to survey the many varied results which have been proved for continuous actions of S_∞ using the Scott analysis.

0.6 Theorem (Scott, [8]) If $\mathcal{L}$ is a countable language, then the isomorphism type of each countable model of $\mathcal{L}$ is Borel (in the Polish space of $\mathcal{L}$-structures on $\mathbb{N}$ with the topology of quantifier free formulas).

As noted earlier, this follows directly from 0.3 (i), (ii)

This was later generalized in [6] to the actions of general Polish groups – if G is a Polish group and X is a Polish G-space, then each $[x]_G$ is Borel. However the proof was abstract, indirect, and lacked the bottom up fine analysis of Scott's original argument. In fact, his argument critically relied on $\utilde{\Delta}^1_1$ being the class of Borel sets.

0.7 Theorem (folklore) If X is a Polish S_∞ space and every orbit is $\utilde{\Pi}^0_\alpha$, then the orbit equivalence relation E_{S_∞} is $\utilde{\Pi}^0_{\alpha+\omega}$.

It is something of a slight simplification here to describe this result as folklore. It probably *is* folklore in the case of countable models considered up to isomorphism, appealing to Scott's analysis in its original form. The more general case would need to be proved either by a careful reduction of the general case to the model theoretic one, using one of the Becker-Kechris theorems to that effect from [1], or by applying the somewhat retrofitted version of Scott's original analysis present at 0.3: One runs the Scott analysis up to the ordinal $\alpha + 1$, and uses 0.3(iii) to observe that $\hat{\varphi}^x_{\alpha+1}$ is a complete invariant since $[x] \in \utilde{\Pi}^0_\alpha$; then the result follows by 0.3(ii).

Ramez Sami gave an abstract argument in [7] to the effect that if G is a Polish group and X is a Polish G-space with a bound in the complexity of its orbits, then it E_G is Borel. His argument, as with the generalization of 0.6 presented at [6], was not very effective – in particular it could not give an apriori bound on the complexity of E_G in terms of the bound on the complexity of the orbits, but rather argued abstractly that E_G must be $\utilde{\Delta}^1_1$ and hence Borel, in a move parallel to [6] on generalizing 0.6.

0.8 Theorem (Becker-Kechris, [1]) If $\mathcal{L}$ is a countable language, and X is the space of $\mathcal{L}$-structures on $\mathbb{N}$, and $B \subset X$ is a Borel invariant set on which the Borel complexities of the orbits are bounded, then Scott heights are bounded as well.

We can suppose that B is $\utilde{\Pi}^0_\alpha$ and every orbit is $\utilde{\Pi}^0_\alpha$. Then given any $\mathcal{M} \in B$, $\varphi^{\mathcal{M}}_{\alpha+1}$ will be a complete invariant for its orbit by the appropriate analog of 0.3(iii). Now we can observe that since $\varphi^{\mathcal{M}}_{\alpha+1}$ has only a *single* model, there must be an atomic model in the fragment of $\mathcal{L}_{\omega_1,\omega}$ it generates – and that single model, in virtue of being atomic in the indicated fragment, will have Scott height at most $\alpha+\omega$ and must be isomorphic to $\mathcal{M}$.

0.9 Theorem (folklore) If X is a Polish S_∞ space, then there is an *effective* map

$$i : X/S_\infty \to \underset{\sim}{\Pi}^0_{\alpha+1}$$

such that orbits $[x]_G$ and $[y]_G$ with the same image under i will meet exactly the same invariant $\underset{\sim}{\Pi}^0_\alpha$ sets.

Here when I say "effective", I mean effective to our relevant set theoretical assumptions. For instance under $\mathrm{AD}^L(\mathbb{R})$, and $G, X \in L(\mathbb{R})$, we obtain the injection in $L(\mathbb{R})$.

0.9 does not quite literally follow from 0.3 as presented above. However, a tighter analysis of the Scott invariants, as in [4], shows, for instance, that the equivalence relation $\varphi^x_n = \varphi^y_n$ is *potentially* $\underset{\sim}{\Pi}^0_{n+2}$.

0.10 Theorem (Friedman, [2]) Let E be a Borel equivalence relation on a Polish space X and suppose Y is a Polish S_∞ and $E \leq_B E^Y_{S_\infty}$. Then there is another Polish S_∞-space $\hat{Y}$ with $E \leq_B E^{\hat{Y}}_{S_\infty}$ and $E^{\hat{Y}}_{S_\infty}$ Borel.

[1] shows that the *logic actions* are universal among S_∞-spaces, and so we can assume that X is the space of structures of some countable language $\mathcal{L}$ with underlying set $\mathbb{N}$. Let

$$\theta : X \to Y$$

witness $E \leq_B E^Y_{S_\infty}$, in the sense that $x_1 E x_2 \Leftrightarrow \theta(x_1) \cong \theta(x_2)$.

It is a well known, demonstrated by a routine back and forth argument, that if $\mathcal{M}_1, \mathcal{M}_2$ are countable models which are *not* isomorphic, then there is some $\alpha < \omega_1^{\mathrm{ck}(\mathcal{M}_1,\mathcal{M}_2)}$ for which

$$\varphi^{\mathcal{M}_1}_\alpha \neq \varphi^{\mathcal{M}_2}_\alpha.$$

Thus by boundedness, and the fact that E and θ are both Borel, we may find a single countable $\alpha < \omega_1$ such that for all $x_1, x_2 \in X$, if they are E-inequivalent, then

$$\varphi^{\theta(x_1)}_\alpha \neq \varphi^{\theta(x_2)}_\alpha.$$

Now the result follows from 0.3(iv).

The remainder of this paper deals with a spirited attempt to generalize 0.3 to general Polish group actions, and thereby prove tighter generalizations of 0.6-0.10.

Here there is a caveat. Any overly naive attempt the existing Scott analysis to arbitrary groups runs afoul if there is no basis of open subgroups around the identity. More generally, it is known (see [2], [3]) that there are Polish groups whose actions do not admit objects in HC as complete invariants, and thus even more sophisticated attempts must necessarily have a structurally very different form to the original Scott analysis.

In this note we our generalization has a radically different structure. For each $\alpha < \omega_1$ one provides an equivalence relation $\approx_\alpha$ that is coarser than the original orbit equivalence relation, but such that at each point x in the space there will be an α_x such that orbit of x is equal to the set of points $\approx_{\alpha_x}$-equivalent to x. The essential new idea in this – as against simply trying to rewrite what was known specifically for S_∞ – is to define the $\approx_\alpha$ in terms of *transitive non-symmetric relations*, $\leq_\beta$, $\beta < \alpha$.

This presents a departure from the analysis for closed subgroups of S_∞, where the existence of a neighborhood basis of clopen subgroups around the identity allows us to define the equivalence relation "the models $\mathcal{M}$ and $\mathcal{N}$ have the same αth approximation to their Scott sentence" in terms of equivalence relations obtained by looking at the subgroups of S_∞ consisting of all permutations fixing some finite subset of $\mathbb{N}$. In some sense, there is a canonical way to approximate S_∞ by a countable collection of clopen subgroups. This method of approximation lifts to the orbit equivalence relation, and allows us to define "the models $\langle \mathcal{M}, \vec{a}\rangle$ and $\langle \mathcal{N}, \vec{b}\rangle$ have the same αth approximation to their Scott sentence" in terms of the set of βth approximations for $\{\langle \mathcal{M}, \vec{a}c\rangle : c \in \mathbb{N}\}$ and $\{\langle \mathcal{N}, \vec{b}d\rangle : d \in \mathbb{N}\}$ as β ranges over α.

