Mixing actions of groups with HAP *

Greg Hjorth

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Abstract

A countable group Γ has the Haagerup approximation property if and only if the mixing actions are dense in the space of all actions of Γ.

1 Introduction

Given a countable group Γ and an atomless standard Borel probability space (X, μ), we can form the space of all measure preserving actions of Γ. In the usual weak topology, as discussed below, this becomes a Polish space. In the lines below, every action is assumed to be measurable, measure preserving, on some such (X, μ).

The present paper is part of a series of results which relate representation theoretic properties of a group to the properties of its space of all actions on some such (X, μ).

Theorem 1.1 *(Ornstein-Weiss; [9]) A countable group Γ is amenable if and only if every free measure preserving action has almost invariant sets.*

[Here, we say that an action of Γ on (X, μ) has *almost invariant sets* if there exists a sequence of measurable sets, $(A_n)_{n\in\mathbb{N}}$, with measure bounded away from 0 and 1, such that for any $\gamma \in \Gamma$

$$\text{limsup}_{n\to\infty}\mu(A_n \Delta \gamma \cdot A_n) \to 0.]$$

Theorem 1.2 *(Connes-Weiss, Schmidt; [3], [11]) A countable group Γ has property (T) if and only if no ergodic action has almost invariant sets.*

Theorem 1.3 *(Glasner-Weiss, [4]) A countable group Γ has property (T) if and only if the set of ergodic actions is closed in the space of all actions.*

In answer to a question of Bergelson and Rosenblatt:

Theorem 1.4 *(Kerr-Pichot; [8]) A countable group Γ does not have property (T) if and only if the weak mixing actions on (X, μ) are a dense G_δ in the space of all actions.*

In the present paper we answer a question due to Alexander Kechris in [7] by showing:

Theorem 1.5 *If a countable group Γ has the Haagerup approximation property (HAP) then the mixing actions are dense in the space of all actions.*

The converse direction was known and is a straight forward consequence of the definitions – see for instance 12.7 [7]. Thus we have

Corollary 1.6 *A countable group Γ has the Haagerup approximation property if and only if the mixing actions are dense in the space of all actions.*

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3 Representation theoretic notions

For the reader's convenience we recall the relevant notions arising from the theory of unitary representations.

Definition For H a Hilbert space, we let $U(H)$ be the unitary operators. Given a unitary representation

$$\pi : \Gamma \rightarrow U(H),$$

$$\gamma \mapsto \pi_\gamma$$

of a countable group Γ, we say that π has *almost invariant vectors* if for all $\epsilon > 0$ and $F \subset \Gamma$ finite there exists $v \in H$ with $\|h\| = 1$ and for all $\gamma \in F$

$$\|\pi_\gamma(v) - v\| < \epsilon.$$

We say that the representation is *mixing* if for all $u, v \in H$

$$\text{limsup}_{\gamma \rightarrow \infty} \langle \pi_\gamma(v), u \rangle \rightarrow 0.$$

Definition Given a countable group Γ, we the *regular left representation* of Γ is defined by

$$\lambda : \Gamma \rightarrow \ell^2(\Gamma),$$

$$\gamma \mapsto \lambda_\gamma,$$

where

$$(\lambda_\gamma(f))(x) = f(\gamma^{-1}x).$$

With these concepts, we can define the classes of groups from the introduction. Note that the definition we give for amenability is an equivalence of the more traditional definitions of amenability; there are manifold different characterizations of amenability, most of which, including the one used here, the reader can find discussed in [10].

Definition A countable group Γ is *amenable* if the left regular representation has almost invariant vectors. A countable group Γ has *property (T)* if whenever a unitary representation has almost invariant vectors, there is an invariant vector of norm one. A countable group has the *Haagerup approximation property,* HAP for short, if there exists a mixing action with almost invariant vectors.

Examples (i) The group $\mathbb{Z}$ is amenable, as is any abelian group. More generally any solvable group is amenable. See [10].

(ii) $\mathbb{F}_2$, the free group on two generators, is *non-amenable*, but it does have the Haagerup approximation property. See the appendix of [6] for an entirely elementary proof of this fact first due to Haagerup.

