Lightening review of Los' theorem

Definition Let S be a set, and use $\mathcal{P}(S)$ to denote the *power set of* S – that is to say, the set of all subsets of S, or $\{A : A \subset S\}$. $\mu \subset \mathcal{P}(S)$ is said to be an *ultrafilter* on S if:

1. if $A, B \in \mu$, the $A \cap B \in \mu$;
2. if $A \in \mu$ and $B \supset A$ then $B \in \mu$;
3. the empty set is not an element of μ;
4. for any $A \in \mathcal{P}(S)$, either A or $A^c (=_{\rm df} S \setminus A)$ is in μ.

We say that an ultrafilter μ is *non-principal* if $\{a\} \notin \mu$ for any $a \in S$.

In the above, the first two points are the definition of a filter; the last two the definition of an ultrafilter.

Exercise Use the compactness theorem for propositional logic to show that for any infinite set S there is a non-principal ultrafilter on S.

Exercise Show that an ultrafilter is closed under finite intersections.

Lemma 0.1 *For μ an ultrafilter on a set S, show that for any $A, B \subset S$ we have*

$$A \cap B \in \mu$$

if and only if both A and B are in μ.

Proof Exercise. □

Definition Let $\mathcal{M}$ be a model for some language $\mathcal{L}$. For notational simplicity I will assume $\mathcal{L}$ is relational, however the more general case where $\mathcal{L}$ contains function symbols is fundamentally the same, and in fact can be reduced to the relational case.

Let μ be an ultrafilter on a set S. For

$$f, g : S \to \mathcal{M}$$

set

$$f \sim_\mu g$$

if there is some $A \in \mu$ such that

$$f(a) = g(a)$$

all $a \in A$. We then let

$$[f]_\mu = \{g : g \sim_\mu f\}.$$

Lemma 0.2 *In this situation, $\sim_\mu$ is an equivalence relation.*

I will again leave the proof of this as an exercise, but please let me know if there are any problems.

Lemma 0.3 *Let $\phi(x_1, x_2, ..., x_n)$ be a first order formula in $\mathcal{L}$. Suppose $f_1, ..., f_n : S \to \mathcal{M}$. Suppose at each $i \leq n$ we have*

$$g_i, h_i \in [f_i]_\mu.$$

Then there is $A \in \mu$ such

$$\forall a \in A(\mathcal{M} \models \phi(g_1(a), ..., g_n(a)))$$

if and only if there is $B \in \mu$ such that

$$\forall a \in B(\mathcal{M} \models \phi(h_1(a), ..., h_n(a))).$$

Proof First let

$$X = \{a \in S : \mathcal{M} \models \phi(g_1(a), ..., g_n(a))\},$$
$$Y = \{a \in S : \mathcal{M} \models \phi(h_1(a), ..., h_n(a))\},$$

and for $i \leq n$ let

$$Z_i = \{a \in S : g_i(a) = h_i(a)\}.$$

By assumptions of lemma, each Z_i is in μ, and hence so is

$$Z = \bigcap_{i \leq n} Z_i.$$

For a contradiction, suppose that $X \in \mu, Y^c \in \mu$. Then consider

$$Z \cap X \cap Y^c \in \mu.$$

For any a in this set we have $a \in X$ and hence

$$\mathcal{M} \models \phi(g_1(a), ..., g_n(a));$$

similarly $a \notin Y$ so

$$\mathcal{M} \models \neg(h_1(a), ..., h_n(a)).$$

However at each i we have $a \in Z_i$, which yields $g_i(a) = h_i(a)$, and thus steers us in to a contradiction. □

Definition For $\mathcal{M}, S, \mu$ as above, we let $\text{Ult}_\mu\mathcal{M}$ denote the collection

$$\{[f]_\mu | f : S \to \mathcal{M}\}.$$

For R an n-ary relational symbol in $\mathcal{L}$ we let

$$\text{Ult}_\mu\mathcal{M} \models R([f_1]_\mu, ..., [f_n]_\mu)$$

if the set

$$\{a \in S : \mathcal{M} \models R(f_1(a), ..., f_n(a))\} \in \mu.$$

By the last lemma, this is well defined – which is to say, it does not depend on the *choice* of $f_i \in [f_i]_\mu$. This resulting $\mathcal{L}$-structure is called the *ultraproduct* of $\mathcal{M}$ by μ.

Theorem 0.4 *(Los) Let $\mathcal{M}, \mathcal{L}, S, \mu$ be as above. For*

$$f_1, ..., f_n : S \to \mathcal{M}$$

and $\varphi(x_1, ..., x_n)$ a first order formula of $\mathcal{L}$,

$$\text{Ult}_\mu\mathcal{M} \models \varphi([f_1]_\mu, ..., [f_n]_\mu)$$

if and only if

$$\{a \in S : \mathcal{M} \models \varphi(f_1(a), ..., f_n(a))\} \in \mu.$$

Proof We prove this by induction on the logical complexity of φ. The case of φ atomic follows by the definition of the ultraproduct.

For passage through the inductive step related to negation, note that

$$\{a \in S : \mathcal{M} \models \varphi(f_1(a), ..., f_n(a))\} \notin \mu$$

if and only if

$$\{a \in S : \mathcal{M} \models \neg\varphi(f_1(a), ..., f_n(a))\} \in \mu$$

by condition 2 in the definition of an ultrafilter, and for passage through conjunction note that

$$\{a \in S : \mathcal{M} \models \varphi_1(f_1(a), ..., f_n(a)) \wedge \varphi_2(f_1(a), ..., f_n(a))\} \in \mu$$

if and only if *both*

$$\{a \in S : \mathcal{M} \models \varphi_1(f_1(a), ..., f_n(a))\}$$

and

$$\{a \in S : \mathcal{M} \models \varphi_2(f_1(a), ..., f_n(a))\}$$

are in μ by 0.1.

Finally, the for the introduction of existential quantifiers, suppose, by inductive hypothesis, the theorem is true for $\varphi(x_1, ..., x_n, y)$. Clearly if there is some g such that

$$\{a \in S : \mathcal{M} \models \varphi(f_1(a), ..., f_n(a), g(a))\} \in \mu$$

then

$$\{a \in S : \mathcal{M} \models \exists y \varphi(f_1(a), ..., f_n(a), y)\} \in \mu.$$

Conversely, suppose

$$A = \{a \in S : \mathcal{M} \models \exists y \varphi(f_1(a), ..., f_n(a), y)\}$$

and we have $A \in \mu$. Then using the axiom of choice we may choose some

$$g : S \to \mathcal{M}$$

such that at each $a \in A$

$$\mathcal{M} \models \varphi(f_1(a), ..., f_n(a), g(a)).$$

□

We used the axiom of choice in the proof of Los' theorem. This is use is unavoidable.

More details on the proofs and generalizations can be found in any standard text on model theory, such as [1], or §12 [2].

References

[1] C.C. Chang, H.J. Keisler, **Model theory,** Third edition. Studies in Logic and the Foundations of Mathematics, 73. North-Holland Publishing Co., Amsterdam, 1990.

[2] T. Jech, **Set theory,** The third millennium edition, revised and expanded. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2003.