Thus, while the original analysis was in terms of refining and coarsening of the orbit equivalence relation into a structured hierarchy of equivalence relations of the form $\varphi^x_{\alpha,\vec{a}} = \varphi^y_{\alpha,\vec{b}}$, and collecting together the various approximations using the set theoretical notion of *set of*, the analysis here is in terms of refining and coarsening transitive, *non-symmetric* relations.

The chief results obtained in this new analysis are as follows:

We first give a new proof of the generalization of 0.6 in [6], presenting a *ground up, ordinal analysis* that for any Polish group G and Polish G-space X, every $[x]_G$ is Borel.

0.11 Theorem Let G be a Polish group and X a Polish G-space. If every orbit is $\mathbf{\Pi}^0_\alpha$ then the orbit equivalence relation is $\mathbf{\Pi}^0_{\alpha+\omega}$.

One might hope that the kinds of change in topologies arguments at §3 below may make lead to a generalization of 0.8, but that remains open, due to a certain lack of clarity in the generalization of the notion of *atomic model.* On the other hand, 0.9 directly extends:

0.12 Theorem (folklore) If G is a Polish group and X is a Polish S_∞ space, then there is an effective map

$$i : X/S_\infty \to \mathbf{\Pi}^0_{\alpha+1}$$

such that orbits $[x]_G$ and $[y]_G$ with the same image under i will meet exactly the same invariant $\mathbf{\Pi}^0_\alpha$ sets.

The situation of 0.10 for general Polish groups remains totally open. The issue here is that the analysis below does not yield 0.3(iv), which played a critical part in Friedman's argument.

In fact, §2 shows that 0.3(iv) fails in for some Polish groups, under any reasonable generalization of the Scott analysis. On the other hand, §3 presents some evidence that a suitable form of this may hold for universal Polish groups – that is to say, those Polish groups which induce a continuous action on a Polish space which is $\leq_B$-universal among all such orbit equivalence relations.

§1. Some proofs

From now on let G be a Polish group and X a Polish G-space.

1.1 Definition For $V_0, V_1 \subset G$ open and non-empty, $x_0, x_1 \in X$, write

$$(x_0, V_0) \leq_1 (x_1, V_1)$$

if

$$\overline{V_0 \cdot x_0} \subset \overline{V_1 \cdot x_1}.$$

Write

$$(x_0, V_0) \leq_{\alpha+1} (x_1, V_1)$$

if for all $W_0 \subset V_0$ open non-empty there is $W_1 \subset V_1$ non-empty and open with

$$(x_1, W_1) \leq_\alpha (x_0, W_0).$$

For λ a limit

$$(x_0, V_0) \leq_\lambda (x_1, V_1)$$

expresses that at every $\alpha < \lambda$

$$(x_0, V_0) \leq_\alpha (x_1, V_1).$$

1.2 Lemma For $W_0 \subset V_0$, $W_1 \supset V_1$, all open and non-empty,

$$(x_0, V_0) \leq_\alpha (x_1, V_1)$$

implies

$$(x_0, W_0) \leq_\alpha (x_1, W_1).$$

Proof. Immediate from the definitions. □

From now on let $\mathcal{B}$ be a countable basis for G and let $\mathcal{B}_0$ be the non-empty elements of $\mathcal{B}$. Assume $G \in \mathcal{B}$.

1.3 Lemma

$$(x_0, V_0) \leq_{\alpha+1} (x_1, V_1)$$

if for all $W_0 \subset V_0$ in $\mathcal{B}_0$ there is $W_1 \subset V_0$ in $\mathcal{B}_0$ with

$$(x_1, W_1) \leq_\alpha (x_0, W_0).$$

Proof. From 1.2. □

1.4 Corollary Let $(\hat{V}_n)_{n\in\mathbb{N}}$ enumerate $\mathcal{B}_0$. Then for all $\alpha < \omega_1$

$$\{(x_0, x_1, n, m) : (x_1, \hat{W}_n) \leq_\alpha (x_0, \hat{W}_m)\}$$

is a $\utilde{\Pi}^0_{\alpha+k}$ set for some $k \in \mathbb{N}$.

1.5 Lemma (a) Each $\leq_\alpha$ is transitive.

(b) $\alpha < \beta$ implies $\leq_\alpha \supset \leq_\beta$ (in the sense that $(x_1, W_1) \leq_\beta (x_0, W_0)$ implies $(x_1, W_1) \leq_\alpha (x_0, W_0)$.

Proof. (a) is almost immediate from the definitions. (b) routinely by transfinite induction once we observe the slightly tricky fact that $\leq_1 \supset \leq_2$. □

1.6 Definition Set $x_0 \approx_\gamma x_1$ if for all $i \in \{0, 1\}$, $\beta < \gamma$, $W_i \in \mathcal{B}_0$, there exists $\hat{W}_i \subset W_i$ with $\hat{W}_i \in \mathcal{B}_0$ and $W_{1-i} \in \mathcal{B}_0$ with

$$(x_i, \hat{W}_i) \leq_\beta (x_{1-i}, W_{1-i}) \leq_\beta (x_i, W_i).$$

Note that $x_0 E^X_G x_1$ implies $x_0 \approx_\gamma x_1$.

1.7 Lemma For $V_0, V_1 \in \mathcal{B}_0$ and $\alpha < \omega_1$, $\alpha \neq 0$, there is an "potentially $\utilde{\Pi}^0_\alpha$ relation" (in the sense of [4]) R such that if

$$x_0 \approx_\alpha x_1$$

then $(x_0, V_0) \leq_\alpha (x_1, V_1)$ if and only if $R(x_0, x_1)$.

Proof. (I say that R *potentially* has a certain complexity if there is a Polish topology on X compatible with the Borel structure in which this complexity is achieved.)

For the base case of $\alpha = 1$, we assign to x_0, x_1 the closures $\overline{V_0 \cdot x_0}, \overline{V_1 \cdot x_1}$ in $\mathcal{F}(X)$ (the standard Effros Borel space of closed sets), and note that it is potentially closed to assert in $\mathcal{F}(X)$ that one set includes another. At limit α the lemma resolves into triviality in light of 1.4.