(iii) $\mathrm{SL}_3(\mathbb{Z})$ has property (T). See for instance [5]. It is not amenable, since no countably infinite group can have an invariant unit vector in its regular left representation. Similarly it does not have HAP, since mixing actions cannot have invariant unit vectors.

(iv) Under the natural semi-direct product provided by linear action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{Z}^2$, the group

$$\Delta = \mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z},$$

is non-amenable, since it has a subgroup isomorphic to $\mathbb{F}_2$. It does not have property (T), since it has $\mathrm{SL}_2(\mathbb{Z})$ as a homomorphic image, and this group in turn has the free group on two generators as a finite index subgroup. On the other hand, as shown in for example in [2], it does have a kind of relative property (T), which is sufficient to exclude the Haagerup approximation property.

4 Measure theoretic notions

We follow the notation of [7].

Notation For (X, μ) a standard Borel space, we let $\mathrm{Aut}(X, \mu)$ be the collection of invertible measure preserving transformations. Given a countable group Γ, $A(\Gamma, X, \mu)$ denotes the collection of homomorphisms from Γ to $\mathrm{Aut}(X, \mu)$. Given such an automorphism

$$a : \Gamma \rightarrow \mathrm{Aut}(X, \mu)$$

and given a group element $\gamma \in \Gamma$, we write denote the invertible measure preserving transformation associated to γ by γ^a.

Notation The *weak topology* on $\mathrm{Aut}(X, \mu)$ is the topology with subbasic open sets of the form

$$\{\pi \in \mathrm{Aut}(X, \mu) : \mu(B \Delta \pi(A)) < \epsilon\},$$

for $A, B \subset X$ measurable, $\epsilon > 0$.

Provided we identify measure preserving transformations which agree on a conull set, $\mathrm{Aut}(X, \mu)$ becomes a Polish space in the weak topology. Then $A(\Gamma, X, \mu)$ becomes a closed subset of

$$\prod_{\Gamma} \mathrm{Aut}(X, \mu),$$

and is again Polish in its own right. These and other preliminary facts can be found at §1 and §10 of [7].

Definition Given a countable group Γ and an action $a \in A(\Gamma, X, \mu)$, we say that the action is *mixing* if for all measurable $A, B \subset X$

$$\mathrm{limsup}_{\gamma \rightarrow \infty} |\mu(A \Delta \gamma^a \cdot B) - \mu(A)\mu(B)| = 0.$$

5 Proof

In everything below, Γ is a countable group with the Haagerup approximation property. The next lemma, with its cutting and splicing along almost invariant sets, has obvious parallels with the arguments of [8].

Lemma 5.1 *Let (X, μ) be an atomless standard Borel probability space. Let $a \in A(\Gamma, X, \mu)$, $B_1, ..., B_n$ be measurable subsets of X, $F \subset \Gamma$ finite, $\epsilon > 0$.*

Then there exists an atomless standard Borel probability space $(\hat{X}, \hat{\mu})$ and $b \in A(\Gamma, \hat{X}, \hat{\mu})$ and measurable subsets $\hat{B}_1, ..., \hat{B}_n$ such that:-

(a) for any $S \subset \{1, 2, ...n\}$

$$\hat{\mu}(\bigcap_{i \in S}(\hat{B}_i \setminus (\bigcap_{i \notin S}(\hat{B}_i)))) = \mu(\bigcap_{i \in S}(B_i \setminus (\bigcap_{i \notin S}(B_i))));$$

(b) for all $m \leq n$ and $\gamma_1, ..., \gamma_m \in F$ and $i_1, ..., i_m \leq n$

$$|\mu(\gamma_1^a \cdot B_{i_1} \cap \gamma_2^a \cdot B_{i_2} \cap ... \cap \gamma_m^a \cdot B_{i_m}) - \hat{\mu}(\gamma_1^b \cdot \hat{B}_{i_1} \cap \gamma_2^b \cdot \hat{B}_{i_2} \cap ... \cap \gamma_m^b \cdot \hat{B}_{i_m})| < \epsilon;$$

(c) $\forall \gamma \in \Gamma, i, j \leq n$

$$|\hat{\mu}(\hat{B}_i \cap \gamma^b \cdot (\hat{B}_j)) - \hat{\mu}(\hat{B}_i) \cdot \hat{\mu}(\hat{B}_j)| \leq |\mu(B_i \cap \gamma^a \cdot (B_j)) - \mu(B_i) \cdot \mu(B_j)|;$$

