Brief comment on HW 1, 223s, Fall 2010

Q1 *Let δ be a regular cardinal and $C \subset \delta$ a club of indiscernibles for L_δ.*

(i) (1pt) Show that if $f : \delta \to \delta$ is definable (without the use of parameters) over L_δ and regressive on C ($\forall \alpha \in C(f(\alpha) < \alpha)$) then it is constant on C.

(ii) (2pts) Show that if $\alpha \in C$, then α is a cardinal in L_δ. (Hint: If not, then no $\alpha \in C$ will be a cardinal, and we can define over L_δ a regressive function from C, $\alpha \mapsto |\alpha|^L$, which will be constant by the last exercise. But then we can let $g(\alpha)$ be the $<_L$-least element of $(\mathcal{P}(|\alpha| \times |\alpha|))^L$ which gives a well ordering of $|\alpha|$ isomorphic to α. Show that g will constant on C, and derive a contradiction.)

No one had any trouble with (i). After reviewing a few different answers, it seems to me that the following might be the most elegant solution for (ii).

Suppose for a contradiction not every element of C is a cardinal. Then by indiscernibility and part (i) we obtain some fixed $\gamma^* < \delta$ such that for every $\alpha \in C$

$$|\alpha|^L = \gamma^*.$$

At each $\alpha \in C$, let $g(\alpha) \subset \gamma^*$ be the α^{th} subset of γ^* in L_δ. Note that g is definable over L_δ.

Then at each $\alpha < \beta$ let $h(\alpha, \beta)$ be the least $\eta \in \gamma^*$ such that

$$\eta \in g(\alpha) \Delta g(\beta).$$

Case (1): For some $\alpha_1 < \beta_1 < \alpha_2 < \beta_2 \in C$ we have

$$h(\alpha_1, \beta_1) = (\alpha_2, \beta_2).$$

Then by indiscernibility we have that at *every* $\alpha_1 < \beta_1 < \alpha_2 < \beta_2 \in C$

$$h(\alpha_1, \beta_1) = (\alpha_2, \beta_2).$$

Then given any $\alpha_1 < \beta_1 \in C, \alpha_2 < \beta_2 \in C$ we can consider some $\alpha_3 < \beta_3 \in C$ with $\alpha_1 < \beta_1, \alpha_2 < \beta_2 \in C$ to conclude from indiscernibility

$$h(\alpha_1, \beta_1) = (\alpha_3, \beta_3),$$
$$h(\alpha_2, \beta_2) = (\alpha_3, \beta_3),$$

and hence

$$h(\alpha_1, \beta_1) = (\alpha_2, \beta_2).$$

Now in particular if we let $\gamma_1 < \gamma_2 < \gamma_3 \in C$ we have

$$h(\gamma_1, \gamma_2) = h(\gamma_2, \gamma_3) = h(\gamma_1, \gamma_3).$$

Call this common value α_0. We have $\alpha_0 \in g(\gamma_1)\Delta g(\gamma_2)$ *and* $\alpha_0 \in g(\gamma_2)\Delta g(\gamma_3)$ *and* $\alpha_0 \in g(\gamma_1)\Delta g(\gamma_3)$ – which is quickly seen to be a contradiction.

Case (2): F For some $\alpha_1 < \beta_1 < \alpha_2 < \beta_2 \in C$ we have

$$h(\alpha_1, \beta_1) \neq (\alpha_2, \beta_2).$$

Then by indiscernibility we have that at *every* $\alpha_1 < \beta_1 < \alpha_2 < \beta_2 \in C$

$$h(\alpha_1, \beta_1) \neq (\alpha_2, \beta_2).$$

Let $(\alpha_\gamma)_{\gamma \in \delta}$ and $(\beta_\gamma)_{\gamma \in \delta}$ be increasing sequences of ordinals through C with $\alpha_\gamma < \beta_\gamma$ all γ. Then

$$\gamma \mapsto h(\alpha_\gamma, \beta_\gamma)$$

gives an injection from δ to γ^*, with a contradiction to δ being a cardinal.

(iii) (2pt) Show that every element of C will be inacessible in L. (Assume the last exercise.)

In parallel to the last exercise, we need to show that they are not singular cardinals and for a contradiction if we assume that they are we get a single $\gamma^* < \delta$ such that each $\alpha \in C$ has $\mathrm{Cof}(\alpha)^L = \gamma^*$. We let $g(\alpha)$ be the $<_L$-least cofinal function from γ^* to α. g is definable over L_δ. Note that at each $\gamma < \gamma^*$

$$\alpha \mapsto g(\alpha)(\gamma)$$

is regressive. If we mimic the argument above and let $h(\alpha, \beta)$ be the least $\gamma < \gamma^*$ with $g(\alpha)(\gamma) \neq g(\beta)(\gamma)$, then the argument from (ii) gives that $h(\alpha, \beta)$ is constant for $\alpha < \beta \in C$. Let γ_0 be common constant value.

$\alpha \mapsto g(\alpha)(\gamma_0)$ is now a regressive function on C, which must be constant on a stationary set, with a contradiction to the assumptions on γ_0.