So let us assume the lemma is true at α and attempt to step up to $\alpha+1$. Then $(x_0, V_0) \leq_{\alpha+1} (x_1, V_1)$ if and only if for all $\hat{W}_0 \subset W_0 \subset V_0$ with $\hat{W}_0, W_0 \in \mathcal{B}_0$, for all $W_1 \in \mathcal{B}_0$

$$[(x_0, \hat{W}_0) \leq_\alpha (x_1, W_1) \leq_\alpha (x_0, W_0)] \Rightarrow \exists \hat{W}_1 \subset V_1 [\hat{W}_1 \in \mathcal{B}_0 \wedge (x_1, \hat{W}_1) \leq_\alpha (x_1, W_1)].$$

Since the relation $\exists \hat{W}_1 \subset V_1 [\hat{W}_1 \in \mathcal{B}_0 \wedge (x_1, \hat{W}_1) \leq_\alpha (x_1, W_1)]$ is unary and Borel, it is potentially closed, and the lemma for α implies it for $\alpha + 1$. □

1.8 Lemma For $\alpha < \omega_1$, $\alpha \neq 0$, the relation

$$x_0 \approx_\alpha x_1$$

is potentially $\utilde{\Pi}^0_{\alpha+1}$.

Proof. We prove this by transfinite induction on α, with the base and limit stages again trivial. So assume true at β and try to wrench up to $\alpha = \beta + 1$. The relation $x_0 \approx_\beta x_1$ is potentially $\utilde{\Pi}^0_{\beta+1} = \utilde{\Pi}^0_\alpha$ by inductive assumption. Then we can apply 1.7 to just read off that the definition of $x_0 \approx_\alpha x_1$ is potentially $\utilde{\Pi}^0_{\beta+2}$, as required. □

1.9 Lemma Let $B \subset X$ be $\utilde{\Pi}^0_\alpha$ ($\alpha \in [1, \omega_1)$). Suppose $(x_0, V_0) \leq_\alpha (x_1, V_1)$ and $x_1 \in B^{*V_1}$.

Then $x_0 \in B^{*V_0}$.

Proof. This is trivial at the base case of $\alpha = 1$, since it amounts to calculating $\overline{V_0 \cdot x_0}$ and $\overline{V_1 \cdot x_1}$. Now suppose it is true at all $\beta < \alpha$ and that

$$B = \bigcap_{i\in\mathbb{N}} B_i$$

with each $B_i \in \utilde{\Sigma}^0_{\beta(i)}$ some $\beta(i) < \alpha$. Then assuming

$$(x_0, V_0) \leq_\alpha (x_1, V_1)$$

and $i \in \mathbb{N}$ and $W_0 \subset V_0$ with $W_0 \in \mathcal{B}_0$. We just need to show that $x_0 \in (B_i)^{\Delta W_0}$.

So choose $W_1 \subset V_1$ with $W_1 \in \mathcal{B}_0$ and

$$(x_0, W_0) \geq_{\beta(i)} (x_1, W_1).$$

Now if x_0 is *not* in $(B_i)^{\Delta W_0}$, then for $C = X \setminus B_i$ we have

$$x_0 \in C^{*W_0}$$

and so by assumption on $\beta(i) < \alpha$ we have

$$x_1 \in C^{*W_1},$$

with a contradiction to $x_1 \in (B_i)^{\Delta W_1}$. □

Thus sharpening Sami's theorem:

1.10 Corollary If every orbit is $\underset{\sim}{\Pi}^0_\alpha$ then E^X_G is $\underset{\sim}{\Pi}^0_{\alpha+\omega}$.

Proof. Since $x_0 E^X_G$ implies $x_0 \approx_{\alpha+\omega} x_1$ implies $x_0 \in ([x_1]_G)^{*G}$ by 1.9 and the assumption on the Borel complexity of $[x_1]_G$, and hence $x_0 E^X_G x_1$. Thus $x_0 E^X_G$ if and only if $x_0 \approx_{\alpha+\omega} x_1$, and so the corollary follows 1.4. □

The rest of this note may seem like a heck of a lot of puff and furry to reprove a well known result – that every orbit is Borel. I believe that the techniques here may have further applications for the "fine analysis of the Borel complexity of orbits", and moreover I simply find it philosophically interesting to give a proof resembling the argument for the logic action that appeals to the Scott analysis, in as much as we may see the Borel complexity of the orbit $[x]_G$ as being a direct result (1.18 below) of a countable length analysis of the stabilizer of the point x. It seems that these calculations would also play an essential role in any attempt to complete the generalizations of 0.8 and 0.10.

From now on assume that d is a bounded right invariant metric on G. (I do *not* ask that it also be complete – so that such a metric always exists). Let G_0 be a countable dense subgroup of G and let $\mathcal{B}_0$ consist of all sets of the form $B_q(g) =_{df} \{d(h, g) < q\}$ for $g \in G_0$ and q a positive rational.

1.11 Definition For $x \in X$ let $\delta(x)$ be least such that for all $g_0, g_1 \in G_0$, $n \in \mathbb{N}$, $q_0, q_1 \in \mathbb{Q}$ with $q_0, q_1 > 1/n$, if

$$(x, B_{q_0}(g_0)) \leq_\alpha (x, B_{q_1}(g_1))$$

then

$$(x, B_{q_0-1/n}(g_0)) \leq_{\alpha+1} (x, B_{q_1+1/n}(g_1)).$$

By 1.5(b) and countability of $\mathcal{B}_0$, such a $\delta(x) < \omega_1$ must exist. Note that $x_0 E^X_G x_1$ implies $\delta(x_0) = \delta(x_1)$; the fact that $\mathcal{B}_0$ is not necessarily translation invariant requires us to give a little ground in 1.11 so as to make $x \mapsto \delta(x)$ G-invariant.

1.12 Lemma Suppose $\delta(x) = \delta$ and $(x, V_0) \leq_{\delta+1} (x, V_1)$. Then $(x, V_0) \leq_{\delta+2} (x, V_1)$.

Proof. Fix $W_0 \subset V_0$ with $W_0 = B_q(g)$ some $q \in \mathbb{Q}^+$, $g \in G_0$. We need $W_1 \subset V_1$ with $(x, W_1) \leq_{\delta+1} (x, W_0)$.

Choose $\bar{q} < q$ with $\bar{q} \in \mathbb{Q}^+$ and let $\hat{W}_0 = B_{\bar{q}}(g)$. Then by definition of $(x, V_0) \leq_{\delta+1} (x, V_1)$ we can find $\hat{W}_1 \in \mathcal{B}_0$ with

$$(x, \hat{W}_0) \geq_\delta (x, \hat{W}_1).$$

By assumption on δ we can find $W_1 \subset \hat{W}_1$ with

$$(x, W_0) \geq_{\delta+1} (x, W_1).$$

□

1.13 Corollary If $\delta(x) \leq \delta$ and $(x, V_0) \leq_{\delta+1} (x, V_1)$, then $(x, V_0) \leq_{\delta+2} (x, V_1)$.

Proof. Induction on δ, starting with the base case $\delta(x) = \delta$ discussed at 1.12. □

1.14 Lemma Suppose $\delta(x_1) \leq \delta$, $x_0 \approx_{\delta+1} x_1$, $V_0, V_1 \in \mathcal{B}$ with

$$(x_0, V_0) \leq_{\delta+1} (x_1, V_1).$$

Then $(x_0, V_0) \leq_{\delta+2} (x_1, V_1)$.

