Math 223s, Fall 2010
HW3

Due date: Wednesday, December 1st, 1pm at start of class

Definition For $x \in 2^\omega$, let $\in_x = \{(n,m) : x(2^n 3^m) = 1\}$. Let $M_x = (\omega; \in_x)$. Let ZFC* denote the first one billion axioms of ZFC. A model $\mathcal{M} = (M; E)$ of ZFC* is said to be an ω-model if its version of ω is genuinely isomorphic to true ω – which is to say, for every $a \in M$ with $\mathcal{M} \models a$ "is a natural number", we have some $k \in \omega$ such that

$$\mathcal{M} \models \exists b (b \text{ is the empty set and } S(S(S^{k\text{times}}....(b))) = a).$$

Q1 (i) Show that the set of pairs $(x,a) \in \omega^\omega \times \omega$ such that $x \in 2^\omega$ and

$$M_x \models \text{ZFC}^* \wedge \text{"}a \text{ is an ordinal"}$$

is a Δ^1_1 subset of $\omega^\omega \times \omega$.

(This basically follows from our observing that truth is Δ^1_1 in class, and please don't trouble yourself with trying to repeat that entire proof. Really, I want you to think about this just enough to convince yourself it is true, and write down just enough to convince me that you have indeed convinced yourself. We will need these and similar calculations in the proofs remaining, and I am disinclined to go through them all carefully in class.)

Definition For $M_x \models$ ZFC*, we let Ord^{M_x} denote the set of $a \in \omega$ with

$$M_x \models \text{"}a \text{ is an ordinal"}.$$

Let LO denote the set of $x \in 2^\omega$ such that $\in_x$ is a linear ordering of ω.

(ii) Show that the set of $x \in 2^\omega$ such that M_x is an ω-model of ZFC* is a Δ^1_1 subset of ω^ω.

(iii) Show that the set of $(x,y) \in 2^\omega \times 2^\omega$ such that $y \in$ LO, $x \models$ ZFC*, and

$$(\omega; \in_y) \cong (\text{Ord}^{M_x}; \in_x)$$

is Σ^1_1 as a subset of $\omega^\omega \times \omega^\omega$.

Q2 (i) Show that the set of $(x,y) \in 2^\omega \times 2^\omega$ such that $x, y \in$ LO and $(\omega; \in_y)$ is isomorphic to an initial segment of $(\omega; \in_x)$ is Σ^1_1.

(ii) Show that the set of $(x,y) \in 2^\omega \times 2^\omega$ such that $x, y \in$ LO and $\in_x$ is a well order and $\in_y$ is isomorphic to an initial segment of $(\omega; \in_x)$ is Π^1_1.

Q2(i), (ii) shows that comparison of lengths is *relatively* Δ^1_1 on the well orderings. This is a short, but very important, calculation. For instance, it is implicit in say the proof from Moschovakis' **Descriptive Set Theory** that Π^1_1 has the *prewellordering property.*