Math 223s, Fall 2010
Comments on HW2

The first three problems gave no difficulties whatsoever, but there was a systematic confusion regarding the definition of Σ_0, and by implication Σ_n. Since *every single* person made exactly the same mistake, I decided it must have been a failure of communication on my part and didn't take off any marks, but let us make sure it is corrected *now*.

Definition In the language of set theory, the Σ_0 formulas are the smallest class containing the atomic formulas, closed under the boolean connectives $\wedge, \vee$, and $\neg$, and closed under *bounded quantification* of the form $\exists x \in y$ and $\forall x \in y$.

Note that in the proof of the covering lemma this is indeed the form of the definition which we needed. Practically everyone seemed to think the Σ_0 formulas were the atomic formulas, which isn't the definition in say Jech, not the definition I (perhaps erroneously)thought I had presented in class, and definitely not the version one would need to say that Σ_1 elementary substructures of some L_δ are isomorphic to some L_γ.

Q4 We needed to show that the direct limits of Σ_n elementary maps give rise to a Σ_n-elementary map into the direct limit.

The answer most people gave would have been fine, if there was first a proper proof for Σ_0, so let us attend to this. We let F be the class of Σ_0-formulas whose truth value is preserved under $\rho_{d,\infty}$ for all choice of parameters in M_d. This class is trivially seen to be true under the boolean connectives. The main issue is to see that if $\vec{b}, c \in M_d$ and $\psi(\vec{x}, z) \in F$ then

$$M_d \models \exists z \in c\psi(\vec{b}, z) \Leftrightarrow M_\infty \models \exists z \in \rho_{d,\infty}\psi(\rho_{d,\infty}(b_0), ..., z).$$

The $\Rightarrow$ direction is more or less trivial, so suppose $M_\infty \models \exists z \in \rho_{d,\infty}\psi(\rho_{d,\infty}(b_0), ..., z)$. Then by the definition of the direct limit we can find some $d' \in D, a \in M_{d'}$ such that

$$M_\infty \models \rho_{d',\infty}(a) \in \rho_{d,\infty} \wedge \rho_{d,\infty}\psi(\rho_{d,\infty}(b_0), ..., \rho_{d',\infty}(a)).$$

Without loss of generality, $d' \geq d$ (otherwise just replace d' by some $d^* \geq d, d'$ and chase the arrows as usual). But then by elementarity of $\rho_{d,d'}$ we have

$$M_d \models \exists z \in c\psi(\vec{b}, z),$$

as required.

Q5 People did fine on this, usually taking $D = \omega$ with its usual ordering and letting M_n be the first n many natural numbers, $\rho_{n,n+1} : k \mapsto k + 1$. This is fine for obtaining embeddings which are required to preserve quantifier free formulas, but let me point out how one can do better.

One idea is if there is a measurable, or even just $0^\sharp$. Take some large regular δ and

$$j : L_\delta \to L_\delta$$

an elementary map with cp(j)= $\kappa < \delta$. let $D = \omega$ as above, let $\rho_{n,n+1} = j$, and note that the well founded part of the resulting M_∞ is exactly κ.

Actually, although some kind of modest large cardinal assumptions are necessary to do a version of this example for models of ZFC, one doesn't really need a measurable to do something like this with embeddings between ordinals, though it is a little bit of a pain to describe the proof in precise mathematical detail. Basically, if we consider the first ω many uncountable cardinals, then they all have the same theory in structure $(\aleph_\omega; \in)$. They don't form indiscernibles over L, if for instance $V = L$, but they do form indiscernibles over Ord. So again let $D = \omega$ in its usual ordering. Let each $M_n = (\aleph_\omega; \in)$. Let $j_{n,n+1}|_{\aleph_1}$ be the identity, and then $j_{n,n+1}(\aleph_m + \alpha) = \aleph_{m+1} + \alpha$.

Hidden in the background there is a lengthy, though ultimately quite routine, induction to show that with k-many alterations quantifiers we cannot tell apart any two sequences of ordinals which have the same length and have the same distance apart in terms of k many iterations of taking limit points. The difficulty in this is not the proof, but rather the statement of the theorem – one of those things where the statement of the theorem is longer than its proof, which is why I am omitting it altogether.