Proof. Fix $W_0 \subset V_0$ in $\mathcal{B}_0$. By $x_0 \approx_{\delta+1} x_1$ we may find $\hat{W}_1 \in \mathcal{B}_0$ with $(x_1, \hat{W}_1) \leq_{\delta+1} (x_0, W_0)$. Transitivity of $\leq_{\delta+1}$ yields $(x_1, \hat{W}_1) \leq_{\delta+1} (x_1, V_1)$. Assumption on δ and 1.13 implies $(x_1, \hat{W}_1) \leq_{\delta+2} (x_1, V_1)$.

Thus we may find $W_1 \in \mathcal{B}_0$ with $W_1 \subset V_1$ and $(x_1, \hat{W}_1) \geq_{\delta+1} (x_1, W_1)$. Then transitivity of $\leq_{\delta+1}$ gives $(x_0, W_0) \geq_{\delta+1} (x_1, W_1)$ as required. □

1.15 Corollary Suppose that $\delta(x_0) = \delta(x_1) \leq \delta$ and that

$$x_0 \approx_{\delta+1} x_1.$$

Then for all ordinals α

$$x_0 \approx_\alpha x_1.$$

Proof. A routine induction on α shows that for $i \in \{0, 1\}$ and $V_i, V_{1-i} \in \mathcal{B}_0$

$$(x_i, V_i) \leq_{\delta+1} (x_{1-i}, V_{1-i})$$

implies

$$(x_i, V_i) \leq_\alpha (x_{1-i}, V_{1-i}).$$

□

1.16 Lemma Suppose $\delta(x_1) \leq \delta$ and $x_0 \approx_{\delta+2} x_1$. Then $\delta(x_0) \leq \delta + 2$.

Proof. Suppose $(x_0, W_0) \leq_{\delta+2} (x_0, V_0)$. It suffices to produce $\hat{W}_0 \subset V_0$ in $\mathcal{B}_0$ with

$$(x_0, W_0) \geq_{\delta+2} (x_0, \hat{W}_0).$$

Since $x_0 \approx_{\delta+2} x_1$ find W_1 with

$$(x_1, W_1) \leq_{\delta+2} (x_0, W_0).$$

By transitivity of $\leq_{\delta+2}$

$$(x_1, W_1) \leq_{\delta+2} (x_0, V_0).$$

So there exists $\hat{W}_0 \subset V_0$ with

$$(x_0, \hat{W}_0) \leq_{\delta+2} (x_1, W_1).$$

Then by lemma 1.14,

$$(x_0, \hat{W}_0) \leq_{\delta+2} (x_1, W_1) \leq_{\delta+2} (x_0, W_0),$$

and we finish by the transitivity of $\leq_{\delta+2}$. □

1.17 Corollary If $\delta(x_1) \leq \delta$ and $x_0 \approx_{\delta+3} x_1$, then for all α

$$x_0 \approx_\alpha x_1.$$

Proof. Let $\bar{\delta} = \delta + 2$. By lemma 1.16, $\delta(x_0) \leq \bar{\delta}$. Since $x_0 \approx_{\bar{\delta}} x_1$ we have the conclusion by lemma 1.14. □

1.18 Proposition Suppose $\delta(x_0), \delta(x_1) \leq \delta$ and $V_0, V_1 \in \mathcal{B}_0$ with

$$(x_0, V_0) \leq_{\delta+1} (x_1, V_1).$$

Then there exist $g \in \overline{V_0}$ and $h \in \overline{V_1}$ with

$$g \cdot x_0 = h \cdot x_1.$$

Proof. In what follows note that if $Wgh \subset V$ then for all $x \in X$ and $\alpha \in \omega_1$

$$(Wg, h \cdot x) \leq_\alpha (V, x).$$

We already fixed a compatible right invariant metric d for G, and let us know add to this a compatiable *complete* metric $\hat{d}$. We will now build group elements $g_{0,0}, g_{0,1}, g_{0,2}, ..., h_{0,0}, h_{0,1}, h_{0,2}, ...g_{1,0}, g_{1,1}, g_{1,2}, ..., h_{1,0}, h_{1,1}, h_{1,2}, ...$ and open neighbourhoods of the identity $V_{0,0}, V_{0,1}, V_{0,2}, ...V_{1,0}, V_{1,1}, V_{1,2}, ...$ such that

(I) $h_{i,j} = g_{i,j}g_{i,j-1}...g_{i,o}$ for $i = 0, 1$, $j \in \mathbb{N}$; and so $h_{i,j+1} = g_{i,j+1}h_{i,j}$;
(II) for all $g, g' \in V_{i,j}$ $(i = 0, 1, j \in \mathbb{N})$

$$\hat{d}(gh_{i,j}, h_{i,j}) < 2^{-j}),$$

$$\hat{d}(g, g') < 2^{-j};$$

(III) each $g_{i,j+1} \in V_{i,j}$ $(i = 0, 1, j \in \mathbb{N})$;
(IV) each $V_{i,j}h_{i,j} \subset V_i$ $(i = 0, 1, j \in \mathbb{N})$;
(V) each $(h_{0,j} \cdot x_0, V_{0,j}) \leq_{\delta+1} (h_{1,j} \cdot x_1, V_{1,j}$.

If we suceed with all this, then we can find elements

$$g = \lim_{j\to\infty} h_{0,j},$$

$$h = \lim_{j\to\infty} h_{1,j}.$$

From (V) we may obtain group elements $\delta_i, \epsilon_i \to 1_G$ with

$$(\delta_i g \cdot x_0, V_{0,j}) \leq_1 (\epsilon_i h \cdot x_1, V_{1,j}),$$

$$\therefore \overline{V_{0,j}\delta_i g x_0} \subset \overline{V_{1,j}\epsilon_i h \cdot x_1}$$

and thus (by (II) $(V_{0,j}\delta_i)_{i\in\mathbb{N}}$ and $(V_{1,j}\epsilon_i)_{i\in\mathbb{N}}$ are neighbourhood bases for the identity) we have $g \cdot x_0 = h \cdot x_1$.

So suppose we have produced the above elements and open sets for $j \leq N$. Then

$$(h_{0,N} \cdot x_0, V_{0,N}) \leq_{\delta+1} (h_{1,N} \cdot x_1, V_{1,N}).$$

Then we may find $W \subset V_{1,N}$ with

$$(h_{0,N} \cdot x_0, V_{0,N}) \geq_{\delta+1} (h_{1,N} \cdot x_1, W)$$

by lemma 1.14. Choose $g_{1,N+1} \in W$ and let $h_{1,N+1} = h_{1,N+1}g_{1,N+1}$. We can then let $V_{1,N+1}$ be a small enough neighbourhood of the identity to ensure (II) and (IV) with $V_{1,N+1}g_{1,N+1} \subset W$. Repeating exactly the same step on the other side we pass from

$$(h_{0,N} \cdot x_0, V_{0,N}) \geq_{\delta+1} (h_{1,N+1} \cdot x_1, V_{1,N+1})$$

to some suitable

$$(h_{0,N+1} \cdot x_0, V_{0,N+1}) \leq_{\delta+1} (h_{1,N+1} \cdot x_1, V_{1,N+1}).$$

and continue. □

In particular:

1.19 Corollary $y \approx_{\delta(x)+1} x \Rightarrow y E_G^X x$ for all $x, y \in X$.

And thus a different proof of the well known fact:

1.20 Corollary For all $x \in X$, $[x]_G$ is Borel.

Here seems as good as place as any to collect together some calculations regarding the $\leq_\alpha$ relations.