(d) for all $i, j \leq n$

$$\mathrm{limsup}_{\gamma \rightarrow \infty} |\hat{\mu}(\gamma^b \cdot (\hat{B})_i \cap \hat{B}_j) - \hat{\mu}(\hat{B}_i)\hat{\mu}(\hat{B}_j)| < \epsilon.$$

Proof To begin with let us fix some large $N \in \mathbb{N}$. (It will be clear after the later calculations that any sufficiently large N will succeed; in fact $N > \frac{2}{\epsilon}$ suffices.) We equip X^N with the diagonal action, c, where

$$\gamma^c \cdot (x_1, x_2, ...x_n) = (\gamma^a \cdot x_1, \gamma^a \cdot x_2, ...)$$

and the product measure μ^N. For each $i \leq n$ we let

$$\pi_i : X^N \to X$$

$$(x_1, x_2, ..., x_n) \mapsto x_i.$$

Now let $\delta > 0$ be very small. (Again, it will be clear from the later calculations that any sufficiently large δ will fill our needs; in fact $\delta < \frac{\epsilon}{N}$ suffices.) Using that Γ has HAP, let (Y, ν) be a standard Borel probability space with a partition

$$Y = \bigcup_{i \leq N} A_i$$

and an action d of Γ on X such that:

(i) the action d is measure preserving and mixing;

(ii) for each $\gamma \in F$ and $i \leq n$

$$\nu(A_i \Delta \gamma^d \cdot A_i) < \delta.$$

We let $\hat{X} = X^N \times Y$ equipped with the product measure, which we denote now by $\hat{\mu}$, and the action

$$\gamma^b \cdot (\vec{x}, y) = (\gamma^c \cdot (\vec{x}), \gamma^d \cdot y).$$

We then let $\hat{B}_j$ be the set of $(\vec{x}, y) \in X^N \times Y$ such that for all $i \leq N$

$$y_i \in A_i \Rightarrow \pi_i(\vec{x}) \in B_j.$$

Property (a) from the lemma should be obvious from the construction. Property (b) follows by making δ sufficiently small.

For each $\gamma \in \Gamma$ we let

$$C_\gamma = \{y \in Y : \forall i(y \in A_i \Rightarrow \gamma^d \cdot y \notin A_i)\}.$$

Since the action of d is mixing we obtain that as $\gamma \to \infty$

$$\nu(A_i \cap (\gamma^d)^{-1} \cdot (Y \setminus A_i)) \to \frac{1}{N^2},$$

and hence

$$\nu(C_\gamma) \to \frac{N(N-1)}{N^2} = \frac{N-1}{N}.$$

Claim: For all γ outside a finite set, and for all $j, \ell \leq n$

$$\frac{N-2}{N}\mu(B_j) \cdot \mu(B_\ell) < \hat{\mu}(\hat{B}_j \cap \gamma^b \cdot (\hat{B}_\ell)) < \frac{N-2}{N}\mu(B_j) \cdot \mu(B_\ell) + \frac{2}{N}.$$

Proof of Claim: Outside a finite set of γ's we obtain $\nu(C_\gamma) > \frac{N-2}{N}$. Then for $y \in C_\gamma$, $y \in A_i, \gamma^d \cdot y \in A_k, z \in A_k$ we have firstly that $i \neq k$ and then

$$\mu^N((\hat{B}_j)^z \cap (\gamma^b \cdot \hat{B}_\ell)^{\gamma^d \cdot y}) = \mu^N((\hat{B}_j)^z \cap (\gamma^c \cdot (\hat{B}_\ell^y)))$$

$$= \mu(\pi_k^{-1}(B_j) \cap \pi_i^{-1}(\gamma^a \cdot B_\ell)) = \mu(B_j) \cdot \mu(B_\ell).$$

And thus for $z \in \gamma^d \cdot C_\gamma$ we have

$$\mu^N((\hat{B}_j)^z \cap (\gamma^b \cdot \hat{B}_\ell)^z) = \mu(B_j) \cdot \mu(B_\ell).$$

For z outside this set,

$$0 \leq \mu^N((\hat{B}_j)^z \cap (\gamma^b \cdot \hat{B}_\ell)^z) \leq 1.$$

Since $\nu(C_\gamma) = \nu(\gamma^d \cdot C_\gamma) > \frac{N-2}{N}$ the claim follows by integrating z over Y. (□Claim)

Thus for N with $\frac{2}{N} < \epsilon$ we obtain (d) from the statement of the lemma.