1.21 Lemma (i)

$$(Wg, g^{-1} \cdot x) \leq_\alpha (W, x)$$

all $W \in \mathcal{B}_0$, $x \in X$, $g \in G$, $\alpha \in \omega_1$.

(ii) If $Wgh \subset V$, W, V open, non-empty, then for all $x \in X$, $\alpha \in [1, \omega_1)$

$$(Wg, h \cdot x) \leq_\alpha (V, x).$$

(iii)

$$(V, y) \leq_\alpha (W, x)$$

if and only if

$$(gV, y) \leq_\alpha (gW, x).$$

Proof. (i) is trivial and (ii) follows (i). (iii) is proved by transfinite induction on α (starting with the base case $\alpha = 1$, where one notes that $\overline{gV \cdot y} = g \cdot \overline{V \cdot y}$ and $\overline{gW \cdot x} = g \cdot \overline{W \cdot x}$). □.

§2. A counterexample

The essential technical fact that is used in Friedman's theorem, in his proof of a positive solution to for $G = S_\infty$, is that the α^{th}-approximations of the Scott analysis give rise to an equivalence relation which is induced by a continuous action of S_∞ on a Polish space.

This essential fact is false for general Polish groups, and we demonstrate this usual model theoretic ideas based on early work by Julia Knight. On the other hand, there is some reason for thinking the appropriate analogue might be true for any sufficiently universal Polish group: There is a way of approximating the generalized Scott analysis above, as shown in §3, so that the equivalence relation induced at the α^{th} level is *never* $\leq_B$ above E_1. Since the only method we have for showing that a general Borel equivalence relation is not Borel reducible to a Polish group action is by reducing E_1.

2.1 Theorem (Knight) There is a countable model $\mathcal{M}$ with a linear ordering $<$ and unary functions $(f_n)_{n\in\omega}$ such that:

(a) $\forall a \in \mathcal{M}(\{f_n(a) : n \in \omega\} = \{b \in \mathcal{M} : b < a\})$;

(b) there is an $\mathcal{L}_{\omega_1,\omega}$-elementary embedding

$$\rho : \mathcal{M} \to \mathcal{M}$$

which is not onto.

2.2 Notation From now on let $\mathcal{M}$ denote a model described by 2.1.

2.3 Lemma There is a $\psi \in \mathcal{L}_{\omega_1,\omega}$ such that for all $a \in \mathcal{M}$

$$\mathcal{M} \models \psi(a)$$

if and only if there is an $\mathcal{L}_{\omega_1,\omega}$-elementary $\rho : \mathcal{M} \to \mathcal{M}$ with $a \notin \text{Ran}(\rho)$.

Proof. Since the property of having some such $\rho : \mathcal{M} \to \mathcal{M}$ with $a \notin \text{Ran}(\rho)$ is invariant under isomorphism and $\mathcal{M}$ is countable, it must be describable by an $\mathcal{L}_{\omega_1,\omega}$ formula. □

2.4 Notation For ψ as in the last lemma, let $\mathcal{M}^+$ denote the expansion of $\mathcal{M}$ by adding a unary predicate Q holding exactly of $\{a : \mathcal{M} \models \psi(a)\}$.

2.5 Lemma $\mathcal{M}^+$ has a model of size $\aleph_1$ of its Scott sentence.

Proof. Knight explicitly observes this for $\mathcal{M}$, and now it follows for $\mathcal{M}^+$ because the expansion is $\mathcal{L}_{\omega_1,\omega}$ definable. □

2.6 Lemma Let $a < b$ be in $\mathcal{M}^+$ with $\mathcal{M}^+ \models Q(a)$. Then there is an automorphism

$$j : \mathcal{M}^+ \to \mathcal{M}^+$$

with $j(a) > b$.

Proof. Let $\varphi^a_{\mathcal{M}^+}$ be the Scott sentence of a in $\mathcal{M}$ – i.e. the canonical $\varphi \in \mathcal{L}_{\omega_1,\omega}$ that describes the orbit of a under the automorphism group of $\mathcal{M}$. It suffices to see that the set of $c \in \mathcal{M}$ with $\mathcal{M} \models \varphi^a_{\mathcal{M}^+}(c)$ is unbounded in the ordering $<$.

For a contradiction, suppose there is some d such that

$$\mathcal{M} \models \forall c > d(\neg\varphi^a_{\mathcal{M}^+}).$$

Then let $\rho : \mathcal{M} \to \mathcal{M}$ be $\mathcal{L}_{\omega_1,\omega}$-elementary with a outside its range. Let $\bar{\mathcal{M}} = \rho[\mathcal{M}^+]$. Note that

$$\bar{\mathcal{M}} \prec_{\mathcal{L}_{\omega_1,\omega}} \mathcal{M}^+$$

and $\bar{\mathcal{M}}$ forms an initial segment of $\mathcal{M}$ with respect to $<$. Then by elementarity or ρ,

$$\bar{\mathcal{M}} \models \forall c > \rho(d)(\neg\varphi^1_{\mathcal{M}^+}(c)),$$

which in turn contradicts $\bar{\mathcal{M}} \prec_{\mathcal{L}_{\omega_1,\omega}} \mathcal{M}^+$, $\mathcal{M}^+ \models \varphi^a_{\mathcal{M}^+}(a) \wedge a > \rho(d)$. □

2.7 Definition Let $\mathcal{N}_0$ be a model with a two sorts,

$$P_0, P_1,$$

a binary relation,

$$E,$$

along with the language of $\mathcal{M}^+$, satisfying:

1. $\forall a \in P_1 \exists! b \in P_0(aEb)$;
2. $(P_0^{\mathcal{N}_0}; <, (f_n)_{n\in\omega}, Q) \cong \mathcal{M}^+$;
3. at each $b \in P_0^{\mathcal{N}_0}$
$$(\{a \in P_1^{\mathcal{N}_0} : aEb\}; < (f_n)_{n\in\omega}, Q) \cong \mathcal{M}^+;$$
4. in all other cases, not described by these conditions above, neither of the relations E or $<$ hold.

Let $\varphi_{\mathcal{N}_0}$ be the Scott sentence of $\mathcal{N}_0$.

2.8 Lemma There exists $\mathcal{N}_1 \models \varphi_{\mathcal{N}_0}$ such that:

1. $|(P_0)^{\mathcal{N}_1}| = \aleph_1$;
2. for each $b \in (p_0)^{\mathcal{N}_1}$
$$|\{a \in \mathcal{N}_1 : \mathcal{N}_1 \models aEb\}| = \aleph_1.$$
3. for all $b_1, b_2 \in (P_0)^{\mathcal{N}_1}$
$$\{a \in \mathcal{N}_1 : \mathcal{N}_1 \models aEb_1\} \cong \{a \in \mathcal{N}_1 : \mathcal{N}_1 \models aEb_2\}.$$

Proof. This follows from $\mathcal{M}^+$ having a size of $\aleph_1$ and the construction of $\mathcal{N}_0$. The essential point is that the various "fibers" of the form $\{a \in \mathcal{N}_1 : \mathcal{N}_1 \models aEb\}$ have no relations holding between them. □

2.9 Notation Fix $\mathcal{N}_1$ as in the last lemma, and fix $\hat{\mathcal{M}}$ of size $\aleph_1$ satisfying the Scott sentence of $\mathcal{M}^+$ and isomorphic to $(P_0)^{\mathcal{N}_1}$. We may in fact assume that the underlying set of $\mathcal{N}_1$ has the form

$$\hat{\mathcal{M}} \dot{\bigcup} \hat{\mathcal{M}} \times \hat{\mathcal{M}},$$

with

1. $\hat{\mathcal{M}} = (P_0)^{\mathcal{N}_1}$;
2. $(a', b')Eb$ if and only if $b' = b$;
3. $a \mapsto (a, b)$ provides an isomorphism of $\hat{\mathcal{M}}$ and $\{x \in \mathcal{N}_1 : \mathcal{N}_1 \models xEb\}$ for any $b \in (P_0)^{\mathcal{N}_1}$.

2.10 Definition We let $\mathbb{P}_0$ denote the partial order of all finite partial functions from $\hat{\mathcal{M}} \times \aleph_2 \times \omega$ to $\{0, 1\}$. In other words, $\mathbb{P}_0$ is the forcing to add $\hat{\mathcal{M}} \times \aleph_2$ many Cohen reals.

We then let $\mathbb{P}_1$ be the partial order of finite partial functions from $(Q)^{\hat{\mathcal{M}}}$ to $\aleph_2$.

Given $H_0 \times H_1 \subset \mathbb{P}_0 \times \mathbb{P}_1$ V-generic, $a \in \hat{\mathcal{M}}$, $\alpha \in \aleph_2$, we let

$$c^{H_0}_{(a,\alpha)} : \omega \to \{0, 1\}$$

be given by

$$c_{(a,\alpha)}^{H_0}(n)=i$$

if and only if

$$(a,\alpha,n)\mapsto i$$

is an element of H_0. We let

$$g^{H_1}:(Q)^{\hat{\mathcal{M}}}\to\aleph_2$$

be the union of the finite partial functions in H_1.

From these functions, we will define an expansion of $\mathcal{N}_1$.

We fix unary predicates $(U_n)_{n\in\omega}$. We then let $\mathcal{N}_{(H_0,H_1)}$ be the expansion of $\mathcal{N}_1$ by the unary predicates $(U_n)_{n\in\omega}$ given by the specification that for all $(a,b)\in\hat{\mathcal{M}}\times\hat{\mathcal{M}}$

$$\mathcal{N}_{(H_0,H_1)}\models U_n((a,b))$$

if an only if

$$c_{(a,g^{H_1}(b))}^{H_0}(n)=1.$$

In all cases not covered by the above specifications, we simply let the U_n's be void.

In other words, we first use H_0 to give us $\aleph_2$ many different ways of expanding $\hat{\mathcal{M}}$ by the addition of countably many fresh unary predicates. We then use a generic function from $\hat{\mathcal{M}}$ to $\aleph_2$ to choose for each fiber of the form $\{a:\mathcal{N}_1\models aEb\}$, $b\in(Q\cap P_0)^{\mathcal{N}_1}$, one of those possible expansions.

2.11 Lemma Let $H_0\subset\mathbb{P}_0$ be V-generic, and let $H_1,H_1'\subset\mathbb{P}_1$ be $V[H_0]$-generic. Then

$$\varphi_1^{\mathcal{N}_{(H_0,H_1)}}=\varphi_1^{\mathcal{N}_{(H_0,H_1')}}.$$

Proof. Note that the construction of $\mathcal{N}_{(H_0,H_1)}$ has the property that for any $(a_0,b_0),(a_1,b_1),...,(a_n,b_n)\in\hat{\mathcal{M}}\times\hat{\mathcal{M}}$

$$\varphi_{0,((a_0,b_0),....,(a_n,b_n))}^{\mathcal{N}_{(H_0,H_1)}}$$

depends solely on H_0 and $g^{H_1}(b_0),...,g^{H_1}(b_n)$, while

$$\varphi_{0,(b_0,...,b_n)}^{\mathcal{N}_{(H_0,H_1)}}$$

depends solely on $b_0,...,b_n$.

Now let $H\subset\mathrm{Coll}(\omega,\aleph_2)$ be $V[H_0]$-generic. It suffices to show that any finite $F,F'\subset(Q)^{\hat{\mathcal{M}}}$ and

$$p:F\to\aleph_2,$$

$$p':F'\to\aleph_2,$$

F,F' finite subsets of $(Q)^{\hat{\mathcal{M}}}$, there is an automorphism

$$j:\hat{\mathcal{M}}\cong\hat{\mathcal{M}}$$

such that $j[F]\cap F'=\emptyset$.

This in turn follows from 2.6. □

Definition 2.12 Let G be the automorphism group of $\mathcal{N}_0$. Let X be the Polish space of expansion of $\mathcal{N}_0$ by the unary predicates $(U_n)_{n\in\omega}$. Let G act on X in the obvious way: For $\mathcal{B}\in X$ and $\sigma\in G$

$$\sigma\cdot\mathcal{B}\models U_n(a)$$

if and only if

$$\mathcal{B}\models U_n(\sigma^{-1}(a)).$$

Theorem 2.13 There is no Polish G-space Y and Borel

$$\theta : X \to Y$$

such that

$$\varphi_1^{\mathcal{B}_1} = \varphi_2^{\mathcal{B}_2}$$

if and only if

$$\theta(\mathcal{B}_1) E_G \theta(\mathcal{B}_2).$$

Proof. Suppose otherwise. Then following [1] we may assume Y is the space of expansions of $\mathcal{N}_0$ to some language $\mathcal{L}^+$ in the canonical logic action of $G = \mathrm{Aut}(\mathcal{N}_0)$ on the expansions of $\mathcal{N}_0$. Note that by Shoenfield absoluteness we continue to have through all generic extensions

$$V[H]$$

of V that $\forall \mathcal{B}_1, \mathcal{B}_2 \in X$

$$\varphi_1^{\mathcal{B}_1} = \varphi_2^{\mathcal{B}_2} \Leftrightarrow \theta(\mathcal{B}_1) E_G \theta(\mathcal{B}_2).$$

Now take $H_0 \times H_1 \subset \mathbb{P}_0 \times \mathbb{P}_1$ as before to be V-generic. Consider $\varphi^{\theta(\mathcal{N}_{(H_0,H_1)})}$, the Scott sentence of $\mathcal{N}_{(H_0,H_1)}$. Since $\varphi_1^{\mathcal{N}_{(H_0,H_1)}}$ already exists in $V[H_0]$, by 2.11, we have that $\varphi^{\theta(\mathcal{N}_{(H_0,H_1)})}$ already exists in $V[H_0]$.