Note then that for all $i, \ell \leq n$ and $\gamma \in \Gamma$ we have, as above,

$$\hat{\mu}(\hat{B}_i \cap \gamma^b \cdot (\hat{B}_\ell)) = \int_Y \mu^N((\hat{B}_i)^z \cap (\gamma^b \cdot \hat{B}_\ell)^z d\nu(z).$$

For $z \in \gamma^d \cdot C_\gamma$ again

$$\mu^N((\hat{B}_j)^z \cap (\gamma^b \cdot \hat{B}_\ell)^z) = \mu(B_j) \cdot \mu(B_\ell),$$

whilst for for $z \notin \gamma^d \cdot C_\gamma$ we obtain

$$\mu^N((\hat{B}_j)^z \cap (\gamma^b \cdot \hat{B}_\ell)^z) = \mu(B_j \cap \gamma^a \cdot B_\ell).$$

This yields (c). □

Corollary 5.2 *Let (X, μ) be an atomless standard Borel probability space. Let $a \in A(\Gamma, X, \mu)$, $C_1, ..., C_k$ be measurable subsets of X, $F \subset \Gamma$ finite, $\epsilon > 0$.*

Then there exists $c \in A(\Gamma, X, \mu)$ such that:-

(i) for all $m \leq k$ and $\gamma_1, ..., \gamma_m \in F$ and $i_1, ..., i_m \leq k$

$$|\mu((\gamma_1^a \cdot C_{i_1} \cap \gamma_2^a \cdot C_{i_2} \cap ... \cap \gamma_m^a \cdot C_{i_m})\Delta(\gamma_1^c \cdot C_{i_1} \cap \gamma_2^c \cdot C_{i_2} \cap ... \cap \gamma_m^c \cdot C_{i_m}))| < \epsilon;$$

(ii) $\forall \gamma \in \Gamma, i, j \leq n$

$$|\mu(C_i \cap \gamma^c \cdot (C_j)) - \mu(C_i) \cdot \mu(C_j)| \leq |\mu(C_i \cap \gamma^a \cdot (C_j)) - \mu(C_i) \cdot \mu(C_j)|;$$

(iii) for all $i, j \leq n$

$$\text{limsup}_{\gamma \to \infty} |\mu(\gamma^c \cdot (C_i) \cap C_j) - \mu(C_i) \cdot \mu(C_j)| < \epsilon.$$

Proof Let $\mathcal{B}$ be the finite Boolean algebra generated by $\{\gamma \cdot C_i : i \leq k, \gamma \in F\}$. Let $B_1, ..., B_n$ enumerate the elements of $\mathcal{B}$. Apply the last lemma to obtain b and $(\hat{X}, \hat{\mu})$ for $B_1, ..., B_n$, $a \in A(\Gamma, X, \mu)$, $\epsilon > 0$, and the indicated F.

Now we can find an measure preserving bijection

$$\psi : \hat{X} \to X$$

with $\psi[\hat{B}_i] = B_i$. Taking

$$\gamma^c \cdot (x) = \psi(\gamma^b \cdot \psi^{-1}(x))$$

is as required. □

Theorem 5.3 *Let Γ be a countable group with the Haagerup approximation property. Let (X, μ) be an atomless standard Borel probability space. Then the mixing actions are dense in the space $A(\Gamma, X, \mu)$.*

Proof Fix $a_0 \in A(\Gamma, X, \mu)$. Fix $\epsilon > 0$. Fix $\mathcal{B}$ a finite Boolean algebra of measurable subsets of X. Fix F a finite subset of Γ. We need to find a mixing action b of Γ such that for all $A \in \mathcal{B}$

$$\mu(\gamma^b \cdot A \Delta \gamma^a \cdot A) < \epsilon.$$

Let $\mathcal{B}_0 = \mathcal{B} \subset \mathcal{B}_1 \subset \mathcal{B}_2 \subset .... \subset \mathcal{B}_n \subset \mathcal{B}_{n+1} \subset ...$ be an increasing sequence of finite Boolean algebras of measurable subsets of X whose union is dense in $M(X, \mu)$, the metric space of all measurable subsets of X equipped with the metric $d(A, B) = \mu(A \Delta B)$. Let $F_0 = F \subset F_1 \subset F_2 \subset ....$ be an increasing sequence of finite subsets of Γ whose union equals all of Γ.