Note however that the structure of $\mathcal{N}_0$ ensures that $\varphi^{\theta(\mathcal{N}_{(H_0,H_1)})}$ can have cardinality at most $\aleph_1$ in $V[H_0]$. Since $\mathbb{P}_0$ has ccc, and is in fact equivalent as a forcing notion to the usual iteration of Cohen forcing by finite conditions, we have that $\aleph_2^{V[H_0]} = \aleph_2^V$, and hence there is some $\alpha < \aleph_2$ such that

$$\varphi^{\theta(\mathcal{N}_{(H_0,H_1)})} \in V[H_0|_{\hat{m}\times\alpha\times\omega}].$$

But the nature of the reduction θ, and its persistence as a reduction through all forcing extensions, $\varphi_1^{\mathcal{N}_{(H_0,H_1)}}$ is uniquely determined by $\varphi^{\theta(\mathcal{N}_{(H_0,H_1)})}$, and hence

$$\varphi_1^{\mathcal{N}_{(H_0,H_1)}} \in V[\varphi^{\theta(\mathcal{N}_{(H_0,H_1)})}] \subset V[H_0|_{\hat{\mathcal{M}}\times\alpha\times\omega}],$$

which contradicts $c_{(a,\alpha)}^{H_0}$ being generic over $V[H_0|_{\hat{\mathcal{M}}\times\alpha\times\omega}]$ for any $a \in \hat{\mathcal{M}}$. □

§3. The generalized Scott analysis as a change in topologies

The counterexample of the last section would seem to rule out any naive attempt to generalize the method of Friedman's argument. Although it was presented there as a counterexample simply for the usual model theoretic Scott analysis, a version of this counterexample will persist through all minor modifications. For the group $\mathrm{Aut}(\mathcal{N}_0)$, whatever variant of the φ_1 invariants we care to contemplate, it should be able to detect the invariant $\underset{\sim}{\Sigma}_2^0$ sets met by an orbit, or not even merit being considered as a rival to the traditional analysis. In particular, for each $c : \omega \to \{0,1\}$, $\varphi_1^{\mathcal{B}}$ should determine whether

$$\exists a \in \mathcal{B} \forall n \in \omega (\mathcal{B} \models U_n(a) \Leftrightarrow c(n) = 1).$$

The basic structure of the counterexample was to introduce $\aleph_2$ many disjoint and mutually generic $\underset{\sim}{\Sigma}_2^0$ sets of this form, without collapsing $\aleph_2$, and thereby overwhelm the ability of any equivalence relation induced by $\mathrm{Aut}(\mathcal{N}_0)$ to carry this information.

On the other hand, the counterexample is completely specific to the Polish group $\mathrm{Aut}(\mathcal{N}_0)$. One might hope that for universal Polish groups, there will always be a version of the φ_α invariants which are reducible to some Polish group action, and in particular to any universal Polish group action.

This section provides some limited evidence, perhaps, for this more optimistic scenario. We show that a modest variant of the $\approx_\alpha$ equivalence relations does not reduce E_1.

The modest variant at level α is a finer equivalence relation than $\approx_\alpha$, but coarser than $\approx_{\alpha+\omega}$.

3.1 Notation Fix G a Polish group, X a Polish space, and d a compatible right invariant. For $a > 0$ let

$$V_a = \{g \in G : d(1,g) < a\}.$$

Let G_0 be a countable dense subgroup of G. Fix a Polish G-space, X.

3.2 Definition For $x \in X$, $a < b$ two positive rationals, let

$$U^{x,\alpha,a,b} = \{z \in X : \exists\delta > 0(a + \delta < b - \delta \wedge (z, V_{a+\delta}) \leq_\alpha (x, V_{b-\delta})\},$$

and

$$W^{x,\alpha,a,b} = \{z \in X : \exists\delta > 0(a + \delta < b - \delta \wedge (x, V_{a+\delta}) \leq_\alpha (z, V_{b-\delta})\}.$$

We then let $\tau_{x,\alpha}$ be the topology generated by the sets of the form

$$h \cdot U^{g\cdot x,\alpha,a,b}, h \cdot W^{g\cdot x,\alpha,a,b},$$

where g, h range over G_0 and $a < b$ range over positive rationals.

3.3 Lemma If $y \in U^{x,\alpha,a,b}$ then there exists an open neighborhood U of 1 such that

$$\forall g \in U(g \cdot y \in U^{x,\alpha,a,b}).$$

Similarly, if $y \in W^{x,\alpha,a,b}$ then there exists an open neighborhood U of 1 such that

$$\forall g \in U(g \cdot y \in W^{x,\alpha,a,b}).$$

Proof. Assume $y \in U^{x,\alpha,a,b}$, as witnessed by $\delta > 0$ with

$$(y, V_{a+\delta}) \leq_\alpha (x, V_{b-\delta}).$$

Now we can let $U = V_{\frac{\delta}{2}}$. Observe that

$$UV_{a+\frac{\delta}{2}} \subset V_{a+\delta},$$

since if $d(hg, 1) \leq d(hg, g) + d(g, 1) = d(h, 1) + d(g, 1)$ by right invariance of d, and hence for any $h \in U$ we have

$$(h \cdot y, V_{a+\frac{\delta}{2}}) \leq_\alpha (x, V_{b-\delta}).$$

The case of $y \in W^{x,\alpha,a,b}$, which can equally be rephrased as $x \in U^{y,\alpha,a,b}$, is entirely similar. □

3.4 Definition We then let $X_{x,\alpha}$ be the collection of $y \in X$ such that $[y]_G$ and $[x]_G$ have the same closure with respect to $\tau_{x,\alpha}$.

It follows from the lemma that $y \in X_{x,\alpha}$ if and only if $[y]_{G_0}$ and $[x]_{G_0}$ have the same closure under $\tau_{x,\alpha}$, and hence $X_{x,\alpha}$ is a Borel set. Note moreover that the topology $\tau_{x,\alpha}$ on $X_{x,\alpha}$ refines the subspace topology given by the inclusion $X_{x,\alpha} \subset X$ – given $y \in O$, we can find open $O' \subset O$ and $a > 0$ with $V_a \cdot y \subset O'$ and $V_{2a}^{-1} \cdot O' \subset O$. Then we can find some $g \cdot x$ with $y \in U^{g\cdot x,\alpha,a,2a}$, and it follows that $U^{g\cdot x,\alpha,a,2a} \subset O$ since $(y, V_a) \leq_\alpha (g \cdot x, V_{2a})$ implies

$$\overline{V_a \cdot y} \subset \overline{V_{2a} \cdot g \cdot x}.$$

3.4 Lemma If $y \in X_{x,\alpha}$ then every $\tau_{x,\alpha}$ open set is $\tau_{y,\alpha}$ open.

Proof. Say

$$(z, V_{a+\delta}) \leq_\alpha (x, V_{b-\delta})$$

witnesses $z \in U^{x,\alpha,a,b}$. (The cases involving translations of z or x under G_0 are entirely similar, and in fact follow from this one.)