Applying the above corollary repeatedly we can find actions

$$a_0 = a, a_1, a_2, ...$$

such that at each $n \geq 0$ and $C_1, ..., C_k \in \mathcal{B}_n$

(i) for all $m \leq k$ and $\gamma_1, ..., \gamma_k \in F_n$

$$|\mu((\gamma_1^{a_n} \cdot C_1 \cap \gamma_2^{a_n} \cdot C_2 \cap ... \cap \gamma_k^{a_n} \cdot C_k)\Delta(\gamma_1^{a_{n+1}} \cdot C_1 \cap \gamma_2^{a_{n+1}} \cdot C_2 \cap ... \cap \gamma_k^{a_{n+1}} \cdot C_k))| < 2^{-n-1}\epsilon;$$

(ii) $\forall \gamma \in \Gamma, i, j \leq k$

$$|\mu(C_i \cap \gamma^{a_{n+1}} \cdot (C_j)) - \mu(C_i) \cdot \mu(C_j)| \leq |\mu(C_i \cap \gamma^{a_n} \cdot (C_j)) - \mu(C_i) \cdot \mu(C_j)|;$$

(iii) for all $i, j \leq k$

$$\text{limsup}_{\gamma\to\infty}|\mu(\gamma^{a_{n+1}} \cdot (C_i) \cap C_j) - \mu(C_i) \cdot \mu(C_j)| < 2^{-n-1}\epsilon.$$

For each $\gamma \in \Gamma$ we can define the corresponding sequence of transformations

$$\psi_n^\gamma : x \mapsto \gamma^{a_n} \cdot x.$$

In light of (i) directly above, these converge in the Polish topology on $\text{Aut}(X, \mu)$ and hence we have a transformation

$$\psi_\gamma : X \to X$$

such that for all $A \subset X$ measurable

$$\mu(\gamma^{a_n} \cdot A\Delta\psi_\gamma(A)) \to 0$$

as

$$n \to \infty.$$

Thus we obtain a fresh action $b \in A(\Gamma, X, \mu)$, with each $\gamma^b = \psi_\gamma$, such that for all $A \subset X$ measurable

$$\mu(\gamma^{a_n} \cdot A\Delta\gamma^b \cdot A) \to 0$$

as $n \to \infty$. Note in particular that for all $A \in \mathcal{B} = \mathcal{B}_0$ and $\gamma \in F = F_0$ we have

$$\mu(\gamma^b \cdot A\Delta\gamma^a \cdot A) < \epsilon,$$

and so it only remains to show the action is mixing.

Given any n and $A, B \in \mathcal{B}_n$ and any $\delta > 0$ we can find some $m > n$ with

$$2^{-m-1}\epsilon < \delta.$$

Then for all γ outside some finite set we have

$$|\mu(A\Delta\gamma^{a_{m+2}} \cdot B) - \mu(A)\mu(B)| < \frac{\delta}{2}$$

by (iii). Then by (ii) we have for all $k \geq m + 2$

$$|\mu(A\Delta\gamma^{a_k} \cdot B) - \mu(A)\mu(B)| < \frac{\delta}{2},$$

and hence

$$|\mu(A\Delta\gamma^b \cdot B) - \mu(A)\mu(B)| \leq \frac{\delta}{2} < \delta.$$

Since $\bigcup_{n\in\mathbb{N}} \mathcal{B}_n$ is dense in $M(X, \mu)$, this suffices to show the action is mixing. □

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greg.hjorth@gmail.com

Department of Mathematics and Statistics
University of Melbourne
Parkville, 3010
Melbourne
Victoria
Australia