Since $y \in X_{x,\alpha}$ we can find rational $b' < b'' < b'''$ in $\mathbb{Q} \cap (b - \delta, b - \frac{\delta}{2})$ such that

$$(x, V_{b-\delta}) \leq_\alpha (y, V_{b'}) \leq_\alpha (y, V_{b''}) \leq_\alpha (x, V_{b'''}).$$

Hence $U^{y,\alpha,a,b''} \subset U^{x,\alpha,a,b}$ and $z \in U^{y,\alpha,a,b}$. □

3.5 Corollary If $y \in X_{x,\alpha}$ then

$$\tau_{x,\alpha} = \tau_{y,\alpha},$$

and hence

$$X_{x,\alpha} = X_{y,\alpha},$$

and in particular $x \in X_{y,\alpha}$.

3.6 Lemma For x, α as above, there is a Polish space $Y_{x,\alpha}$ which can be identified with a dense G_δ subset of $(X_{x,\alpha}, \tau_{x,\alpha})$.

Proof. Let $\mathcal{P}_{x,\alpha}$ be the partial order of non-empty open sets in $(X_{x,\alpha}, \tau_{x,\alpha})$. It suffices to see that for any sufficiently generic filter $\mathcal{G} \subset \mathcal{P}_{x,\alpha}$ there is a unique $z \in \bigcap \mathcal{G}$.

First of all, it is completely standard that there will be a unique $z_{\mathcal{G}}$ such that z is in the intersection of all $U \subset X$ such that U is open in the original topology on X and there is some $V \in \mathcal{G}$ with $V \subset U$.

Claim: For all $\beta \leq \alpha$, $a < b \in \mathbb{Q}$ positive, $g \in G_0$:

(i) if $h \cdot U^{g \cdot x, \alpha, a, b} \in \mathcal{G}$, then $(h^{-1} \cdot z, V_a) \leq_\beta (x, V_b)$;

(ii) if $h \cdot W^{g \cdot x, \alpha, a, b} \in \mathcal{G}$, then $(x, V_a) \leq_\beta (h^{-1} \cdot z, V_b)$.

Proof of Claim: We prove this by transfinite induction on β. For notational simplicity I am going to assume $g = h = 1$ throughout – in light of 1.8, this is a harmless simplification.

For the base case, $\beta = 0$, (i) is trivial – since $U^{x,\alpha,a,b} \in \mathcal{G}$ implies that for any open set $O \subset X$, if some $W \in \mathcal{G}$ has $W \subset O$, then the fact that $\mathcal{G}$ is a *filter* of non-empty sets implies that for all $g' \in V_a \cap G_0$ we must have $V_b cdotx \cap g' \cdot O \neq \emptyset$. For (ii), notice that if $O \cap V_a \cdot x \neq \emptyset$, then at every $W \leq W^{x,\alpha,a,b} \in \mathcal{G}$ in $\mathcal{P}_{x,\alpha}$ we have some $W' \leq W$ in $\mathcal{P}_{x,\alpha}$ and $g' \in G_0 \cap V_b$ such that $g' \cap W' \subset O$.

The inductive step through limits is trivial, so now suppose $\beta < \alpha$ and we have proved (i) and (ii) at β and wish to lever up to $\beta + 1$.

For (i), suppose $W \subset U^{x,\alpha,a,b}$, $W \in \mathcal{G}$. Then given $g' \in G_0$ and $V_c g' \subset V_a$ we can choose some $g \cdot x \in W$, $g \in G_0$. Given $(g \cdot x, V_a) \leq_\alpha (x, V_b)$ we can find some $h \in G_0$ $d > 0$ such that $V_d h \subset V_b$ and

$$(x, V_d h) \leq_\beta (g \cdot x, V_{\frac{c}{2}} g'),$$

$$\therefore (h \cdot x, V_d) \leq_\beta (g' g \cdot x, V_{\frac{c}{3}}).$$

Then

$$W \cap (g')^{-1} \cdot W^{g \cdot x, \alpha, \frac{c}{2}, \frac{2c}{3}} \neq \emptyset.$$

Since $\mathcal{G}$ is sufficiently generic, we may assume it contains some such $\tau_{x,\alpha}$ open set, and then we have by the inductive assumption that

$$(g' g \cdot x, V_{\frac{c}{3}}) \leq_\beta (g' z_{\mathcal{G}}, V_{\frac{2c}{3}}),$$

$$\therefore (g' g \cdot x, V_{\frac{c}{3}}) \leq_\beta (z_{\mathcal{G}}, V_{\frac{2c}{3}} g') \leq_\beta (z_{\mathcal{G}}, V_c g')$$

we have

$$(x, V_d h) \leq_\beta (g' g \cdot x, V_{\frac{c}{3}}) \leq_\beta \leq_\beta (z_{\mathcal{G}}, V_c g').$$

Since we can complete this argument for any such basic open $V_c g' \subset V_a$, this entails $(z_{\mathcal{G}}, V_a) \leq_{\beta+1} (x, V_b)$.

The argument for (ii), is entirely similar. (□ Claim)

We now let $Y_{x,\alpha}$ be space of all sufficiently generic filters on $\mathcal{P}_{x,\alpha}$, which can be identified with a G_δ subset of $2^{\mathcal{P}_{x,\alpha}}$ in the product topology, and hence is Polish. For $\mathcal{G} \subset \mathcal{P}_{x,\alpha}$ and $W \in \mathcal{P}_{x,\alpha}$ one clearly has exactly one of

$$W \in \mathcal{G}$$

or for some $U \in \mathcal{P}_{x,\alpha}$

$$U \in \mathcal{G} \wedge U \cap W = \emptyset.$$

Thus

$$Y_{x,\alpha} \mapsto X_{x,\alpha}$$

$$\mathcal{G} \mapsto z_{\mathcal{G}}$$

provides a homeomorphism of $Y_{x,\alpha}$ with a dense G_δ subset of $X_{x,\alpha}$. $\square$

3.7 Definition Let $\approx_\alpha^*$ be the equivalence relation on X given by

$$x \approx_\alpha^* y$$

if and only if $X_{x,\alpha} = X_{y,\alpha}$.

3.8 Theorem For $\alpha \in \omega_1$ we have the following:
(i) $\approx_\alpha^*$ is $\underset{\sim}{\Pi}^0_{\alpha+k}$ for some $k \in \omega$;
(ii) $\approx_{\alpha+1} \subset \approx_\alpha^*$;
(iii) $\approx_\alpha^* \subset \approx_{\alpha+\omega}$;
(iv) E_1 is not Borel reducible to $\approx_\alpha^*$.

Proof. (i) follows by the earlier calculations for $\leq_\alpha$ and the observation following 3.3 to the effect that

$$x \approx_\alpha^* y$$

if and only if

$$\overline{[x]_{G_0}}^{\tau_{x,\alpha}} = \overline{[y]_{G_0}}^{\tau_{x,\alpha}}.$$

(ii) is more or less immediate from the definitions.

(iii) follows from (i) and 1.9 – which among other things implies that if $x \approx_{\alpha+\omega} y$ then they meet the same invariant $\underset{\sim}{\Pi}^0_{\alpha+k}$ sets at every $k \in \omega$.

(iv) follows from 3.6 and [5] 4.1. $\square$

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